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Constructive method for unbiased extremum seeking of static maps in the presence of delays[☆]

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ABSTRACT

In this paper, we study the unbiased extremum seeking (ES) algorithm for n-dimensional uncertain quadratic static maps in the presence of time-varying measurement delays. For the first time, we present a quantitative analysis of the unbiased ES. We consider delays with a large known constant part and a small time-varying uncertainty. Such delays may arise when measurements together with a time stamp are transmitted to ES controller via communication network. For the quantitative bounds, we assume that the Hessian is uncertain from a known range. By applying a delay-free transformation, explicit quantitative conditions in terms of simple scalar inequalities depending on the tuning parameters are established which ensure the exponential unbiased convergence of the ES system. Moreover, the corresponding results for the classical ES are presented. For globally quadratic maps, our results are semi-global, whereas for locally quadratic static maps, we provide a bound for the region of convergence. Appropriate ES parameters can be found for any large known delay and small enough delay uncertainty. Two numerical examples from the literature illustrate the efficiency of the proposed method.

1. Introduction

Extremum Seeking (ES) is a model-free adaptive control method for optimizing an unknown non-linear output map in real time under the premise of the existence of extremum value [1]. In [2], the rigorous stability analysis of extremum seeking was shown by using averaging theory and singular perturbations. Since then, various ES theoretical results and applications have emerged including semi-global and global ES control [3,4], time-varying ES control [5], ES in the presence of known delays with delay compensation, ES for PDE systems [6–9] and ES by using delay [10]. A detailed survey on ES control can be found in [11]. All the above results are qualitative that work for the static maps provided the dithers are fast enough, but the quantitative bounds on the ES controller parameters are missing.

The existing methods for ES in the presence of delays mostly treat known time-invariant (constant or distributed) delays and employ the known qualitative results on averaging for the time-delay systems (referring to [12] applicable to time-invariant delays). Robustness with respect to constant small delay uncertainties in the output of static quadratic scalar maps was studied in [13] by using [12] as well. To the best of authors' knowledge, there are no ready to be used (even qualitative) results on averaging that can be employed for the stability analysis of the ES algorithms in the presence of fast-varying delay uncertainties (which may be piecewise continuous and without any constraints on the delay derivatives). Such delays appear e.g. in the case of discrete-time delayed measurements (via time-delay modeling [14]). Recent constructive methods for periodic averaging that are based on time-delay [15] or delay-free transformations [16] give important tools for robustness of ES algorithms with respect to unavoidable fast-varying delay uncertainties along with quantitative bounds on the controller parameters.

The first constructive methods for ES with quantitative bounds on the tuning parameters (dither frequencies and amplitudes) were suggested in [17–19] for the quadratic static maps under approximate knowledge of the Hessian by using the time-delay approach (based on time-delay transformation) to averaging [15]. The time-delay approach was extended to sampled-data ES of static quadratic maps [20] and to non-quadratic maps [21]. Bounded extremum seeking of static

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quadratic maps with small uncertain measurement delays and quantitative bounds was studied in [22] via the time-delay approach and in [23] via a delay-free transformation and strict Lyapunov functions.

In the present paper we extend the new constructive method to averaging based on delay-free transformation [16,24] to the suggested in [25,26] unbiased ES with exponential unbiased convergence. Our objective is the first constructive unbiased ES algorithm for the quadratic static maps with numerical bounds on the controller tuning parameters in the presence of time-varying measurement delays. We consider the unbiased ES of static nD quadratic map in the presence of measurement delays with a large known constant part and a small time-varying uncertainty. Such delays may arise when measurements together with a time stamp are transmitted to ES controller via communication network. Due to uncertain delay, accurate estimate of the Hessian seems to be not possible. To achieve efficient quantitative results, we assume that uncertain Hessian is from a known range. Explicit quantitative conditions in terms of simple scalar inequalities are established which ensure the exponential unbiased convergence of the ES system. For globally quadratic maps, our results are semi-global, whereas for locally quadratic static maps, we provide a bound on the region of convergence. Appropriate ES parameters can be found for any large known delay and small enough delay uncertainty. We also present the corresponding results for the classical ES. Two numerical examples illustrate the efficiency of the suggested approach, whereas our results for the classical ES are favorably compared to the existing ones [17–19].

A conference version for 1D static maps in the presence of large known constant delay via a delay-free transformation was presented in [27].

Notation: The notation used in this paper is fairly standard. $\mathbb{N}$ refers to the set of positive integers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices with the induced matrix norm $\|\cdot\|$. The notation $e_i\in\mathbb{R}^n$, $(i=1,2,\dots,n)$ denotes the column vector with a 1 in the ith coordinate and 0's elsewhere. The notation $P>0$ for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The superscript T denotes matrix transposition. For $0<P\in\mathbb{R}^{n\times n}$ and $x\in\mathbb{R}^n$, we write $|x|_P^2=x^TPx$.

Consider a multi-variable quadratic map $Q(\theta(t))$

$$Q(\theta(t))=Q^*+\frac{1}{2}|\theta(t)-\theta^*|_H^2,\ 0\le t\in\mathbb{R}, \tag{1}$$

where $\theta(t)\in\mathbb{R}^n$ is the vector input, $\theta^*\in\mathbb{R}^n$ and $Q^*\in\mathbb{R}$ are uncertain, and H is an unknown Hessian matrix. Without loss of generality, we consider a minimum seeking with $H>0$, where (1) has a minimum value $Q(t)=Q^*$ at $\theta=\theta^*$. The delayed measurements are given by

$$y(t)=\begin{cases}0, & t\in[0,D(0)),\\ Q(\theta(t-D(t))), & t\ge D(0).\end{cases} \tag{2}$$

where $D(t)$ is a time-varying delay.

For simplicity we adopt a quadratic output map following seminal literature [8,9], but our results (as well as results of [8,9]) can be applied to any output map that is a C^3 function in the vicinity of its extremum points, as any such function can be locally approximated by the quadratic one. Differently from the globally quadratic case, where we provide semi-global results in Theorem 1 below, in the locally quadratic we present regional results with a bound on the domain of convergence (see Remark 6 below).

We will employ the unbiased ES algorithm as introduced for the non-delayed case in [25,26]. By using the measurements only, this algorithm constructs an input $\theta(t)$ that exponentially converges to θ^*.

In this paper we consider an uncertain time-varying delay $D(t)$ subject to the following assumption:

Assumption 1. We consider uncertain piecewise-continuous delay of the form

$$D(t)=D_0+\Delta_{\varepsilon\mu}(t),\quad |\Delta_{\varepsilon\mu}(t)|\le\varepsilon\mu,\quad t\ge 0, \tag{3}$$

with the known $D_0\ge 0$ (that may be large) and small $\mu>0$ and $\varepsilon>0$.

The delay uncertainty may appear e.g. due to sampling and delays if the measurements are transmitted to the controller by using communication network [14]. The upper bound on the delay $D(t)$ is given by $D_M=D_0+\varepsilon\mu$, where $D_M\to D_0$ as $\varepsilon\mu\to 0$.

We assume

Assumption 2. The extremum point θ^* to be sought is uncertain from a known ball B with radius σ_0 where its elements satisfy $\theta_i^*\in[\underline{\theta}_i^*,\overline{\theta}_i^*]$, $i=1,\dots,n$ with $\sum_{i=1}^n(\overline{\theta}_i^*-\underline{\theta}_i^*)^2=\sigma_0^2$. The extremum value Q^* is uncertain subject to $|Q^*-Q_0|\le\Delta_Q$ with known Q_0 and $\Delta_Q>0$.

Remark 1. Note that if the delay $D(t)$ is known (constant or continuously differentiable with $\dot{D}(t)\le\bar{d}<1$), an ES algorithm for finding extremum point of a quadratic map is not needed, since θ^* can be easily found in the finite time as follows. Consider for simplicity a known constant delay D_0. Fix any $\varepsilon>0$ and denote $\omega_{i,0}=\frac{2\pi l_i}{\varepsilon}$ with $l_i\in\mathbb{N}$ satisfying

$$l_i\notin\{l_j,\frac{1}{2}(l_j+l_k),l_j+2l_k,l_j+l_k\pm l_m\}, \tag{4}$$

for all distinct i, j, k and m. Define the vector functions

$$S_0(t)=[\sin(\omega_{1,0}t),\dots,\sin(\omega_{n,0}t)]^T,\quad M_0(t)=[2\sin(\omega_{1,0}t),\dots,2\sin(\omega_{n,0})]^T,$$

and matrix function $N(t)$ with elements

$$N_{i,i}(t)=16(\sin^2(\omega_{i,0}t)-\tfrac{1}{2}),\quad N_{i,j}(t)=4\sin(\omega_{i,0}t)\sin(\omega_{j,0}t),\quad i\ne j. \tag{5}$$

Choose any $\theta(0)\in\mathbb{R}^n$. Apply the input

$$\theta(t)=\theta(0)+S_0(t+D_0),\ t\in[0,D_0+2\varepsilon].$$

According to [28], the unknown Hessian and gradient can be found in the finite time $D_0+2\varepsilon$ as follows:

$$\frac{1}{\varepsilon}\int_{D_0}^{D_0+\varepsilon}N(s)y(s)\,ds=H,$$
$$G:=\frac{1}{\varepsilon}\int_{D_0+\varepsilon}^{D_0+2\varepsilon}M_0(s)y(s)\,ds=H(\theta(0)-\theta^*).$$

Then, the extremum point is given by

$$\theta^*=\theta(0)-H^{-1}G.$$

We thank the anonymous reviewer for bringing this idea to our attention. Note that a similar idea of the Hessian and gradient estimate on the initial time interval (for $D(t)=0$) was suggested in [29], whereas the estimate of the Hessian only and known constant delay was suggested in [27].

1.1. Unbiased ES in the presence of uncertain delay

Define the perturbation and demodulation signals as

$$\begin{aligned}S(t)&=[a_1\sin(\omega_1 t),\dots,a_n\sin(\omega_n t)]^T,\\ M(t)&=[\tfrac{2}{a_1}\sin(\omega_1 t),\dots,\tfrac{2}{a_n}\sin(\omega_n t)]^T,\end{aligned} \tag{6}$$

where amplitudes a_i are non-zero real numbers and the frequencies have a form

$$\omega_i=\frac{2\pi i}{\varepsilon},\quad \varepsilon>0,\ i=1,\dots,n. \tag{7}$$

It is worth noting that Assumption 1 dictates that the unknown delay term $\Delta_{\varepsilon\mu}(t)$ should be much smaller than the periods of $S(t)$ and $M(t)$. Without this condition, the influence of the delay would become too significant to preserve the robustness of the performance under ES controller considered below. For the network-based implementation this means that maximum transmission interval (sampling) should be essentially smaller than the dither period whereas the sum of network-induced and computational delays should be almost constant to preserve the performance.

1.1.1. Unbiased ES algorithm

Inspired by [25,26], we employ an unbiased ES algorithm. Define an exponentially decaying function $\alpha(t)$ as

$$\alpha(t) = \alpha_0 e^{-\lambda t}, \quad 0 \le t \in \mathbb{R}, \tag{8}$$

where α_0 and λ are tuning positive parameters. Choose a scalar adaptation gain $k > 0$. Given $\varepsilon > 0$ (its numerical value will be found later from conditions of Theorem 1) and the corresponding frequencies ω_i given by (7), define the input $\theta(t)$ with a real-time estimate $\hat{\theta}(t)$ of θ^* as follows:

$$\theta(t) = \begin{cases} \theta(0), & 0 \le t < D_M, \\ \hat{\theta}(t) + \alpha(t + D_0)S(t + D_0), & t \ge D_M, \end{cases} \tag{9}$$

with $S(t)$ and $\alpha(t)$ given by (6) and (8), respectively.

As illustrated in Fig. 1, the unbiased ES algorithm has the following form

$$\begin{aligned} &\dot{\hat{\theta}}(t) = -\tfrac{kM(t)}{\alpha(t)}[y(t) - \eta(t)], && t \ge D_M, \\ &\hat{\theta}(t) = \theta(0) - \alpha(D_M + D_0)S(D_M + D_0), && t \in [0, D_M], \end{aligned} \tag{10}$$

where the high-pass filter state $\eta(t)$ is governed by

$$\begin{aligned} &\dot{\eta}(t) = -\omega_h \eta(t) + \omega_h y(t), \quad t \ge D_M, \\ &\eta(D_M) = Q_0, \end{aligned} \tag{11}$$

with some $\omega_h > 0$.

Our objective is to derive constructive quantitative conditions for the semi-global exponential convergence in the presence of the time-varying delays: given an uncertain time-varying delay $D(t)$ and a ball for the initial state $\theta(0)$, find appropriate exponentially decaying gains, high-pass filters and perturbation frequencies that guarantee the exponential convergence of the estimation error. The results presented below establish conditions (in terms of simple scalar inequalities) which ensure exponential stability of the estimation error and provide quantitative bounds on the controller parameters.

Remark 2. When the Hessian H is unknown, an alternative version of the unbiased ES algorithm with a delay compensator of the known part D_0 of the delay (similar to [30] with constant delay) can be used. Let frequencies $\omega_{i,0}$ subject to (4), vector functions $M_0(t), S_0(t)$ and matrix function $N(t)$ be defined as in Remark 1. Consider $\theta(t) = \hat{\theta}(t) + \alpha(t + D_0)S_0(t + D_0)$ for $t \ge D_M$ and $\theta(t) = \theta(0)$ for $t \in [0, D_M]$. The unbiased ES has a form

$$\begin{aligned} \hat{H}(t) &= \tfrac{N(t)}{\alpha(t)}[y(t) - \eta(t)], \\ \dot{\hat{\theta}}(t) &= -\tfrac{kM_0(t)}{\alpha(t)}[y(t) - \eta(t)] - \underbrace{k\hat{H}(t)[\hat{\theta}(t) - \hat{\theta}(t - D_0)]}_{\text{delay compensation term}}, \\ \dot{\eta}(t) &= -\omega_h \eta(t) + \omega_h y(t). \end{aligned} \tag{12}$$

Here $\hat{H}$ is an estimator of the Hessian H. Our stability analysis presented below can be applied here leading to new qualitative result - exponential convergence of θ to θ^* for appropriate a_i, k, ω_h, λ and small enough ε and μ. However, the delay compensation term with $\hat{H}(t)$ causes additional perturbations to be bounded and leads to very small quantitative bounds on ε.

Note that our constructive method for finding ε (related to the dither frequencies according to (7)) leads to $\Delta_{\varepsilon\mu}$ of the order of $O(\varepsilon\mu)$ (similarly to [13] for constant delay uncertainty and our preliminary result [31] via the time-delay approach). Since the upper bound of ε depends on H, estimation of H from the measurements on the initial interval similar to Remark 1 below seems to be not possible in the presence of delay uncertainty. Therefore, to have efficient quantitative bounds on ε, we assume that uncertain H is from the known range:

Assumption 3. The Hessian H is uncertain and subject to $H_m I \le H \le H_M I$, where H_m and H_M are two known positive scalars.

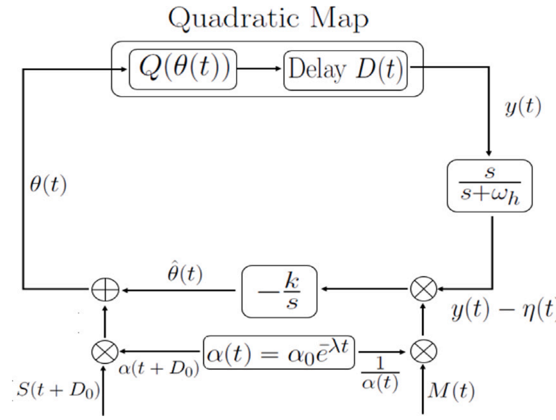


Fig. 1. Unbiased ES algorithm with time-varying measurement delay $D(t)$.

Following [26,30,32] (see Lemma 2 in the Appendix), given a desirable decay rate $\lambda > 0$, we choose positive tuning parameters k and ω_h that satisfy the following inequalities:

$$kH_m > \lambda, \quad \omega_h > 2\lambda, \quad e^{-1} \ge kH_M D_0. \tag{13}$$

The first two inequalities mean that adaptation (learning) rate should surpass the decay rate λ of the perturbation (exploration) signal. The third inequality yields that for given H_m and H_M and a larger D_0, a smaller k should be chosen, which leads to a smaller decay rate λ.

Remark 3. In contrast to the conventional ES design [2], the unbiased ES algorithm (10), (11) is equipped with high-pass filter, and exponentially decaying perturbation and growing demodulation signals. Despite the exponentially growing signal in the algorithm, the high-pass filter has a crucial role in guaranteeing the unbiased convergence by ensuring the exponential decay of $y(t) - \eta(t)$ to zero at the rate of 2λ (see (67) below).

Define the estimation errors $\tilde{\theta}(t)$ and $\tilde{\eta}(t)$ as

$$\begin{aligned} \tilde{\theta}(t) &= \hat{\theta}(t) - \theta^*, \quad t \ge 0, \\ \tilde{\eta}(t) &= \eta(t) - Q^*, \quad t \ge 0. \end{aligned} \tag{14}$$

We will further present (10) as

$$\begin{aligned} \dot{\tilde{\theta}}(t) = -\tfrac{kM(t)}{\alpha(t)} \Big[&\tfrac{1}{2}|\tilde{\theta}(t - D(t)) + \alpha(t - \Delta_{\varepsilon\mu}(t)) \times \\ &S(t - \Delta_{\varepsilon\mu}(t))|_H^2 - \tilde{\eta}(t) \Big], \quad t \ge D_M, \\ \tilde{\theta}(t) = \theta(0) - \theta^* - &\alpha(D_M + D_0)S(D_M + D_0), \quad t \in [0, D_M], \end{aligned} \tag{15}$$

and

$$\begin{aligned} &\dot{\tilde{\eta}}(t) = -\omega_h \tilde{\eta}(t) + \omega_h [y(t) - Q^*], && t \ge D_M, \\ &\tilde{\eta}(t) = Q_0 - Q^*, && t = D_M. \end{aligned} \tag{16}$$

Taking into account

$$\begin{aligned} &\tilde{\theta}(t - D(t)) = \tilde{\theta}(t - D_0) - \textstyle\int_{t-D(t)}^{t-D_0} \dot{\tilde{\theta}}(s)\, ds, \\ &\alpha(t - \Delta_{\varepsilon\mu}(t))S(t - \Delta_{\varepsilon\mu}(t)) = \alpha(t)S(t) - \textstyle\int_{t-\Delta_{\varepsilon\mu}(t)}^{t} \tfrac{d}{ds}\alpha(s)S(s)\, ds, \end{aligned}$$

system (15), (16) can be further expressed as

$$\begin{aligned} &\dot{\tilde{\theta}}(t) = -\tfrac{kM(t)}{\alpha(t)} \Big[\tfrac{1}{2}|\tilde{\theta}(t - D_0) + \alpha(t)S(t)|_H^2 - \tilde{\eta}(t) \Big] + w(t), \quad t \ge D_M, \\ &\dot{\tilde{\eta}}(t) = -\omega_h \tilde{\eta}(t) + \omega_h [y(t) - Q^*], \quad t \ge D_M, \end{aligned} \tag{17}$$

where

$$\begin{aligned} w(t) &= -\tfrac{1}{2}\tfrac{kM(t)}{\alpha(t)}|\Gamma(t)|_H^2 - \tfrac{kM(t)}{\alpha(t)}[\tilde{\theta}(t - D_0) + \alpha(t)S(t)]^T H \Gamma(t), \\ \Gamma(t) &= -\underbrace{\int_{t-D(t)}^{t-D_0} \dot{\tilde{\theta}}(s)\, ds}_{O(d) = O(\varepsilon\mu)} - \underbrace{\int_{t-\Delta_{\varepsilon\mu}(t)}^{t} \left(\frac{d}{ds}\alpha(s)S(s) \right) ds}_{O(\frac{d}{\varepsilon}) = O(\mu)}. \end{aligned} \tag{18}$$

The order of $\Gamma(t)$ follows from $\dot{S}(s) = O(\varepsilon)$ and $\dot{\tilde{\theta}}(s) = O(1)$. One can easily show that

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} M(s)ds = 0, \qquad \frac{1}{\varepsilon}\int_t^{t+\varepsilon} M(s)S^T(s)ds = I_n, \\ \frac{1}{\varepsilon}\int_t^{t+\varepsilon} M(s)S^T(s)HS(s)ds = 0. \tag{19}$$

Thus, from (19), (17) and (18), the averaged system corresponding to (15) with $\mu \to 0$ is given by

$$\dot{\tilde{\theta}}_{av}(t) = -kH\tilde{\theta}_{av}(t - D_0), \tag{20}$$

which is exponentially stable for kH subject to (13) (see Lemma 2 in the Appendix).

Remark 4. Compared to the classical ES, high-pass filter and exponential perturbation/demodulation signals in the unbiased ES lead to more challenging averaging-based stability analysis. Additional restrictions (13) on the tuning parameters are needed along with the convergence proof of the additional error term $\tilde{\eta}(t)$. To manage with the exponential convergence we assume and later prove that $|\tilde{\theta}(t)| \leq \sigma e^{-\lambda t}$ (see (65)) for some $\sigma > 0$ instead of $|\tilde{\theta}(t)| \leq \sigma$ for the classical ES.

1.1.2. Stability analysis

We will perform stability analysis of (15), (16) (also (17)) via a delay-free transformation inspired by [16]. Define

$$\begin{aligned}
\rho_1(t) &:= -\frac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)[-kM(s)S^T(s)H + kH]ds,\\
\rho_2(t) &:= -\frac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)[kM(s)]ds,\\
\rho_3(t) &:= -\frac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)\left[-\frac{1}{2}kM(s)S^T(s)HS(s)\right]ds,\\
\rho_4(t) &:= -\frac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)\left[-\frac{1}{2}kM(s)\right]ds.
\end{aligned} \tag{21}$$

The following Lemma provides accurate upper-bounds on each function in (21):

Lemma 1. *The functions $\rho_1(t), \dots, \rho_4(t)$ are bounded as follows for $t \geq 0$:*

$$\begin{aligned}
\|\rho_1(t)\| &\leq \varepsilon\bar{\rho}_1 := \frac{\varepsilon kH_M}{2\pi}\left[\frac{1}{2}\sum_{i=1}^n\frac{1}{i} + \sum_{1\leq i\neq j\leq n}\frac{a_j}{a_i}\left(\frac{1}{|i-j|}+\frac{1}{i+j}\right)\right],\\
|\rho_2(t)| &\leq \varepsilon\bar{\rho}_2 := \frac{\varepsilon k}{\pi}\left(\sum_{i=1}^n\frac{1}{ia_i}\right),\\
|\rho_3(t)| &\leq \varepsilon\bar{\rho}_3 := \varepsilon kH_M\left[\sum_{j=1}^n a_j^2\right]\left(\sum_{i=1}^n\frac{1}{a_i}\left(\frac{1}{6}+\frac{1}{8\pi i}\left[1+\frac{1}{4\pi i}\right]\right)^{\frac{1}{2}}\right),\\
|\rho_4(t)| &\leq \varepsilon\bar{\rho}_4 := \frac{\varepsilon k}{2\pi}\left(\sum_{i=1}^n\frac{1}{ia_i}\right).
\end{aligned} \tag{22}$$

Proof. See Appendix. □

Differentiating (21), we have

$$\dot{\rho}_1(t) = -kM(t)S^T(t)H + kH, \quad \dot{\rho}_2(t) = kM(t), \\ \dot{\rho}_3(t) = -\frac{1}{2}kM(t)S^T(t)HS(t), \quad \dot{\rho}_4(t) = -\frac{1}{2}kM(t). \tag{23}$$

Introduce the delay-free transformation $z(t)$ as in (25) where

$$G(t) = \begin{cases} 0, & t \in [0, D_M],\\ \rho_1(t)\tilde{\theta}(t-D_0) + \rho_2(t)\alpha^{-1}(t)\tilde{\eta}(t) & \\ +\rho_3(t)\alpha(t) + \rho_4(t)\alpha^{-1}(t)|\tilde{\theta}(t-D_0)|_H^2, & t > D_M. \end{cases} \tag{24}$$

Consider the following transformation

$$z(t) = \tilde{\theta}(t) - G(t), \quad t \geq 0. \tag{25}$$

By employing (25) and (17), we obtain

$$\dot{z}(t) = -kHz(t-D_0) + Y(t) + w(t), \quad t \geq D_M, \\ z(t) = \theta(0) - \theta^* - \alpha(D_M + D_0)S(D_M + D_0), \quad t \in [0, D_M], \tag{26}$$

where

$$Y(t) = -kHG(t-D_0) - \rho_1(t)\dot{\tilde{\theta}}(t-D_0) - \rho_2(t)\left(\frac{d\alpha^{-1}(t)}{dt}\tilde{\eta}(t) + \alpha^{-1}(t)\dot{\tilde{\eta}}(t)\right) \\ -\rho_3(t)\dot{\alpha}(t) - \rho_4(t)\frac{d\alpha^{-1}(t)}{dt}|\tilde{\theta}(t-D_0)|_H^2 - 2\rho_4(t)\alpha^{-1}(t)\tilde{\theta}^T(t-D_0)H\dot{\tilde{\theta}}(t-D_0). \tag{27}$$

The terms $G(t)$, $Y(t)$ and $w(t)$ are of the order of $O(\max\{\varepsilon,\mu\})$ provided $\tilde{\theta}(t), \dot{\tilde{\theta}}(t)$ and $\tilde{\eta}(t), \dot{\tilde{\eta}}(t)$ are of the order of $O(1)$. Hence, (26) can be regarded as an $O(\varepsilon)$ perturbation of the averaged system (20) for small enough ε.

The bound on z will be found by utilizing the variation of constants formula for the time-delay system (26). Then the bound on $\tilde{\theta}$ will be found by employing (25). We are in a position to formulate our main result on semi-global exponential stability of the estimation error system (15):

Theorem 1. *Let Assumptions 1–3 hold. Given any $D_0 > 0$, let ω_h, λ and k satisfy (13). Consider the estimation error system (15), (16) with uncertain delay $D(t)$. The functions $M(t)$, $S(t)$ and $\alpha(t)$ are defined by (6), (8) with tuning parameters α_0 and a_i. Let $\bar{\rho}_j$, $j = 1,2,3,4$ are the bounds defined in (22). Given any $\sigma_0 > 0$, let the tuning parameter σ is subject to*

$$[\sigma_0 + \alpha_0 e^{-2\lambda D_0}\sqrt{\textstyle\sum_{i=1}^n a_i^2}] \\ \times\left[e^{2kH_mD_0} + \max\{kH_M D_0 e^{3\lambda D_0}, \frac{H_M}{H_m}e^{2kH_mD_0}[e^{kH_mD_0}-1]\}\right] < \sigma. \tag{28}$$

Let there exist ε^* and μ^* that satisfy

$$\left[\sigma_0 + \alpha_0 e^{-\lambda(2D_0+\varepsilon^*\mu^*)}\sqrt{\textstyle\sum_{i=1}^n a_i^2}\right]e^{kH_m(2D_0+\varepsilon^*\mu^*)} + \varepsilon^*\Delta_G \\ +(\varepsilon^*\Delta_Y + \mu^*\Delta_w)\left[D_0 + \frac{1}{(kH_m-\lambda)}\right]e^{\lambda D_0} + \left[\sigma_0 + \alpha_0 e^{-\lambda(2D_0+\varepsilon^*\mu^*)}\sqrt{\textstyle\sum_{i=1}^n a_i^2}\right] \\ \times\max\{kH_M D_0 e^{\lambda(3D_0+\varepsilon^*\mu^*)}, \frac{H_M}{H_m}e^{kH_m(2D_0+\varepsilon^*\mu^*)}[e^{kH_mD_0}-1]\} < \sigma, \tag{29}$$

where

$$\begin{aligned}
\Delta_G &= \bar{\rho}_1\sigma e^{\lambda D_0} + \bar{\rho}_2\frac{\sigma_\eta}{\alpha_0} + \bar{\rho}_3\alpha_0 + \bar{\rho}_4\frac{\sigma^2 H_M}{\alpha_0}e^{2\lambda D_0},\\
\sigma_\eta &= \Delta_Q e^{\omega_h(D_0+\varepsilon^*\mu^*)} + \frac{\sigma_y\omega_h}{\omega_h - 2\lambda},\\
\sigma_y &= \frac{H_M}{2}\left(\sigma e^{\lambda(D_0+\varepsilon^*\mu^*)} + \alpha_0 e^{\lambda\varepsilon^*\mu^*}\sqrt{\textstyle\sum_{i=1}^n a_i^2}\right)^2,\\
\Delta_Y &= kH_M\Delta_G e^{\lambda D_0} + \bar{\rho}_1\Delta e^{\lambda D_0} + \bar{\rho}_2\alpha_0^{-1}\lambda\sigma_\eta + \bar{\rho}_2\alpha_0^{-1}\omega_h(\sigma_\eta+\sigma_y)\\
&\quad +\bar{\rho}_3\alpha_0\lambda + \bar{\rho}_4 H_M\alpha_0^{-1}\sigma e^{2\lambda D_0}(2\Delta + \lambda\sigma),\\
\Delta &= \frac{2k}{\alpha_0}[\sigma_\eta + \sigma_y]\sqrt{\textstyle\sum_{i=1}^n \frac{1}{a_i^2}},\\
\Delta_w &= \frac{kH_M}{\alpha_0}\Delta_\Gamma\left[\mu^*\Delta_\Gamma + 2\left(\sigma e^{\lambda D_0} + \alpha_0\sqrt{\textstyle\sum_{i=1}^n a_i^2}\right)\right]\sqrt{\textstyle\sum_{i=1}^n \frac{1}{a_i^2}},\\
\Delta_\Gamma &= \varepsilon^*\Delta e^{\lambda(D_0+\varepsilon^*\mu^*)} + \varepsilon^*\lambda\alpha_0\sqrt{\textstyle\sum_{i=1}^n a_i^2}e^{\lambda\varepsilon^*\mu^*} + 2\pi\alpha_0\sqrt{\textstyle\sum_{i=1}^n (ia_i)^2}e^{\lambda\varepsilon^*\mu^*}.
\end{aligned} \tag{30}$$

Then, for all $\varepsilon \in (0,\varepsilon^*]$, $\mu \in [0,\mu^*)$ and $\tilde{\theta}(0) \leq \sigma_0$ the following inequalities hold for $D(t)$ subject to (3):

$$|\tilde{\theta}(t)| < \sigma e^{-\lambda t}, \quad |\tilde{\eta}(t)| \leq \sigma_\eta e^{-2\lambda t}, \quad t \geq D_M, \tag{31}$$

i.e. meaning that the estimation error system (15), (16) is exponentially stable with a decay rate λ. Moreover, for any D_0 and σ_0, (28) and (29) are always feasible for small enough $\lambda, k > 0$, ε^*, μ^* and appropriate $\sigma > \sigma_0$.

Proof. See Appendix. □

Remark 5. The conditions of Theorem 1 impose clear restrictions on the decay rate λ. More precisely, larger delay D_0 and larger μ place a limitation on the decay rate λ, and lead to smaller ε^*, i.e. slower convergence and higher dither frequency. The same holds true for the classical ES and conditions of Corollary 1 in the next section.

Remark 6. In cases where the map is non-quadratic, but is C^3 function, it can be approximated as the quadratic one (1) in a vicinity of θ^* (i.e. for $|\theta(t)-\theta^*| \leq \sigma_1$ with some known σ_1). Then the results of Theorem 1 hold true with $\sigma = \sigma_1 - \alpha_0\sqrt{\sum_{i=1}^n a_i^2}$. By arguments of Theorem 1, it can be shown that if (28), (29) hold, then for all $\varepsilon \leq \varepsilon^*$, $\mu \leq \mu^*$ and $D(t)$ subject to (3) all the solutions of (15), (16) with $|\tilde{\theta}(0)| \leq \sigma_0$ satisfy (31) (meaning regional exponential stability). Similar results hold true for the classical ES in the next section.

1.2. Classical ES: uncertain time-varying delay

In this subsection, we consider the classical ES algorithm in the presence of uncertain time-varying delay. Consider the ES algorithm (9)–(10) with $\alpha(t) \equiv 1$ and $\eta(t) \equiv 0$ for $t \geq 0$, which results in the following equation for the real-time estimate $\hat{\theta}(t)$ of $\theta(t)$:

$$\begin{aligned} &\dot{\hat{\theta}}(t) = -kM(t)\left(Q^* + \tfrac{1}{2}|\hat{\theta}(t-D(t)) - \theta^* + S(t)|_H^2\right), \quad t \geq D_M, \\ &\hat{\theta}(t) = \theta(0) - S(D_M + D_0), \qquad t \in [0, D_M], \end{aligned} \tag{32}$$

where $S(t)$, $M(t)$ are defined in (6) and k is a positive gain. Then, the estimation error $\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*$ is governed by

$$\begin{aligned} &\dot{\tilde{\theta}}(t) = -kM(t)\left(Q^* + \tfrac{1}{2}|\tilde{\theta}(t-D(t)) + S(t)|_H^2\right), \quad t \geq D_M, \\ &\tilde{\theta}(t) = \theta(0) - \theta^* - S(D_M + D_0), \qquad t \in [0, D_M]. \end{aligned} \tag{33}$$

Note that (33) with $D(t) = 0$ is the estimation error in the classical ES algorithm which was analyzed in [18,19] via the time-delay approach. The following corollary follows by arguments of Theorem 1:

Corollary 1. *Let Assumptions 1–3 hold, and k satisfy $e^{-1} \geq kH_M D_0$. Consider the estimation error system (33) with uncertain delay $D(t)$. Let $\bar{\rho}_j$, $j = 1,2,3,4$ are the bounds defined in (22). The functions $M(t)$ and $S(t)$ are defined by (6) with tuning parameters a_i. Given any $\sigma_0 > 0$, let the tuning parameter σ satisfy*

$$[\sigma_0 + \sqrt{\textstyle\sum_{i=1}^n a_i^2}][e^{2kH_m D_0} + \tfrac{H_M}{H_m} e^{2kH_m D_0}[e^{kH_m D_0} - 1]] < \sigma. \tag{34}$$

Let there exist ε^* and μ^* that satisfy

$$\begin{aligned} &[\sigma_0 + \sqrt{\textstyle\sum_{i=1}^n a_i^2}]e^{kH_m(2D_0 + \varepsilon^*\mu^*)}\left(1 + \tfrac{H_M}{H_m}[e^{kH_m D_0} - 1]\right) \\ &\qquad + \varepsilon^*\Delta_G + (\varepsilon^*\Delta_Y + \mu^*\Delta_w)\left[D_0 + \tfrac{1}{kH_m}\right] < \sigma, \end{aligned} \tag{35}$$

where

$$\begin{aligned} &\Delta_G = \bar{\rho}_1\sigma + \bar{\rho}_2(Q_0 + \Delta_Q) + \bar{\rho}_3 + \bar{\rho}_4\sigma^2 H_M, \\ &\sigma_y = \tfrac{H_M}{2}\left(\sigma + \sqrt{\textstyle\sum_{i=1}^n a_i^2}\right)^2, \quad \Delta_Y = kH_M\Delta_G + \bar{\rho}_1\Delta + 2\bar{\rho}_4 H_M \Delta\sigma, \\ &\Delta = 2k\left[Q_0 + \Delta_Q + \sigma_y\right]\sqrt{\textstyle\sum_{i=1}^n \tfrac{1}{a_i^2}}, \\ &\Delta_w = kH_M\Delta_\Gamma\left[\mu^*\Delta_\Gamma + 2\left(\sigma + \sqrt{\textstyle\sum_{i=1}^n a_i^2}\right)\right]\sqrt{\textstyle\sum_{i=1}^n \tfrac{1}{a_i^2}}, \\ &\Delta_\Gamma = \varepsilon^*\Delta + 2\pi\sqrt{\textstyle\sum_{i=1}^n (ia_i)^2}. \end{aligned} \tag{36}$$

Then, for all $\varepsilon \in (0, \varepsilon^*]$, $\mu \in [0, \mu^*)$ and $\tilde{\theta}(0) \leq \sigma_0$ the solutions of (33) satisfy $|\tilde{\theta}(t)| < \sigma$, $t \geq D_M$, provided that $D(t)$ satisfies (3). Furthermore, these solutions are exponentially attracted to the set

$$\begin{aligned} &\Theta = \left\{\tilde{\theta}(t) \in \mathbb{R}^n : |\tilde{\theta}(t)| < \Delta_\theta\right\}, \\ &\Delta_\theta = \varepsilon\left(\Delta_G + \tfrac{\Delta_Y}{kH_m} + D\Delta_Y\right) + \mu\left(\tfrac{\Delta_w}{kH_m} + D\Delta_w\right), \end{aligned} \tag{37}$$

with a decay rate kH_m. Moreover, for any D_0 and σ_0, (34) and (35) are always feasible for small enough ε^*, μ^* and k and appropriate $\sigma > \sigma_0$.

2. Examples

To illustrate the efficiency of our approach, we will consider below two examples from the literature [9,33].

2.1. GPS-denied 2D autonomous vehicle

Consider an autonomous vehicle in an environment without GPS orientation. The vehicle has a velocity-controlled model (single integrator)

$$\dot{\theta}(t) = u(t), \tag{38}$$

where $\theta(t) \in \mathbb{R}^2$ is the state (position of the vehicle), $u(t) \in \mathbb{R}^2$ is the control input, and the measurement $y(t)$ is defined by (2) with Q given by (1) and delay $D(t)$ of the form (3). Let Assumptions 1–3 hold. By using the measurements only, our objective is to design a control law $u(t)$ that drives the position $\theta(t)$ to the extremum point θ^* for $t \to \infty$.

To construct $u(t)$, we differentiate (9). It is seen that the unbiased ES algorithm leads to the exponential convergence of $\theta(t)$ to θ^*:

$$\begin{aligned} &\dot{\theta}(t) = -\tfrac{kM(t)}{\alpha(t)}[y(t) - \eta(t)] + \dot{\alpha}(t + D_0)S(t + D_0) \\ &\qquad\qquad + \alpha(t + D_0)\dot{S}(t + D_0), \quad t \geq D_M, \\ &\theta(t) = \theta(0), \qquad t \in [0, D_M], \end{aligned} \tag{39}$$

with the high-pass filter (11).

Following [17–19], we consider the $2D$ quadratic map (1) with

$$\theta^* = [0,0]^T, \; Q_0 = 0, \; H = diag\{2,2\}. \tag{40}$$

We assume that Q^* and H are uncertain satisfying Assumptions 2, 3, and we consider the following cases:

$$\text{Case I: } \Delta_Q = 0, \quad H_m = 2, \quad H_M = 2. \tag{41}$$

$$\text{Case II: } \Delta_Q = 0.1, \quad H_m = 1.9, \quad H_M = 2.1. \tag{42}$$

$$\text{Case III: } \Delta_Q = 1, \quad H_m = 1.1, \quad H_M = 3. \tag{43}$$

We choose the following tuning parameters for all ES algorithms:

$$a_1 = a_2 = 0.2, \quad 2\omega_1 = \omega_2 = \frac{4\pi}{\varepsilon},$$

whereas σ is tuned to achieve a larger ε^*.

Let $D(t)$ be an uncertain time-varying delay satisfying (13). Choose $D(0) = D_0$, and D_0, μ, k will be selected later.

ES with exponential stability: Consider the unbiased ES algorithm (10), (11) with $\alpha_0 = 1$ and $\omega_h = 0.03$. For $\sigma_0 = \sqrt{2}$, the maximum values of ε^* that follow from Theorem 1 are shown in Table 1. Note that for $D_0 \to \infty$ the decay rate approaches zero. Thus, for (42) with $D_0 = 50$ from (13) we obtain that maximum λ should be less than 0.00212.

We further provide simulation of the unbiased ES algorithm (10), (11) for case III (43), with time-varying delay $D(t) = 2 + \varepsilon\mu\sin(t)$, and $\lambda = 0.007$, $\varepsilon = 3.4 \cdot 10^{-3}$, $\mu = 10^{-3}$, and initial condition $\theta(0) = [1,1]^T$. The plot of $|\theta(t)|$ is presented in Fig. 2. It is seen that $|\theta(t)|$ exponentially converges to zero, which demonstrates the efficiency of the method.

We also present a $3D$ plot with color mapping to visualize the sensitivity of ε^* with respect to uncertainties in μ and H_M, with fixed parameters $D_0 = 2$, $\Delta_Q = 1$, $H_m = 1.1$, $\sigma = 3$ and $k = \lambda = 0.01$. The plot of ε^* and its 2D contour plot are presented in Fig. 3.

Let us consider the scenario where $y(t)$ is not quadratic map, but can be approximated as a quadratic map (1) with (40) in a vicinity of θ^* given by $|\theta(t) - \theta^*| \leq \sigma_1 = 1.2$. We apply unbiased ES (15), (16) to Case III (see (43)) and $D_0 = 2$, $\omega_h = 0.03$, $k = \lambda = 0.001$. By using Remark 6, we find $\sigma_0 = 0.33$, $\varepsilon^* = 1.69 \cdot 10^{-2}$ and $\mu^* = 5 \cdot 10^{-4}$ meaning that for $\varepsilon \leq 1.69 \cdot 10^{-2}$ and $\mu \leq 5 \cdot 10^{-4}$, all solutions of (15), (16) with $|\tilde{\theta}(0)| \leq \sigma_0 = 0.33$ satisfy $|\tilde{\theta}(t)| < 0.92e^{-0.001t}$ for $t > 2$.

ES with practical stability: Consider the classical ES algorithm (32). In this case the solutions converge to a small attractive ball with a decay rate $\lambda = kH_m$. The maximum values of ε^* that follow from Corollary 1 are shown in Table 2. It is seen that our results essentially enlarge the value of ε^* (decrease the dither frequency) compared to the previous constructive results via time-delay transformation [17,19].

Table 1
Example 2.1: maximum ε^* for $\sigma_0 = \sqrt{2}$.

Unbiased ES: Uncertain delay	D_0	μ	σ	k	λ	ε^*
Theorem 1, $\Delta_Q = 0$, $H_m = H_M = 2$	0	0	2.7	0.01	0.01	$2.6 \cdot 10^{-2}$
Theorem 1, $\Delta_Q = 0.1$, $H_m = 1.9$, $H_M = 2.1$	2	10^{-3}	3.1	0.01	0.01	$1.43 \cdot 10^{-2}$
Theorem 1, $\Delta_Q = 1$, $H_m = 1.1$, $H_M = 3$	2	10^{-3}	3.1	0.01	0.01	$0.34 \cdot 10^{-2}$
Theorem 1, $\Delta_Q = 1$, $H_m = 1.1$, $H_M = 3$	50	10^{-3}	4.9	0.002	0.002	$0.95 \cdot 10^{-2}$

Table 2
Example 2.1: maximum ε^* for $\sigma_0 = \sqrt{2}$.

Classical ES: Uncertain delay	D_0	μ	σ	k	$\lambda = kH_m$	ε^*
[19], $\Delta_Q = 0$, $H_m = H_M = 2$	0	0	$2\sqrt{2}$	0.01	0.02	$0.17 \cdot 10^{-1}$
[17], $\Delta_Q = 0$, $H_m = H_M = 2$	0	0	$2\sqrt{2}$	0.01	0.02	$0.42 \cdot 10^{-1}$
Corollary 1, $\Delta_Q = 0$, $H_m = H_M = 2$	0	0	3	0.01	0.02	$1.09 \cdot 10^{-1}$
Corollary 1, $\Delta_Q = 0.1$, $H_m = 1.9$, $H_M = 2.1$,	2	10^{-3}	3.2	0.01	0.019	$0.634 \cdot 10^{-1}$
Corollary 1, $\Delta_Q = 1$, $H_m = 1.1$, $H_M = 3$,	2	10^{-3}	3.4	0.01	0.011	$0.148 \cdot 10^{-1}$
Corollary 1, $\Delta_Q = 1$, $H_m = 1.1$, $H_M = 3$,	50	10^{-3}	5	0.002	0.0022	$0.377 \cdot 10^{-1}$

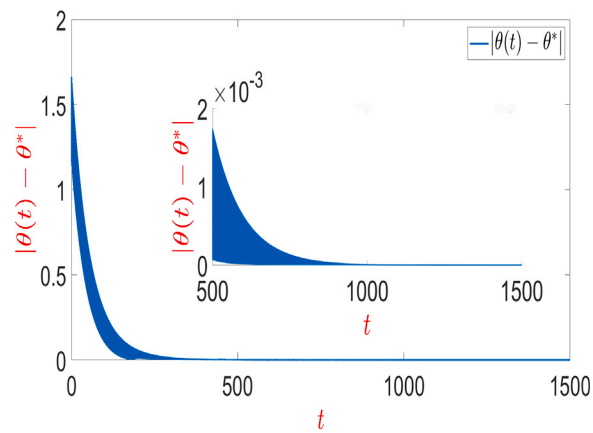
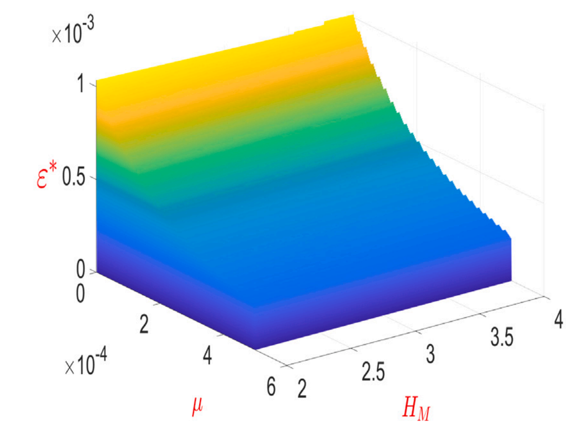
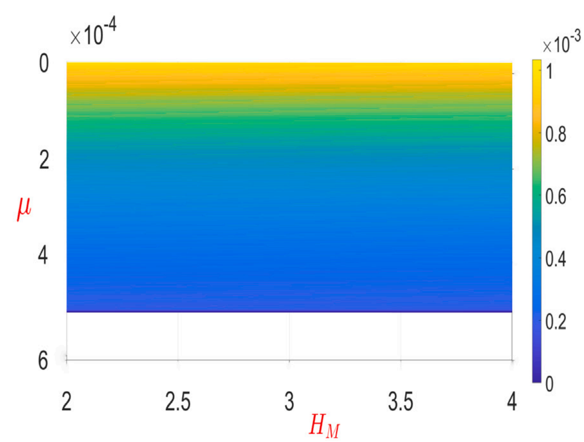


Fig. 2. Section 2.1, Unbiased ES (10), (11): plot of $|\theta(t)|$.



(a) 3D plot of ε^* with respect to μ and H_M.



(b) 2D Contour plot of the above 3D plot.

Fig. 3. Section 2.1, visualization of ε^* as a function of delay uncertainties and the Hessian's upper bound. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

2.2. Source seeking

The goal of the source seeking is to guide a vehicle (we consider here the single integrator model (38)) in the GPS denied environment to a source, which is located in the extremum of the map. Following [6,9], we consider the $2D$ quadratic map (1) with

$$\theta^* = [0, 1]^T, \quad Q_0 = 1, \quad H = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}. \tag{44}$$

Note that the eigenvalues of H are 0.7639 and 5.2361. As explained in 1.1, we assume Q^* and H are uncertain satisfying Assumptions 2, 3, and we consider the following cases:

$$\text{Case I:}\ \Delta_Q = 0.1, \quad H_m = 0.61, \quad H_M = 5.38. \tag{45}$$

$$\text{Case II:}\ \Delta_Q = 1, \quad H_m = 0.5, \quad H_M = 7. \tag{46}$$

We select the parameters as follows:

$$a_1 = a_2 = 0.5, \quad 2\omega_1 = \omega_2 = \frac{4\pi}{\varepsilon}, \tag{47}$$

whereas σ is tuned to achieve a larger ε^*.

ES with exponential stability: Consider the unbiased ES algorithm (10), (11) with $\alpha_0 = 1$ and $\omega_h = 0.03$. The maximum values of ε^* that follow from Theorem 1 are shown in Table 3. Note that here for (46) with $D_0 = 40$ we obtain from (13) that $\lambda < 0.00065$.

We further provide simulation of the unbiased ES algorithm (10), (11) for case II (46), with time-varying delay $D(t) = 2 + \varepsilon\mu\sin(t)$, and $\lambda = 0.007$, $\varepsilon = 0.76 \cdot 10^{-4}$, $\mu = 0.1 \cdot 10^{-4}$, and initial condition $\theta(0) = [1, 1]^T$. The plot of $|\theta(t) - \theta^*|$ is presented in Fig. 4. It is seen that $|\theta(t) - \theta^*|$ exponentially converges to zero, which demonstrates the efficiency of the method.

We also present a $3D$ plot with color mapping to visualize the sensitivity of ε^* with respect to uncertainties in μ and H_M, with fixed parameters $D_0 = 2$, $\Delta_Q = 1$, $H_m = 0.5$, $k = 0.015$, $\sigma = 3.8$ and $\lambda = 0.007$. The plot of ε^* and its 2D contour plot are shown in Fig. 5.

Table 3
Example 2.2: maximum ε^* for $\sigma_0 = 1$.

Unbiased ES: Uncertain delay	D_0	μ	σ	k	λ	ε^*
Theorem 1, $\Delta_Q = 0.1$, $H_m = 0.61$, $H_M = 5.38$	2	$0.01 \cdot 10^{-3}$	3.5	0.015	0.007	$0.743 \cdot 10^{-3}$
Theorem 1, $\Delta_Q = 1$, $H_m = 0.5$, $H_M = 7$	2	$0.01 \cdot 10^{-3}$	4	0.015	0.007	$0.076 \cdot 10^{-3}$
Theorem 1, $\Delta_Q = 1$, $H_m = 0.5$, $H_M = 7$	40	$0.01 \cdot 10^{-3}$	4.1	0.0013	0.0006	$1 \cdot 10^{-3}$

Table 4
Example 2.2: maximum ε^* for $\sigma_0 = 1$.

Classical ES: Uncertain delay	D_0	μ	σ	k	$\lambda = kH_m$	ε^*
[18], $\Delta_Q = 0$, $H_m = 0.61$, $H_M = 5.38$	2	0	2	–	0.0115	$0.53 \cdot 10^{-2}$
[18], $\Delta_Q = 1$, $H_m = 0.61$, $H_M = 5.38$	2	0	2	–	0.0115	$0.19 \cdot 10^{-2}$
Corollary 1, $\Delta_Q = 0$, $H_m = 0.61$, $H_M = 5.38$,	2	0	3.4	0.0188	0.0115	$0.71 \cdot 10^{-2}$
Corollary 1, $\Delta_Q = 1$, $H_m = 0.61$, $H_M = 5.38$,	2	0	3.4	0.0188	0.0115	$0.69 \cdot 10^{-2}$
Corollary 1, $\Delta_Q = 1$, $H_m = 0.5$, $H_M = 7$	2	10^{-3}	9.3	0.0188	0.0094	$0.0267 \cdot 10^{-2}$
Corollary 1, $\Delta_Q = 1$, $H_m = 0.5$, $H_M = 7$	40	10^{-3}	9.4	0.0013	0.0006	$0.32 \cdot 10^{-2}$

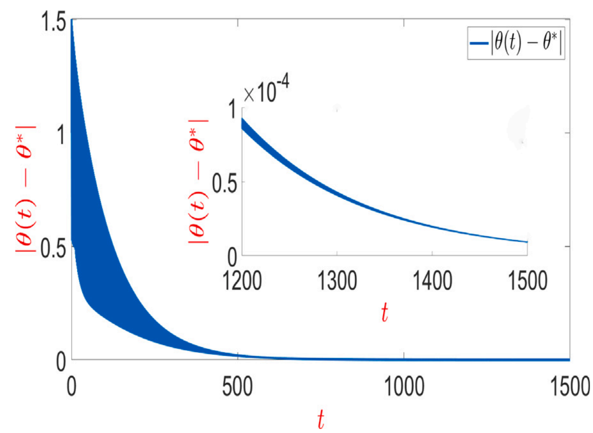
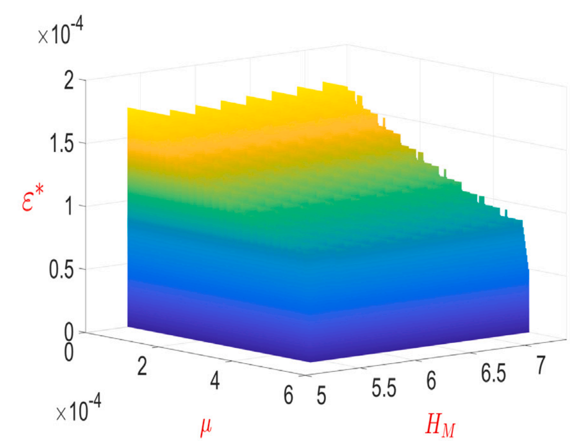
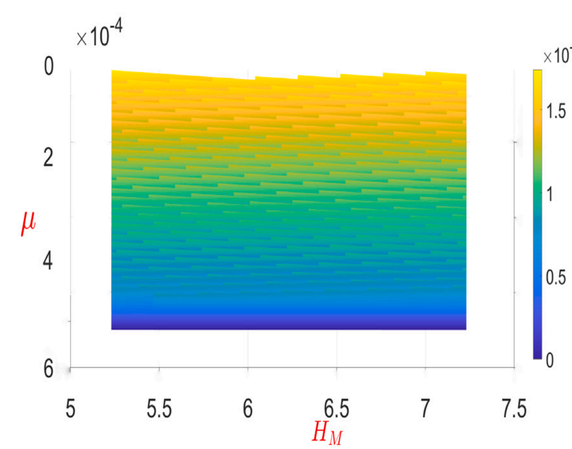


Fig. 4. Section 2.2, Unbiased ES (10), (11): plot of $|\theta(t) - \theta^*|$.



(a) 3D plot of ε^* with respect to μ and H_M.



(b) 2D Contour plot of the above 3D plot.

Fig. 5. Section 2.2, visualization of ε^* as a function of delay uncertainties and the Hessian's upper bound. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Let us consider the scenario where $y(t)$ is not quadratic map, but can be approximated as a quadratic map (1) with (44) in a vicinity of θ^* given by $|\theta(t) - \theta^*| \le \sigma_1 = 2.5$. We apply unbiased ES (15), (16) to Case II (see (46)) and $D_0 = 2$, $\omega_h = 0.03$, $k = 0.0015$, $\lambda = 0.0001$. By using Remark 6, we find $\sigma_0 = 0.32$, $\varepsilon^* = 2.3 \cdot 10^{-3}$ and $\mu^* = 10^{-4}$ meaning that for all $\varepsilon \le 2.3 \cdot 10^{-3}$ and $\mu \le 10^{-4}$, all solutions of (15), (16) with $|\tilde{\theta}(0)| \le \sigma_0 = 0.32$ satisfy $|\tilde{\theta}(t)| < 1.8e^{-0.0001t}$ for $t > 2$.

ES with practical stability: Consider the classical ES algorithm (32). Maximum values of ε^* that follow from Corollary 1 are shown in Table 4. Also in this example our results are favorably compared with the existing ones [18] based on time-delay transformation.

From Tables 1–4, it is seen that larger delay, decay rate λ and uncertainty Δ_Q lead to a smaller ε, i.e. higher dither frequencies. Furthermore, the obtained values of ε^* by the unbiased ES algorithm are smaller than the ones obtained by the classical ES. The latter can be explained by the additional constraints and terms in (28)–(30) compared to their counterparts in (34)–(36). Also from simulations, ε^* by the unbiased ES algorithm are smaller. Additionally, the delay uncertainty decreases the bound on ε^*.

3. Conclusion

This paper studied the unbiased ES algorithm for uncertain n-dimensional quadratic maps in the presence of uncertain time-varying delays via a delay-free transformation. The explicit quantitative conditions in terms of scalar linear inequalities were established which guarantee the exponential stability of the ES control system. In addition, improved practical stability conditions for classical ES were provided that essentially improved the existing results. The results are semi-global for globally quadratic maps. For locally quadratic static maps, we provide a bound on the region of convergence. Appropriate ES parameters can be chosen for any large known part of constant delay to achieve practical/exponential convergence. Future work may include constructive methods for unbiased ES of non-quadratic and dynamic maps in the presence of time-varying delays.

CRediT authorship contribution statement

Adam Jbara: Writing – original draft, Software, Methodology, Data curation. **Emilia Fridman:** Original idea, Writing – review & editing, Methodology, Supervision. **Xuefei Yang:** Writing – review & editing, Resources, Investigation, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

Proof of Lemma 1. The function $\rho_1(t)$ can be rewritten as

$$\rho_1(t) = k\left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)[M(s)S^T(s)-I]ds\right)H, \tag{48}$$

with

$$\begin{aligned}&\tfrac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)[M(s)S^T(s)-I]ds\\ &=-\sum_{i=1}^n\left(\tfrac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)\cos(2\omega_i s)ds\right)e_ie_i^T\\ &\quad+\sum_{1\le i\ne j\le n}\tfrac{2a_j}{a_i}\left(\tfrac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)\sin(\omega_i s)\sin(\omega_j s)ds\right)e_ie_j^T.\end{aligned} \tag{49}$$

Using trigonometric identities and integration by parts we obtain

$$\begin{aligned}&\|\tfrac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)[M(s)S^T(s)-I]ds\|\\ &\qquad\le\tfrac{\varepsilon}{4\pi}\sum_{i=1}^n\tfrac{1}{i}+\tfrac{\varepsilon}{2\pi}\sum_{1\le i\ne j\le n}\tfrac{a_j}{a_i}\left(\tfrac{1}{|i-j|}+\tfrac{1}{i+j}\right).\end{aligned} \tag{50}$$

Thus, from (48)–(50), we get

$$\|\rho_1(t)\|\le\varepsilon\bar\rho_1,\quad t\ge 0, \tag{51}$$

with $\bar\rho_1$ defined in (22). Also, the function $\rho_2(t)$ can be rewritten as

$$\rho_2(t)=\sum_{i=1}^n\frac{2k}{a_i}\left[-\frac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)\sin(\omega_i s)ds\right]e_i. \tag{52}$$

Using integration by parts, we have

$$-\tfrac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)\sin(\tfrac{2\pi i}{\varepsilon}s)\,ds=-\tfrac{\varepsilon}{2\pi i}\cos(\tfrac{2\pi i}{\varepsilon}t). \tag{53}$$

Thus, from (52) and (53), we get

$$|\rho_2(t)|\le\varepsilon\bar\rho_2,\quad t\ge 0, \tag{54}$$

with $\bar\rho_2$ defined in (22). An upper bound on $\rho_4(t)$ can be derived similarly to the bound on $\rho_2(t)$.

The function $\rho_3(t)$ can be rewritten as

$$\rho_3(t)=\sum_{i=1}^n\tfrac{k}{a_i}\left[\tfrac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)\sin(\omega_i s)S^T(s)HS(s)ds\right]e_i. \tag{55}$$

Using Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}&|\tfrac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)\sin(\omega_i s)S^T(s)HS(s)ds|\\ &\le|\tfrac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)^2\sin^2(\omega_i s)ds|^{\frac12}|\tfrac{1}{\varepsilon}\int_t^{t+\varepsilon}(S^T(s)HS(s))^2ds|^{\frac12}.\end{aligned} \tag{56}$$

Repeated integration by parts yields the following bound for the first term on the right side of (56):

$$|\tfrac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)^2\sin^2(\omega_i s)ds|\le\tfrac{\varepsilon^2}{6}+\tfrac{\varepsilon^2}{8\pi l_i}\left[1+\tfrac{1}{4\pi l_i}\right]. \tag{57}$$

The second term on the right side of (56) can be bounded as

$$|\tfrac{1}{\varepsilon}\int_t^{t+\varepsilon}(S^T(s)HS(s))^2ds|\le H_M^2\left[\sum_{j=1}^n a_j^2\right]^2. \tag{58}$$

Finally, from (55)–(58) we have $|\rho_3(t)|\le\varepsilon\bar\rho_3$ with $\bar\rho_3$ defined in (22). □

For the exponential stability analysis of unbiased ES (10) and (11), the following Lemma will be used:

Lemma 2. *Consider the delayed differential equation*

$$\dot x(t)=-kHx(t-D_0),\ \ x(t)\in\mathbb{R}^n, \tag{59}$$

where $k>0$, $D_0>0$ is a constant delay, and H is a positive matrix that satisfies $0<H_mI\le H\le H_MI$ with scalars H_m and H_M. Let $X(t)$ be the $n\times n$ fundamental matrix of (59), meaning that it satisfies (59) for $t>0$ and its initial condition is defined by $X(0)=I$ and $X(t)=0,\ t<0$. If $kH_MD_0\le e^{-1}$, then the following holds:

$$\|X(t)\|\le\begin{cases}1, & 0\le t\le D_0,\\ e^{-kH_m(t-D_0)}, & t>D_0.\end{cases} \tag{60}$$

Proof of Lemma 2. Since $H>0$, there exists an orthogonal matrix $U\in\mathbb{R}^{n\times n}$ (obviously, $\|U\|=1$) such that

$$UHU^{-1}=diag\{h_1,\dots,h_n\}\triangleq\mathcal{H}>0. \tag{61}$$

Denote $e(t)=Ux(t)$, $t\ge D_0+\varepsilon$. Then from (59) we get

$$\dot e(t)=k\mathcal{H}e(t-D_0),\quad t>D_0+\varepsilon, \tag{62}$$

where $k\mathcal{H}=diag\{kh_1,\dots,kh_n\}$. Let $\bar X(t)$ be the fundamental matrix of system (62). Then $\bar X(t)$ has the following form

$$\bar X(t)=diag\{\bar X_1(t),\bar X_2(t),\dots,\bar X_n(t)\}$$

with $\bar X_i(t)$, $(i=1,\dots,n)$ being solutions of

$$\dot e_i(t)=kh_ie_i(t-D_0),\quad e_i(t)=0,\ t<0,\quad e_i(0)=1.$$

Moreover, we have $\|X(t)\|=\|\bar X(t)\|$. By using Theorem 2.7 and Corollary 2.14 in [32], if $kH_MD_0\le e^{-1}$ then the following holds

$$0<\bar X_i(t)\le\begin{cases}1, & 0\le t\le D_0,\\ e^{-kH_m(t-D_0)}, & t>D_0,\end{cases}$$

implying

$$\|\bar X(t)\|=\|X(t)\|\le\begin{cases}1, & 0\le t\le D_0,\\ e^{-kH_m(t-D_0)}, & t>D_0.\end{cases}\quad\square \tag{63}$$

Proof of Theorem 1. Given $\sigma_0>0$, let σ be subject to (28). From (15) we have

$$\begin{aligned}|\tilde\theta(t)|&\le\sigma_0+\alpha_0e^{-\lambda(D_0+D_M)}\sqrt{\textstyle\sum_{i=1}^n a_i^2}\\ &\underbrace{<}_{(29)}\sigma e^{-\lambda D_M}\le\sigma e^{-\lambda t},\quad t\in[0,D_M].\end{aligned} \tag{64}$$

We assume (and further prove) that

$$|\tilde\theta(t)|<\sigma e^{-\lambda t},\quad t>D_M. \tag{65}$$

Then, from (64) and (65), we have

$$|\tilde\theta(t)|<\sigma e^{-\lambda t},\quad t\ge 0. \tag{66}$$

By using (66), it follows from (1) and (2) that

$$\begin{aligned}|y(t)-Q^*|&=|Q(t-D(t))-Q^*|\\ &=\tfrac12|\tilde\theta(t-D(t))+\alpha(t-\Delta_{\varepsilon\mu}(t))S(t-\Delta_{\varepsilon\mu}(t))|_H^2\\ &\underbrace{\le}_{(30)}\sigma_ye^{-2\lambda t},\quad t\ge D_M.\end{aligned} \tag{67}$$

We apply the variation of constants formula to Eq. (16):

$$\tilde\eta(t)=e^{-\omega_h(t-D_M)}\tilde\eta(D_M)+\omega_h\int_{D_M}^t e^{-\omega_h(t-s)}[y(s)-Q^*]ds,\ \ t\ge D_M. \tag{68}$$

Employing the conditions $\omega_h>2\lambda$, $|Q^*-Q_0|\le\Delta_Q$ in Assumption 2, and (67)–(68), we have

$$\begin{aligned}|\tilde\eta(t)|&\le e^{-\omega_h(t-D_M)}|\tilde\eta(D_M)|+\omega_h\int_{D_M}^t e^{-\omega_h(t-s)}|y(s)-Q^*|ds\\ &\le e^{-\omega_h(t-D_M)}\Delta_Q+\tfrac{\omega_h\sigma_y}{\omega_h-2\lambda}e^{-2\lambda t}\underbrace{<}_{(30)}\sigma_\eta e^{-2\lambda t},\qquad t\ge D_M,\end{aligned} \tag{69}$$

which implies the second inequality in (31).

In addition, via (24), (27), (66), (67) and (69), we obtain

$$\begin{aligned} &|G(t)| < \varepsilon\Delta_G e^{-\lambda t}, \quad |Y(t)| < \varepsilon\Delta_Y e^{-\lambda t}, \quad |\dot{\tilde{\theta}}(t)| < \Delta e^{-\lambda t}, \\ &|\Gamma(t)| < \mu\Delta_\Gamma e^{-\lambda t}, \qquad |w(t)| < \mu\Delta_w e^{-\lambda t}, \qquad t \geq D_M, \end{aligned} \tag{70}$$

where Δ_G, Δ_Y, Δ, Δ_Γ and Δ_w are defined in (30).

By variation of constants formula (see Lemma 9.1 in Agarwal et al. [32]), the solution to (26) can be presented as

$$\begin{aligned} z(t) = X(t-D_M)z(D_M) + \int_{D_M}^t X(t-s)[Y(s)+w(s)]ds \\ + \int_{D_M}^t X(t-s)[-kH\psi(s-D_0)]ds, \quad t > D_M, \end{aligned} \tag{71}$$

where $\psi(s-D_0) = 0$ if $s > D_0 + D_M$ and $\psi(s-D_0) = z(s-D_0)$ if $D_0 \leq s \leq D_0 + D_M$. Then it follows from (71) that

$$\begin{aligned} |z(t)| \leq \|X(t-D_M)\| \cdot |z(D_M)| + kH_M \int_{D_M}^t \|X(t-s)\| \cdot |\psi(s-D)|ds \\ + \int_{D_M}^t \|X(t-s)\| \cdot [|Y(s)| + |w(s)|]\, ds, \qquad t > D_M. \end{aligned} \tag{72}$$

We will further employ bounds on $\|X\|$ given in Lemma 2. We consider three intervals $[D_M, D_M+D_0]$, $(D_M+D_0, D_M+2D_0]$ and (D_M+2D_0, ∞).

For $t \in [D_M, D_M+D_0]$, by using (60), (70) and (72) we find

$$\begin{aligned} |z(t)| &\leq |z(D_M)| + kH_M \int_{D_M}^{D_M+D_0} |z(s-D_0)|ds + \int_{D_M}^t [|Y(s)| + |w(s)|]\, ds \\ &\leq (\sigma_0 + \alpha_0 e^{-\lambda(D_0+D_M)}\sqrt{\textstyle\sum_{i=1}^n a_i^2})[1+kH_M D_0] \\ &\quad + \varepsilon\Delta_Y D_0 e^{-\lambda D_M} + \mu\Delta_w D_0 e^{-\lambda D_M}. \end{aligned} \tag{73}$$

By using (25), we further have

$$\begin{aligned} |\tilde{\theta}(t)| &\leq (\sigma_0 + \alpha_0 e^{-\lambda(D_0+D_M)}\sqrt{\textstyle\sum_{i=1}^n a_i^2})[1+kH_M D_0] \\ &\quad + [\varepsilon\Delta_Y + \mu\Delta_w]D_0 e^{-\lambda D_M} + \varepsilon\Delta_G e^{-\lambda t} \\ &\underbrace{<}_{(29)} \sigma e^{-\lambda t}, \qquad D_M \leq t \leq D_M + D_0. \end{aligned} \tag{74}$$

For $t \in (D_M+D_0, D_M+2D_0]$, by using (60), (70) and (72) we find

$$\begin{aligned} |z(t)| &\leq e^{-kH_m(t-D_M-D_0)} \cdot |z(D_M)| \\ &\quad + kH_M \int_{D_M}^{D_M+D_0} \|X(t-s)\| \cdot |z(s-D_0)|ds \\ &\quad + \int_{D_M}^t \|X(t-s)\| \cdot [|Y(s)| + |w(s)|]\, ds \\ &\leq (\sigma_0 + \alpha_0 e^{-\lambda(D_0+D_M)}\sqrt{\textstyle\sum_{i=1}^n a_i^2})[e^{-kH_m(t-D_M-D_0)} + kH_M D_0] \\ &\quad + (\varepsilon\Delta_Y + \mu\Delta_w)\left[D_0 + \frac{1}{(kH_m-\lambda)}\right] e^{-\lambda(t-D_0)}. \end{aligned} \tag{75}$$

By using (25), we further have

$$\begin{aligned} |\tilde{\theta}(t)| &\leq (\sigma_0 + \alpha_0 e^{-\lambda(D_0+D_M)}\sqrt{\textstyle\sum_{i=1}^n a_i^2})[e^{-kH_m(t-D_M-D_0)} + kH_M D_0] \\ &\quad + \varepsilon\Delta_G e^{-\lambda t} + (\varepsilon\Delta_Y + \mu\Delta_w)\left[D_0 + \frac{1}{(kH_m-\lambda)}\right] e^{-\lambda(t-D_0)} \\ &\underbrace{<}_{(29)} \sigma e^{-\lambda t}, \quad D_M + D_0 \leq t \leq D_M + 2D_0. \end{aligned} \tag{76}$$

For $t > D_M + 2D_0$, by using (60), (70) and (72), we obtain

$$\begin{aligned} |z(t)| &\leq e^{-kH_m(t-D_M-D_0)} \cdot |z(D_M)| \\ &\quad + kH_M \int_{D_M}^{D_M+D_0} \|X(t-s)\||z(s-D_0)|ds \\ &\quad + \int_{D_M}^t \|X(t-s)\| \cdot [|Y(s)| + |w(s)|]\, ds \\ &\leq (\sigma_0 + \alpha_0 e^{-\lambda(D_0+D_M)}\sqrt{\textstyle\sum_{i=1}^n a_i^2}) \\ &\quad \times [\frac{H_M}{H_m}[e^{kH_m D_0} - 1] + 1]e^{-kH_m(t-D_M-D_0)} \\ &\quad + (\varepsilon\Delta_Y + \mu\Delta_w)\left[D_0 + \frac{1}{(kH_m-\lambda)}\right] e^{-\lambda(t-D_0)}, \end{aligned} \tag{77}$$

we further have

$$\begin{aligned} |\tilde{\theta}(t)| &\leq +(\sigma_0 + \alpha_0 e^{-\lambda(D_0+D_M)}\sqrt{\textstyle\sum_{i=1}^n a_i^2}) \\ &\quad \times \left[\frac{H_M}{H_m}[e^{kH_m D_0} - 1] + 1\right] e^{-kH_m(t-D_{\varepsilon\mu}-D_0)} + \varepsilon\Delta_G e^{-\lambda t} \\ &\quad + (\varepsilon\Delta_Y + \mu\Delta_w)\left[D_0 + \frac{1}{(kH_m-\lambda)}\right] e^{-\lambda(t-D_0)} \\ &\underbrace{<}_{(29)} \sigma e^{-\lambda t}, \qquad t \geq D_M + 2D_0. \end{aligned} \tag{78}$$

We prove further that inequalities (28) and (29) guarantee the bound (65). From (64) the inequality $|\tilde{\theta}(t)| < \sigma e^{-\lambda t}$ holds for $t \in [0, D_M]$. Then $|\tilde{\theta}(t)| < \sigma e^{-\lambda t}$ holds also for some $t > D_M$ due to continuity of $\tilde{\theta}(t)$. We assume by contradiction that there exists $t > D_M$ such that (65) does not hold. Namely, there exists the smallest $t^* > D_M$ such that $|\tilde{\theta}(t^*)| = \sigma e^{-\lambda t^*}$ and $|\tilde{\theta}(t)| < \sigma e^{-\lambda t}$ when $t \in [D_M, t^*)$. Thus $|\tilde{\theta}(t)| \leq \sigma e^{-\lambda t}$ holds for all $t \in [D_M, t^*]$. There are three possibilities: $t^* \in [D_M, D_M+D_0]$ or $(D_M+D_0, D_M+2D_0]$ or (D_M+2D_0, ∞).

If $t^* \in [D_M, D_M+D_0]$, then under the non-strict inequality $|\tilde{\theta}(t)| \leq \sigma e^{-\lambda t}$ we find that the non-strict version of inequality (73) holds for $t \in [D_M, t^*]$. Then, by using (29), we have

$$\begin{aligned} |\tilde{\theta}(t)| &\leq (\sigma_0 + \alpha_0 e^{-\lambda(D_0+D_M)}\sqrt{\textstyle\sum_{i=1}^n a_i^2})[1+kH_M D_0] \\ &\quad + [\varepsilon\Delta_Y + \mu\Delta_w]D_0 e^{-\lambda D_M} + \varepsilon\Delta_G e^{-\lambda t} \\ &< \sigma e^{-\lambda t}, \qquad t \in [D_M, t^*], \end{aligned} \tag{79}$$

which contradicts to $|\tilde{\theta}(t^*)| = \sigma e^{-\lambda t^*}$.

If $t^* \in (D_M+D_0, D_M+2D_0]$, then under the non-strict inequality $|\tilde{\theta}(t)| \leq \sigma e^{-\lambda t}$ we find that the non-strict version of inequality (75) holds for $t \in (D_M+D_0, t^*]$. Then, by using (29), we have

$$\begin{aligned} |\tilde{\theta}(t)| &\leq (\sigma_0 + \alpha_0 e^{-\lambda(D_0+D_M)}\sqrt{\textstyle\sum_{i=1}^n a_i^2})\left[e^{-kH_m(t-D_M-D_0)} + kH_M D_0\right] \\ &\quad + \varepsilon\Delta_G e^{-\lambda t} + (\varepsilon\Delta_Y + \mu\Delta_w)\left[D_0 + \frac{1}{(kH_m-\lambda)}\right] e^{-\lambda(t-D_0)} \\ &< \sigma e^{-\lambda t}, \qquad t \in (D_M+D_0, t^*], \end{aligned} \tag{80}$$

which contradicts to $|\tilde{\theta}(t^*)| = \sigma e^{-\lambda t^*}$.

If $t^* \in (D_M+2D_0, \infty)$, then under the non-strict inequality $|\tilde{\theta}(t)| \leq \sigma e^{-\lambda t}$ we find that the non-strict version of inequality (77) holds for $t \in (D_M+2D_0, t^*]$. Then, by using (29), we have

$$\begin{aligned} |\tilde{\theta}(t)| &\leq (\sigma_0 + \alpha_0 e^{-\lambda(D_0+D_M)}\sqrt{\textstyle\sum_{i=1}^n a_i^2}) \\ &\quad \times \left[\frac{H_M}{H_m}[e^{kH_m D_0} - 1] + 1\right] e^{-kH_m(t-D_M-D_0)} + \varepsilon\Delta_G e^{-\lambda t} \\ &\quad + (\varepsilon\Delta_Y + \mu\Delta_w)\left[D_0 + \frac{1}{(kH_m-\lambda)}\right] e^{-\lambda(t-D_0)} \\ &< \sigma e^{-\lambda t}, \qquad t \in (D_M+2D_0, t^*], \end{aligned} \tag{81}$$

which contradicts to $|\tilde{\theta}(t^*)| = \sigma e^{-\lambda t^*}$ and completes the proof of (31).

Finally, given any D_0 and σ_0, there always exist small enough ε^*, μ and λ, k, ω_h subject to (13) that satisfy (28), (29) for some $\sigma > \sigma_0$ since for $k = \varepsilon^* = \lambda = 0$ these inequalities are reduced to $\sigma > \sigma_0$. □

Data availability

No data was used for the research described in the article.

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