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Predictor methods for finite-dimensional observer-based control of stochastic parabolic PDEs[☆]

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ABSTRACT

We study output-feedback control of 1D stochastic semilinear heat equations with nonlinear multiplicative noise and uncertain time-varying input/output delays or sawtooth delays (that correspond to network-based control), where the nonlinearities satisfy globally Lipschitz condition. We assume that the input delay has a large constant known part r. We consider Neumann actuation with non-local measurement. To compensate r, we consider a chain of M sub-predictors (conventional sub-predictors as in the deterministic case) and a novel chain of $M+1$ sub-predictors, respectively, both in the form of ODEs that correspond to the delay fraction r/M. For both cases, we construct Lyapunov functionals that depend on the deterministic and stochastic parts of the finite-dimensional part of the closed-loop systems, and employ the corresponding Itô's formulas for stochastic ODEs and PDEs, respectively. We provide the mean-square L^2 exponential stability analysis of the full-order closed-loop system, leading to LMIs that are feasible for any r provided M and the observer dimension are large enough, and Lipschitz constants, as well as the upper bounds of unknown delays, are small enough. For the novel sub-predictors, we add an additional sub-predictor to the chain that leads to the closed-loop system with the stochastic infinite-dimensional tail and the stochastic finite-dimensional part where the delay fraction r/M and the stochastic term appear in separate equations, which essentially simplifies stochastic Lyapunov functional structure and the resulting LMIs. We also consider a classical observer-based predictor for linear heat equations with nonlinear multiplicative noise and show that the corresponding LMI stability conditions are feasible for any r provided the observer dimension is large enough, and the upper bounds of unknown delays and noise intensity are small enough. A numerical example demonstrates that for comparatively large M and upper bound of noise intensity, the introduction of an addition sub-predictor leads to a larger r compared with conventional sub-predictors, whereas for the linear heat equations with nonlinear noise, the classical predictor always allows larger delays.

1. Introduction

In recent years, estimation and control problems for stochastic PDEs become popular due to their wide applications in many areas of science, engineering, and finance. However, control theory for stochastic PDEs is still at its very beginning stage and many tools and methods, which are effective in the deterministic case, do not work anymore in the stochastic setting [1]. Finite-dimensional controllers for parabolic systems via the modal decomposition approach are very attractive in applications [2,3]. Modal decomposition was extended to the stochastic setting in [4,5] for additive noise under state-feedback and output-feedback controllers, respectively, and in [6,7] for multiplicative noise under state-feedback control. However, in [2–6], efficient bounds of the observer or controller dimensions were not provided. In recent paper [8], the first constructive LMI-based method for finite-dimensional observer-based controller of deterministic parabolic PDEs was suggested, where the observer dimension was found from simple LMI conditions. In our recent paper [9], the constructive method in [8] was extended to stochastic parabolic PDEs with nonlinear multiplicative noise under boundary control and observer.

Robustness with respect to small delays and/or sampling intervals for deterministic heat equations was studied in [10] for distributed static output-feedback control, in [11] for boundary state-feedback and in [12] for boundary controller based on PDE observer. Delayed implementation of finite-dimensional observer-based controllers for 1D heat equations was introduced in [13] for deterministic case and in [14] for

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stochastic case. For the estimation of deterministic heat equations with a large input/output delay, a PDE sub-predictor was presented in [15] and a chain of observers was designed in [16]. Finite-dimensional observer-based classical predictors and/or sub-predictors were introduced in [17–19] for linear parabolic PDEs. In [20], finite-dimensional observer-based sub-predictors for semilinear parabolic PDEs with constant input delay were explored. However, for stochastic systems, there are few results on predictor-based control, and most existing results are confined to stochastic linear ODEs (see, e.g., [21,22] for state-feedback case and [23] for observer-based case). In [24], the predictor-based boundary state-feedback control for deterministic linear parabolic PDEs with stochastic input delay was studied. To the best of our knowledge, predictor-based control for stochastic PDEs has not been studied yet.

In the present paper, for the first time, we provide efficient predictor methods for stochastic parabolic PDEs. We consider finite-dimensional observer-based control of 1D stochastic semilinear heat equations with nonlinear multiplicative noise and time-varying input/output delays or sawtooth delays that correspond to network-based control (see [25, Chapter 7]), where the nonlinearities satisfy globally Lipschitz condition. We assume unknown measurement delays and large input/transmission delays that have known constant part r and unknown time-varying part τ_u. We consider Neumann actuation with non-local measurement. To compensate r, we construct a chain of M sub-predictors (conventional sub-predictors as the deterministic case [17, 20]) and a novel chain of $M+1$ sub-predictors, respectively, both in the form of ODEs that correspond to the delay fraction r/M. For both cases, we construct appropriate Lyapunov functionals that depend on the deterministic and stochastic terms of the finite-dimensional part of the closed-loop systems, and employ corresponding Itô's formulas for stochastic ODEs and PDEs, respectively. We provide the mean-square L^2 exponential stability analysis of the full-order closed-loop system, leading to LMI conditions for finding M, the observer dimension N, Lipschitz constants, the known delay r, and upper bounds of unknown delays that preserve the exponential stability. We show that for any r, the LMIs are feasible for large enough M, N, and small enough Lipschitz constants and upper bounds of unknown delays. Note that for the novel sub-predictors, we add an additional sub-predictor to the chain that leads to the closed-loop system with the stochastic infinite-dimensional tail and the finite-dimensional part where the delay fraction r/M and the stochastic term appear in separate equations. Such separation of stochastic term and the delay fraction r/M avoids some stochastic-dependent terms in our Lyapunov functional which has only one stochastic-dependent term that corresponds to measurement delays. This essentially simplifies the Lyapunov-based stability analysis.

We also consider a classical observer-based predictor for linear heat equations with nonlinear multiplicative noise, where differently from the deterministic case [19], the predictor is constructed for the unstable modes only (otherwise the stochastic part explodes for large observer dimension). We show that the corresponding LMI stability conditions are feasible for any r provided the observer dimension is large enough and upper bounds of unknown delays and noise intensity are small enough. A numerical example demonstrates that our sub-predictors with additional sub-predictor and the conventional sub-predictors as studied in [17,20] for deterministic PDEs lead to complementary results, whereas additional sub-predictor for the stochastic case leads to a larger delay for comparatively large M and upper bound of noise intensity. For the linear heat equations with nonlinear noise, a numerical example shows that the classical predictor always allows larger delays.

The novelty of stochastic compared with deterministic case [13,17–20] can be formulated as follows:

1. We have a novel chain of sub-predictors to simplify stochastic Lyapunov functional structure and the resulting LMIs.
2. Differently from the deterministic case, we construct a stochastic classical predictor for the unstable modes only.
3. In the Lyapunov-based stability analysis we use stochastic Lyapunov functionals combined with Halanay's inequality for the expected value of the Lyapunov functional and Itô's formula.

Preliminary results on sub-predictors for stochastic semilinear heat equations with constant input delay were reported in [26].

Notations: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of increasing sub σ-fields of $\mathcal{F}$ and let $\mathbb{E}\{\cdot\}$ be the expectation operator. Denote by $\mathcal{W}(t)$ the 1D standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $L^2(0,1)$ the space of all square integrable functions $f : [0,1] \to \mathbb{R}$ with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\mathrm{d}x$ and induced norm $\|f\|^2_{L^2} = \langle f, f\rangle$. Let $L^2(\Omega; L^2(0,1))$ be the set of all random variables $z \in L^2(0,1)$ with $\mathbb{E}\|z\|^2_{L^2} < \infty$. $H^1(0,1)$ is the Sobolev space of functions $f : [0,1] \longrightarrow \mathbb{R}$ with a square integrable weak derivative. The norm defined in $H^1(0,1)$ is $\|f\|^2_{H^1} = \|f\|^2_{L^2} + \|f'\|^2_{L^2}$. The Euclidean norm is denoted by $|\cdot|$. For $P \in \mathbb{R}^{n\times n}$, $P > 0$ means that P is symmetric and positive definite. The symmetric elements of a symmetric matrix will be denoted by $*$. For $0 < P \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, we write $|x|^2_P = x^{\mathrm{T}}Px$. Denote $\mathbb{N}$ by the set of positive integers and I by the identity matrix of appropriate size. Let $J_{0,M}$ be an upper triangular Jordan block of order M with zero diagonal and $\otimes$ be the Kronecker product. Denote by $\mathcal{O}(N^k)$, $k \in \mathbb{R}$ the order of N^k time complexity.

Consider the Sturm–Liouville eigenvalue problem

$$\phi'' + \lambda\phi = 0, \ x \in (0,1), \quad \phi'(0) = \phi'(1) = 0.$$

This problem induces a sequence of eigenvalues with corresponding eigenfunctions given by:

$$\begin{aligned} &\phi_1(x) = 1, \quad \lambda_1 = 0, \\ &\phi_n(x) = \sqrt{2}\cos(\sqrt{\lambda_n}x), \quad \lambda_n = (n-1)^2\pi^2, \ n \geq 2. \end{aligned} \tag{1.1}$$

The eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ form a complete orthonormal system in $L^2(0,1)$. Given a positive integer N and $h \in L^2(0,1)$ satisfying $h \overset{L^2}{=} \sum_{n=1}^{\infty} h_n\phi_n$, we denote $\|h\|^2_N = \sum_{n=N+1}^{\infty} h_n^2$.

2. Sub-predictors for stochastic semilinear heat equation

Consider the following stochastic semilinear heat equation under delayed Neumann actuation:

$$\begin{aligned} &\mathrm{d}z(x,t) = [\tfrac{\partial^2}{\partial x^2} z(x,t) + g(x, z(x,t))]\mathrm{d}t + \sigma(x, z(x,t))\mathrm{d}\mathcal{W}(t), \\ &z_x(0,t) = 0, \ z_x(1,t) = u(t - r - \tau_u(t)), \\ &z(x,0) = z_0(x), \end{aligned} \tag{2.1}$$

where $z_0 \in L^2(\Omega; L^2(0,1))$, $\mathrm{d}\mathcal{W}(t) := \dot{\mathcal{W}}(t)\mathrm{d}t$ with $\dot{\mathcal{W}}(t)$ formally regarded as the derivative of Brownian motion $\mathcal{W}(t)$ (see P. 2 in [27]). u is the control input to be designed. We assume the input delay with a known constant part r and unknown time-varying part $\tau_u(t) \in [0, \tau_{M,u}]$. In (2.1), $\sigma(x, z(x,t))\mathrm{d}\mathcal{W}(t)$ is a nonlinear multiplicative noise which appears due to the random parameter variation of $g(x, z(x,t))\mathrm{d}t$. Nonlinear functions $g, \sigma : [0,1]\times\mathbb{R} \to \mathbb{R}$ satisfy for all $x \in [0,1]$,

$$\begin{aligned} &\sigma(x,0) = 0, \ |\sigma(x,z_1) - \sigma(x,z_2)| \leq \bar{\sigma}|z_1 - z_2|, \\ &g(x,0) = 0, |g(x,z_1) - g(x,z_2)| \leq \bar{g}|z_1 - z_2|, \ \forall z_1, z_2 \in \mathbb{R}, \end{aligned} \tag{2.2}$$

where $\bar{g}, \bar{\sigma} > 0$ are constants. Here $\bar{\sigma}$ describes the upper bound of noise intensity.

Remark 1. As in [9], our results can be easily extended to a more general Sturm–Liouville operator $\frac{\partial}{\partial}(p(x)\frac{\partial}{\partial x}z(x,t)) + q(x)$ on the right-hand side of (2.1). Note that in (2.1), we consider the white noise which is uniform in the spatial variable. Such white noise appears in many applications including Musiela's equation of the bond market (see [28, Sec. 13.3]) and filtering equations (see [28, Sec. 13.8]). We suggest a nonlinear noise perturbation function $\sigma(x, z)$ to describe the distribution of noise with respect to space and state. A more general case is to consider the space-dependent white noise (see [29, P. 44]). In this case, treatment of the induced term $\langle\sigma(z(\cdot,t))\mathrm{d}\mathcal{W}(\cdot,t),\phi_n\rangle$ is challenging. The extension to space-dependent white noise via modal decomposition may be a topic for future research.

We consider the delayed non-local measurement output:

$$y(t)=\langle c,z(\cdot,t-\tau_y(t))\rangle,\quad t-\tau_y(t)\geq 0,\\ y(t)=0,\ \ t-\tau_y(t)<0, \tag{2.3}$$

where $c\in L^2(0,1)$, $\tau_y(t)$ is unknown measurement delay satisfying $\tau_y(t)\leq\tau_{M,y}$ for some known $\tau_{M,y}>0$. Note that r may be much larger than $\tau_{M,u}$ and $\tau_{M,y}$. Similar to [13], we treat two classes of input and output delays: continuously differentiable delays and sawtooth delays that correspond to network-based control.

For the case of continuously differentiable delays, we assume that $\tau_y(t)$ is lower bounded by $\tau_{m,y}>0$. This assumption is employed for well-posedness only. Following Sec. 3.1 of [18], we assume that there exists a unique $t_u^*\in[r,r+\tau_{M,u}]$ such that $t-r-\tau_u(t)<0$ if $t<t_u^*$ and $t-r-\tau_u(t)\geq 0$ if $t\geq t_u^*$. For the case of sawtooth delays, $\tau_y(t)$ and $r+\tau_u(t)$ are induced by two networks (from sensor to controller and from controller to actuator) with r being large transmission delay from sensor to actuator (see Sec. 7.5 in [25]). Henceforth the dependence of $\tau_y(t)$, $\tau_u(t)$ on t will be suppressed to shorten notations.

Present the solution to (2.1) as

$$z(x,t)=\textstyle\sum_{n=1}^{\infty}z_n(t)\phi_n(x),\ \ z_n(t)=\langle z(\cdot,t),\phi_n\rangle, \tag{2.4}$$

where $\{\phi_n\}_{n=1}^{\infty}$ are given in (1.1). By differentiating z_n in (2.4) and using integration by parts, we arrive at the following infinite stochastic equations

$$\mathrm{d}z_n(t)=[-\lambda_nz_n(t)+g_n(t)+b_nu(t-r-\tau_u)]\mathrm{d}t\\ +\sigma_n(t)\mathrm{d}\mathcal{W}(t),\quad n\geq 1, \tag{2.5}$$

where

$$g_n(t)=\langle g(\cdot,\textstyle\sum_{j=1}^{\infty}z_j(t)\phi_j),\phi_n\rangle,\\ \sigma_n(t)=\langle\sigma(\cdot,\textstyle\sum_{j=1}^{\infty}z_j(t)\phi_j),\phi_n\rangle,\\ b_1=1,\quad b_n=(-1)^{n-1}\sqrt{2},\ \ n\geq 2. \tag{2.6}$$

By (1.1) and the integral convergence test, we have

$$\textstyle\sum_{n=N+1}^{\infty}\frac{b_n^2}{\lambda_n}\leq\frac{2}{\pi^2}(\frac{1}{N^2}+\int_N^{\infty}\frac{1}{x^2}\mathrm{d}x)=\frac{2(N+1)}{\pi^2N^2},\ \ N\geq 1. \tag{2.7}$$

Remark 2. For the Dirichlet actuation $z(1,t)=u(t-r-\tau_u)$, we will have (2.7) with $b_n=-\phi_n'(1)=\mathcal{O}(n)$, $n\to\infty$. In this scenario, the sum $\sum_{n=N+1}^{\infty}b_n^2/\lambda_n$ is unbounded. We cannot get feasible LMIs to guarantee the stability of the closed-loop system. By using the dynamic-extension-based method (see, e.g., [30]), we may study the Dirichlet actuation.

Let $\delta>0$ be a desired decay rate and let $N_0\in\mathbb{N}$ satisfy

$$-\lambda_n+\bar{g}+\bar{\sigma}^2/2<-\delta,\quad n>N_0, \tag{2.8}$$

where N_0 is used for the controller design. Compared with [20] for the deterministic PDEs, the additional term $\sigma^2/2$ in (2.8) is induced by the stochastic perturbations (see Remark 2.2 in [9]). Let $N\in\mathbb{N}$, $N\geq N_0$, where N will be the dimension of the observer.

Introduce the following notations

$$z^{N_0}(t)=[z_1(t),\dots,z_{N_0}(t)]^{\mathrm{T}},\quad B_0=[b_1,\dots,b_{N_0}]^{\mathrm{T}},\\ z^{N-N_0}(t)=[z_{N_0+1}(t),\dots,z_N(t)]^{\mathrm{T}},\ \ A_0=\mathrm{diag}\{-\lambda_n\}_{n=1}^{N_0},\\ \sigma^{N_0}(t)=\mathrm{col}\{\sigma_n(t)\}_{n=1}^{N_0},\ \ \sigma^{N-N_0}(t)=\mathrm{col}\{\sigma_n(t)\}_{n=N_0+1}^{N},\\ G^{N_0}(t)=\mathrm{col}\{g_n(t)\}_{n=1}^{N_0},G^{N-N_0}(t)=\mathrm{col}\{g_n(t)\}_{n=N_0+1}^{N},\\ A_1=\mathrm{diag}\{-\lambda_n\}_{n=N_0+1}^{N},\quad B_1=[b_{N_0+1},\dots,b_N]^{\mathrm{T}}. \tag{2.9}$$

From (2.5) we find that $z^{N_0}(t)$ and $z^{N-N_0}(t)$ satisfy

$$\mathrm{d}z^{N_0}(t)=[A_0z^{N_0}(t)+G^{N_0}(t)\\ +B_0u(t-r-\tau_u)]\mathrm{d}t+\sigma^{N_0}(t)\mathrm{d}\mathcal{W}(t),\\ \mathrm{d}z^{N-N_0}(t)=[A_1z^{N-N_0}(t)+G^{N-N_0}(t)\\ +B_1u(t-r-\tau_u)]\mathrm{d}t+\sigma^{N-N_0}(t)\mathrm{d}\mathcal{W}(t). \tag{2.10}$$

Let $c_n=\langle c,\phi_n\rangle$, $C_0=[c_1,\dots,c_{N_0}]$. Assume that

$$c_n\neq 0,\ \ 1\leq n\leq N_0. \tag{2.11}$$

Then, the pair (A_0,C_0) is observable by the Hautus lemma. Choose $L_0=[l_1,\dots,l_{N_0}]^{\mathrm{T}}$ such that

$$P_o(A_0-L_0C_0)+(A_0-L_0C_0)^{\mathrm{T}}P_o<-2\delta P_o, \tag{2.12}$$

where $0<P_o\in\mathbb{R}^{N_0\times N_0}$. Furthermore, following [8] we let $l_n=0$, $N_0<n\leq N$. Since $b_n\neq 0$, $n\geq 1$ (see (2.6)), the pair (A_0,B_0) is controllable by the Hautus lemma. Let $K_0\in\mathbb{R}^{1\times N_0}$ satisfy

$$P_c(A_0-B_0K_0)+(A_0-B_0K_0)^{\mathrm{T}}P_c\leq-2\delta P_c, \tag{2.13}$$

where $0<P_c\in\mathbb{R}^{N_0\times N_0}$.

2.1. Conventional sub-predictors

In this section, we follow [17,20] to design a chain of sub-predictors (see (2.14) below). Compared with the deterministic PDEs [17,20], the analysis for stochastic PDEs is more challenging: **(i)** in the Lyapunov analysis, we cannot take *generator* term by term in the infinite sum since the mean-square L^2 convergence of the generators sum cannot be guaranteed. Instead, we present the Lyapunov function in the form of the one for the stochastic PDE and the other one for finite-dimensional stochastic ODEs and apply the generator to each part; **(ii)** To prove the mean-square exponential stability, we employ the corresponding Itô's formula for stochastic ODEs and (strong solutions of) PDEs, respectively; **(iii)** The state-derivative-dependent Lyapunov functionals in [17,20] are inapplicable for the stochastic case since solutions to stochastic systems are non-differentiable. We construct a novel Lyapunov functional (see (2.37) and (2.38)) that depends on the deterministic and stochastic terms of the finite-dimensional part of the closed-loop systems.

Consider stochastic systems (2.10). To deal with the constant input delay part $r>0$, we fix $M\in\mathbb{N}$ and divide r into M parts of equal size $\frac{r}{M}$. We design a chain of sub-predictors

$$\hat{z}_1^j(t-r)\mapsto\cdots\mapsto z_i^j(t-\tfrac{M-i+1}{M}r)\mapsto\dots\\ \mapsto\hat{z}_M^j(t-\tfrac{1}{M}r)\mapsto z^j(t),\quad j\in\{N_0,N-N_0\}, \tag{2.14}$$

where $\hat{z}_i^j(t-\frac{M-i+1}{M}r)\mapsto\hat{z}_{i+1}^j(t-\frac{M-i}{M}r)$ means that $\hat{z}_i^j(t)$ predicts the value of $\hat{z}_{i+1}^j(t+\frac{r}{M})$, $\hat{z}_M^j(t-\frac{1}{M}r)\mapsto z^j(t)$ means that $\hat{z}_M^j(t)$ predicts the value of $z^j(t+\frac{r}{M})$. The sub-predictors satisfy

$$\mathrm{d}\hat{z}_M^{N_0}(t)=[A_0\hat{z}_M^{N_0}(t)+\hat{G}_M^{N_0}(t)+B_0u(t-\tfrac{M-1}{M}r)]\mathrm{d}t\\ -L_0[C_0\hat{z}_M^{N_0}(t-\tfrac{r}{M})+C_1\hat{z}_M^{N-N_0}(t-\tfrac{r}{M})-y(t)]\mathrm{d}t,\\ \mathrm{d}\hat{z}_M^{N-N_0}(t)=[A_1\hat{z}_M^{N-N_0}(t)+\hat{G}_M^{N-N_0}(t)+B_1u(t-\tfrac{M-1}{M}r)]\mathrm{d}t,\\ \mathrm{d}\hat{z}_i^{N_0}(t)=[A_0\hat{z}_i^{N_0}(t)+\hat{G}_i^{N_0}(t)+B_0u(t-\tfrac{i-1}{M}r)]\mathrm{d}t\\ -L_0[C_0\hat{z}_i^{N_0}(t-\tfrac{r}{M})+C_1\hat{z}_i^{N-N_0}(t-\tfrac{r}{M})\\ -C_0\hat{z}_{i+1}^{N_0}(t)-C_1\hat{z}_{i+1}^{N-N_0}(t)]\mathrm{d}t,\\ \mathrm{d}\hat{z}_i^{N-N_0}(t)=[A_1\hat{z}_i^{N-N_0}(t)+\hat{G}_i^{N-N_0}(t)\\ +B_1u(t-\tfrac{i-1}{M}r)]\mathrm{d}t,\quad 1\leq i\leq M-1,\quad t\geq 0, \tag{2.15}$$

subject to $\hat{z}_i^j(t)=0,\ t\le 0,\ 1\le i\le M,\ j\in\{N_0,N-N_0\}$, where $y(t)$ is given by (2.3), $C_1=[c_{N_0+1},\dots,c_N]$,

$$\begin{aligned}&\hat{G}_i^{N_0}(t)=\mathrm{col}\{\hat{g}_n^{(i)}(t)\}_{n=1}^{N_0},\quad \hat{G}_i^{N-N_0}(t)=\mathrm{col}\{\hat{g}_n^{(i)}(t)\}_{n=N_0+1}^{N},\\&\hat{g}_n^{(i)}(t)=\langle g(\cdot,\Phi_0(\cdot)\hat{z}_i^{N_0}(t)+\Phi_1(\cdot)\hat{z}_i^{N-N_0}(t)),\phi_n\rangle,\\&\Phi_0(\cdot)=[\phi_1(\cdot),\dots,\phi_{N_0}(\cdot)],\ \Phi_1(\cdot)=[\phi_{N_0+1}(\cdot),\dots,\phi_N(\cdot)].\end{aligned}\tag{2.16}$$

Remark 3. Here we construct the sub-predictors (2.15) without noise, otherwise, the error systems will have double noise that leads to conservative results. Although there is no explicit noise in sub-predictors (2.15), the sub-predictors are still random due to the measurement output $y(t)=\langle c,z(\cdot,t-\tau_y)\rangle$ that appears in system $\hat{z}_M^{N_0}$, where $z(\cdot,t-\tau_y)$ is a stochastic process.

The finite-dimensional observer $\hat{z}(x,t)$ of the state $z(x,t)$, based on (2.15), is given by

$$\hat{z}(x,t)=\Phi_0(x)\hat{z}_1^{N_0}(t-r)+\Phi_1(x)\hat{z}_1^{N-N_0}(t-r).\tag{2.17}$$

The controller is chosen as

$$u(t)=0,t\le 0,\quad u(t)=-K_0\hat{z}_1^{N_0}(t),\quad t>0,\tag{2.18}$$

where $K_0\in\mathbb{R}^{1\times N_0}$ is determined by (2.13).

2.2. Well-posedness of the closed-loop system

We start with the case of continuously differentiable delays. First, we introduce the change of variables (see [31])

$$w(x,t)=z(x,t)-\psi(x)u(t-r-\tau_u),\tag{2.19}$$

where $\psi(x)=-\frac{2}{\pi}\cos(\frac{\pi}{2}x)$ satisfies

$$\psi''(x)=-\mu\psi(x),\ \mu=\tfrac{\pi^2}{4},\ \psi'(0)=0,\psi'(1)=1.\tag{2.20}$$

Then we have the equivalent stochastic PDE

$$\begin{aligned}&\mathrm{d}w(t)=[\mathcal{A}w(t)+g(\cdot,w(t)+\psi(\cdot)u(t-r-\tau_u))]\mathrm{d}t\\&-\psi(\cdot)[\mu u(t-r-\tau_u)+(1-\tau_u')F_u(t-r-\tau_u)]\mathrm{d}t\\&+\sigma(\cdot,w(t)+\psi(\cdot)u(t-r-\tau_u))\mathrm{d}\mathcal{W}(t),\ t\ge 0,\end{aligned}\tag{2.21}$$

where $w(t)=w(\cdot,t)$, $F_u(t)=-K_0[(A_0-B_0K_0)\hat{z}_1^{N_0}(t)+\hat{G}_1^{N_0}(t)-L_0C_0\hat{z}_1^{N_0}(t-\frac{r}{M})-C_1\hat{z}_1^{N-N_0}(t-\frac{r}{M})+L_0C_0\hat{z}_2^{N_0}(t)+L_0C_1\hat{z}_2^{N-N_0}(t)]$ satisfies $\mathrm{d}u(t)=F_u(t)\mathrm{d}t$, and

$$\begin{aligned}&\mathcal{A}:\mathcal{D}(\mathcal{A})\subseteq L^2(0,1)\to L^2(0,1),\quad \mathcal{A}h=h'',\\&\mathcal{D}(\mathcal{A})=\{h\in H^2(0,1)|h'(0)=h'(1)=0\}.\end{aligned}$$

In (2.21), we take $u(t-r-\tau_u)$ (see (2.18)) as the non-homogeneous term. Let $\hat{Z}(t)=\mathrm{col}\{\hat{z}_1^{N_0}(t),\hat{z}_1^{N-N_0}(t),\dots,\hat{z}_{M+1}^{N_0}(t),\hat{z}_{M+1}^{N-N_0}(t)\}$. From (2.3), (2.15), and (2.18), we have

$$\begin{aligned}&\mathrm{d}\hat{Z}(t)=\big[\mathbb{A}\hat{Z}(t)+\hat{G}(t)-\textstyle\sum_{i=1}^M\mathbb{B}_i\mathbb{K}_0\hat{Z}(t-\frac{i-1}{M}r)\\&-(I_M\otimes C_0)\hat{Z}(t-\tfrac{r}{M})+(J_{0,M}\otimes C_0)\hat{Z}(t)+\mathbb{L}_0\langle c,w(\cdot,t-\tau_y)\rangle\\&-\mathbb{L}_0\langle c,\psi\rangle\mathbb{K}_0\hat{Z}(t-\tau_y-r-\tau_u(t-\tau_y))\big]\mathrm{d}t,\ t\ge 0,\end{aligned}\tag{2.22}$$

where

$$\begin{aligned}&\mathbb{A}=I_M\otimes\mathrm{diag}\{A_0,A_1\},\quad \mathbb{K}_0=[K_0,0_{(MN+N-N_0)\times 1}],\\&\mathbb{B}_i=\mathrm{col}\{0_{(i-1)N\times 1},B_0,B_1,0_{(M-i)\times 1}\},\ i=1,\dots,M,\\&\hat{G}(t)=\mathrm{col}\{\hat{G}_1^{N_0},\hat{G}_1^{N-N_0},\dots,\hat{G}_M^{N_0},\hat{G}_M^{N-N_0}\},\\&\mathbb{L}_0=\mathrm{col}\{0_{N(M-1)\times 1},L_0,0_{(N-N_0)\times 1}\},\quad \mathcal{C}_0=\begin{bmatrix}L_0C_0 & L_0C_1\\ 0 & 0\end{bmatrix}.\end{aligned}$$

For (2.22), we take $w(\cdot,t-\tau_y)$ and other delayed terms as the non-homogeneous terms. Therefore, the closed-loop system is the coupled system between (2.21) and (2.22).

Proposition 1. *Consider the closed-loop system (2.1) subject to the control law (2.18) and the corresponding sub-predictor-based observer $\hat{z}(x,t)$ defined by (2.15), (2.17) (i.e., (2.21) and (2.22)). Assume that nonlinear functions g,σ satisfy (2.2). Then the solution to the closed-loop system (2.1), (2.18), (2.15), with initial value $z_0\in\mathcal{D}(\mathcal{A})$ almost surely and $z_0\in L^2(\Omega;L^2(0,1))$ exists uniquely and satisfies*

$$\begin{aligned}&z(\cdot,t)\in L^2(\Omega;C([0,\infty);L^2(0,1)))\\&\qquad\cap L^2(\Omega\times[0,\infty)\backslash\mathcal{J};H^1(0,1)),\end{aligned}\tag{2.23}$$

and $w(\cdot,t)=z(\cdot,t)-\psi(\cdot)u(t-r-\tau_u)\in\mathcal{D}(\mathcal{A})$, $t\ge 0$ almost surely.

Proof. We follow the step method used in [18] for the well-posedness. First, we consider $t\in[0,t_u^*]$. By (2.18) we have $u(t-r-\tau_u)\equiv 0$, $t\in[0,t_u^*]$. Since $w(\cdot,0)=z_0\in\mathcal{D}(\mathcal{A})$ almost surely and g,σ satisfy (2.2), by Theorem 6.7.4 in [29], (2.21) has a unique strong solution $w\in L^2(\Omega;C([0,t_u^*]);L^2(0,1))\cap L^2(\Omega\times[0,t_u^*];H^1(0,1))$ and $w(\cdot,t)\in\mathcal{D}(\mathcal{A})$, $t\in[0,t_u^*]$, almost surely. For $y(t)$ defined in (2.3), we have

$$\mathbb{E}\textstyle\int_0^{t_u^*}|y(s)|^2\mathrm{d}s\le t_u^*\|c\|_{L^2}^2\mathbb{E}\sup_{s\in[0,t_u^*]}\|w(s)\|_{L^2}^2.\tag{2.24}$$

By Theorem 3.6.3 in [29], we obtain

$$\mathbb{E}\sup_{s\in[0,t_u^*]}\|w(s)\|_{L^2}^2\le\bar{\kappa}_1(1+\mathbb{E}\|w(0)\|_{L^2}^2),$$

for some $\bar{\kappa}_1>0$, which together with (2.24) implies

$$\mathbb{E}\textstyle\int_0^{t_u^*}|y(s)|^2\mathrm{d}s\le t_u^*\|c\|_{L^2}^2\bar{\kappa}_1(1+\mathbb{E}\|w(0)\|_{L^2}^2)=:\kappa_1.\tag{2.25}$$

Then we consider (2.22) for $t\in[0,t_u^*]$. Since g satisfies the globally Lipschitz condition (2.2), we have that $\hat{G}_i^j$ defined below (2.15) satisfy the globally Lipschitz condition with respect to $\hat{Z}$. By taking $w(\cdot,t-\tau_y)$ and the delay terms as non-homogeneous terms, and using the step method on $[0,\frac{r}{M}]$, $[\frac{r}{M},\frac{2r}{M}]$, … until $t=t_u^*$, we conclude from Theorem 2.3.1 and Lemma 2.3.2 in [27] that system (2.22) admits a unique solution on $[0,t_u^*]$ which satisfies for some $\kappa_2>0$,

$$\begin{aligned}&\hat{Z}\in L^2(\Omega\times[0,t_u^*];\mathbb{R}^{MN}),\\&\mathbb{E}\textstyle\int_0^{t_u^*}|\hat{Z}(s)|^2\mathrm{d}s\le\tau_u^*\sup_{s\in[0,\tau_u^*]}\mathbb{E}|\hat{Z}(s)|^2\\&\le\tau_u^*e^{3(2\|\mathbb{A}+J_{0,M}\otimes\mathcal{C}_0\|^2+2\bar{g}^2)\tau_u^*(\tau_u^*+4)}=:\kappa_2.\end{aligned}\tag{2.26}$$

In the second step, we consider $t\in[t_u^*,t_u^*+t_m]$, where $t_m=\min\{\tau_{m,y},r/M\}$. We have

$$\begin{aligned}&t-\tfrac{r}{M},t-\tau_y(t)\in[0,t_u^*],\quad t\in[t_u^*,t_u^*+t_m],\\&t-r-\tau_u(t)\in[0,t_u^*],\qquad t\in[t_u^*,t_u^*+t_m].\end{aligned}\tag{2.27}$$

From (2.26) and (2.27), it follows $\mathbb{E}\int_{t_u^*-t_m}^{t_u^*}|u(t-r-\tau_u(t))|^2\mathrm{d}t\le\kappa_3$ and $\mathbb{E}\int_{t_u^*-t_m}^{t_u^*}|F_u(t-r-\tau_u)|^2\mathrm{d}t<\kappa_3$ for some $\kappa_3>0$. By Theorem 6.7.4 in [29], we have that (2.21), with $u(t-r-\tau_u(t))$ and $F_u(t-r-\tau_u)$ as non-homogeneous terms and initial value $w(\cdot,t_u^*)\in\mathcal{D}(\mathcal{A})$ almost surely (obtained at the previous step), has a unique strong solution $w\in L^2(\Omega;C([t_u^*,t_u^*+t_m]);L^2(0,1))\cap L^2(\Omega\times[t_u^*,t_u^*+t_m];H^1(0,1))$ and $w(\cdot,t)\in\mathcal{D}(\mathcal{A})$, $t\in[t_u^*,t_u^*+t_m]$ almost surely. Next, consider (2.22) for $t\in[t_u^*,t_u^*+t_m]$ with initial condition $\hat{Z}(t_u^*)\in L^2(\Omega;\mathbb{R}^{MN})$ obtained at the previous step. We have $\mathbb{E}\int_0^{t_u^*+t_m}|y(s)|^2\mathrm{d}s=\mathbb{E}\int_0^{t_u^*+t_m}|\langle c,w(\cdot,s-\tau_y)\rangle+\langle c,\psi\rangle u(s-\tau_y(s)-r-\tau_u(s-\tau_y(s)))|^2\mathrm{d}s<\kappa_4$ for certain $\kappa_4>0$, which together with (2.26) implies that (2.22), with $w(\cdot,t-\tau_y)$ and other delay terms as non-homogeneous terms, has a unique solution that satisfies $\hat{Z}\in L^2(\Omega\times[t_u^*,t_u^*+t_m];\mathbb{R}^{MN})$ and $\mathbb{E}\int_0^{t_u^*+t_m}|\hat{Z}(s)|^2\mathrm{d}s<\kappa_5$ for some $\kappa_5>0$.

Continuing step-by-step on $[t_u^*+t_m,t_u^*+2t_m]$, $[t_u^*+2t_m,t_u^*+3t_m]$, … we obtain the existence of a unique strong solution to (2.21) satisfying $w\in L^2(\Omega;C([0,\infty);L^2(0,1)))\cap L^2(\Omega\times[0,\infty)\backslash\mathcal{J};H^1(0,1))$ such that $w(\cdot,t)\in\mathcal{D}(\mathcal{A})$, $t\ge 0$, almost everywhere, and a unique solution to (2.22) satisfying $\hat{Z}\in L^2(\Omega\times[0,\infty);\mathbb{R}^{MN})$, where $\mathcal{J}=\{t_u^*+jt_m\}_{j=0}^{\infty}$. Using the change of variables (2.19), we have that (2.1) subject to control law (2.18) admits a unique solution satisfying (2.23).

For sawtooth delays case, we can similarly obtained that the existence of a unique strong solution to (2.21) satisfying $w \in L^2(\Omega; C([0,\infty); L^2(0,1)))\backslash\mathcal{J} \cap L^2(\Omega\times[0,\infty)\backslash\mathcal{J}; H^1(0,1))$ such that $w(\cdot,t)\in\mathcal{D}(\mathcal{A})$, $t\in\mathbb{R}^+$, almost surely, and (2.1) subject to control law (2.18) admits a unique solution satisfying (2.23).

2.3. Mean-square L^2 stability analysis

Define the estimation errors as follows

$$\begin{aligned}&e_i^j(t)=\hat z_{i+1}^j(t-\tfrac{M-i}{M}r)-\hat z_i^j(t-\tfrac{M-i+1}{M}r), 1\le i\le M-1,\\&e_M^j(t)=z^j(t)-\hat z_M^j(t-\tfrac{r}{M}),\quad j\in\{N_0,N-N_0\}.\end{aligned}\tag{2.28}$$

Then the last term on the right-hand-side of differential equation for $\hat z_M^{N_0}(t)$ in (2.15) can be presented as

$$\begin{aligned}&C_0\hat z_M^{N_0}(t-\tfrac{r}{M})+C_1\hat z_M^{N-N_0}(t-\tfrac{r}{M})-y(t)\\&\overset{(2.3)}{=}-[C_0e_M^{N_0}(t)+C_1e_M^{N-N_0}(t)+\zeta(t-\tau_y)]\\&\qquad+C_0Y_{\tau_y}^{N_0}(t)+C_1Y_{\tau_y}^{N-N_0}(t),\\&Y_{\tau_y}^j(t)=z^j(t)-z^j(t-\tau_y),\ j\in\{N_0,N-N_0\},\\&\zeta(t)=\textstyle\sum_{n=N+1}^\infty c_nz_n(t).\end{aligned}\tag{2.29}$$

Furthermore, by (2.28), we get

$$\hat z_1^{N_0}(t-r)+\textstyle\sum_{i=1}^M e_i^{N_0}(t)=z^{N_0}(t).\tag{2.30}$$

In particular, if the errors $e_i^{N_0}(t)$, $1\le i\le M$ converge to zero, from (2.30) we have $\hat z_1^{N_0}(t)\to z^{N_0}(t+r)$, meaning that $\hat z_1^{N_0}(t)$ predicts the future system state $z^{N_0}(t+r)$.

Using (2.10), (2.15), and (2.29), we arrive at

$$\begin{aligned}&\mathrm{d}e_M^{N_0}(t)=[(A_0-L_0C_0)e_M^{N_0}(t)+B_0K_0Y_{\tau_u}^{N_0}(t)+H_M^{N_0}(t)\\&\quad-L_0C_1e_M^{N-N_0}(t)+L_0C_0Y_{M,r}^{N_0}(t)+L_0C_1Y_{M,r}^{N-N_0}(t)\\&\quad+L_0C_0\bar Y_{\bar\tau_y}^{N_0}(t)+L_0C_1\bar Y_{\bar\tau_y}^{N-N_0}(t)\\&\quad-L_0\zeta(t-\tfrac{r}{M}-\bar\tau_y)]\mathrm{d}t+\sigma^{N_0}(t)\mathrm{d}\mathcal{W}(t),\\&\mathrm{d}e_M^{N-N_0}(t)=[A_1e_M^{N-N_0}(t)+H_M^{N-N_0}(t)\\&\quad+B_1K_0Y_{\tau_u}^{N_0}(t)]\mathrm{d}t+\sigma^{N-N_0}(t)\mathrm{d}\mathcal{W}(t),\\&\mathrm{d}e_{M-1}^{N_0}(t)=[(A_0-L_0C_0)e_{M-1}^{N_0}(t)+L_0C_0Y_{M-1,r}^{N_0}(t)\\&\quad+H_{M-1}^{N_0}(t)-L_0C_1e_{M-1}^{N-N_0}(t)+L_0C_1Y_{M-1,r}^{N-N_0}(t)\\&\quad+L_0(C_0e_M^{N_0}(t)+C_1e_M^{N-N_0}(t)+\zeta(t-\tfrac{r}{M}-\bar\tau_y))\\&\quad-L_0C_0\bar Y_{\bar\tau_y}^{N_0}(t)-L_0C_1\bar Y_{\bar\tau_y}^{N-N_0}(t)\\&\quad-L_0C_0Y_{M,r}^{N_0}(t)-L_0C_1Y_{M,r}^{N-N_0}(t)]\mathrm{d}t,\\&\mathrm{d}e_i^{N_0}(t)=[(A_0-L_0C_0)e_i^{N_0}(t)+L_0C_0Y_{i,r}^{N_0}(t)+H_i^{N_0}(t)\\&\quad+L_0[C_0e_{i+1}^{N_0}(t)+C_1e_{i+1}^{N-N_0}(t)]-L_0C_0Y_{i+1,r}^{N_0}(t)\\&\quad-L_0C_1Y_{i+1,r}^{N-N_0}(t)-L_0C_1e_i^{N-N_0}(t)\\&\quad+L_0C_1Y_{i,r}^{N-N_0}(t)]\mathrm{d}t,\ 1\le i\le M-2,\\&\mathrm{d}e_i^{N-N_0}(t)=[A_1e_i^{N-N_0}(t)+H_i^{N-N_0}(t)]\mathrm{d}t, 1\le i\le M-1,\end{aligned}\tag{2.31}$$

where

$$\begin{aligned}&Y_{\tau_u}^{N_0}(t)=\hat z_1^{N_0}(t-r)-\hat z_1^{N_0}(t-r-\tau_u),\\&\bar Y_{\bar\tau_y}^j(t)=z^j(t-\tfrac{r}{M})-z^j(t-\tfrac{r}{M}-\bar\tau_y),\quad \bar\tau_y=\tau_y(t-\tfrac{r}{M}),\\&Y_{i,r}^j(t)=e_i^j(t)-e_i^j(t-\tfrac{r}{M}),\quad 1\le i\le M,\\&H_i^j(t)=\hat G_{i+1}^j(t-\tfrac{M-i}{M}r)-G_i^j(t-\tfrac{M-i+1}{M}r),\ 1\le i\le M-1,\\&H_M^j(t)=G^j(t)-\hat G_M^j(t),\quad j\in\{N_0,N-N_0\}.\end{aligned}\tag{2.32}$$

Introduce the notations

$$\begin{aligned}&X_z(t)=\mathrm{col}\{z^{N_0}(t),z^{N-N_0}(t)\},\ \mathcal{B}=\mathrm{col}\{B_0,B_1\},\\&F_z=\begin{bmatrix}A_0-B_0K_0&0\\-B_1K_0&A_1\end{bmatrix},\quad G(t)=\begin{bmatrix}G^{N_0}(t)\\G^{N-N_0}(t)\end{bmatrix},\\&\bar Y_{\bar\tau_y}(t)=\begin{bmatrix}\bar Y_{\bar\tau_y}^{N_0}(t)\\\bar Y_{\bar\tau_y}^{N-N_0}(t)\end{bmatrix},\\&X_e(t)=\mathrm{col}\{e_1^{N_0}(t),\dots,e_M^{N_0}(t),e_1^{N-N_0}(t),\dots,e_M^{N-N_0}(t)\},\\&\mathcal{I}_0=[I_{N_0},\dots,I_{N_0},0_{N_0\times M(N-N_0)}]\in\mathbb{R}^{N_0\times MN},\ \mathcal{C}=[C_0,\ C_1],\\&\tilde{\mathcal{I}}_0=[I_{N_0},0_{N_0\times(N-N_0)}],\ Y_{e,r}(t)=X_e(t)-X_e(t-\tfrac{r}{M}),\\&H(t)=\mathrm{col}\{H_1^{N_0}(t),\dots,H_M^{N_0}(t),H_1^{N-N_0}(t),\dots,H_M^{N-N_0}(t)\},\\&\mathbf{L}_\zeta=\begin{bmatrix}0_{N_0(M-2)\times1}\\L_0\\-L_0\\0_{M(N-N_0)\times1}\end{bmatrix},\quad\tilde{\mathcal{I}}_1=\begin{bmatrix}0_{(M-1)N_0\times N}\\ [I_{N_0}\quad 0]\\0_{(M-1)(N-N_0)\times N}\\ [0\quad I_{N-N_0}]\end{bmatrix},\\&\bar\sigma(t)=\begin{bmatrix}0_{N_0(M-1)\times1}\\\sigma^{N_0}(t)\\0_{(N-N_0)(M-1)\times1}\\\sigma^{N-N_0}(t)\end{bmatrix},\ \sigma(t)=\begin{bmatrix}\sigma^{N_0}(t)\\\sigma^{N-N_0}(t)\end{bmatrix},\\&F_e=\\&\begin{bmatrix}I_M\otimes(A_0-L_0C_0)+J_{0,M}\otimes L_0C_0&-I_M\otimes L_0C_1+J_{0,M}\otimes L_0C_1\\0&I_M\otimes A_1\end{bmatrix},\\&\Lambda_e=\begin{bmatrix}I_M\otimes L_0C_0-J_{0,M}\otimes L_0C_0&I_M\otimes L_0C_1-J_{0,M}\otimes L_0C_1\\0_{M(N-N_0)\times MN_0}&0\end{bmatrix}.\end{aligned}\tag{2.33}$$

Then from (2.5), (2.10), (2.18), , we have the following closed-loop system for $t\ge0$,

$$\begin{aligned}&\mathrm{d}X_z(t)=[F_zX_z(t)+\mathcal{B}K_0\mathcal{I}_0X_e(t)+G(t)\\&\quad+\mathcal{B}K_0Y_{\tau_u}^{N_0}(t)]\mathrm{d}t+\sigma(t)\mathrm{d}\mathcal{W}(t),\\&\mathrm{d}X_e(t)=[F_eX_e(t)+H(t)+\Lambda_eY_{e,r}(t)-\mathbf{L}_\zeta\mathcal{C}\bar Y_{\bar\tau_y}(t)\\&\quad+\mathcal{I}_1\mathcal{B}K_0Y_{\tau_u}^{N_0}(t)+\mathbf{L}_\zeta\zeta(t-\tfrac{r}{M}-\bar\tau_y)]\mathrm{d}t+\bar\sigma(t)\mathrm{d}\mathcal{W}(t),\\&\mathrm{d}z_n(t)=[-\lambda_nz_n(t)+g_n(t)-b_nK_0[\tilde{\mathcal{I}}X_z(t)-\mathcal{I}_0X_e(t)]\\&\quad+b_nK_0Y_{\tau_u}^{N_0}(t)]\mathrm{d}t+\sigma_n(t)\mathrm{d}\mathcal{W}(t),n>N.\end{aligned}\tag{2.34}$$

For $M=1$, we have closed-loop system (2.34) with F_e, Λ_e, $\mathbf{L}_\zeta$ replaced by

$$F_e=\begin{bmatrix}A_0-L_0C_0&-L_0C_1\\0&A_1\end{bmatrix},\ \mathbf{L}_\zeta=\begin{bmatrix}-L_0\\0_{(N-N_0)\times1}\end{bmatrix},$$
$$\Lambda_e=\begin{bmatrix}L_0\mathcal{C}\\0_{(N-N_0)\times N}\end{bmatrix}.$$

Let

$$\begin{aligned}&X(t)=\mathrm{col}\{X_z(t),X_e(t)\},\ Y_r=X(t)-X(t-\tfrac{r}{M}),\\&\mathbf{I}_1=\begin{bmatrix}I_N\\0_{MN\times N}\end{bmatrix},\ \mathbf{I}_2=\begin{bmatrix}0_{N\times MN}\\I_{MN}\end{bmatrix},\ \Lambda=\mathbf{I}_2\Lambda_e\mathbf{I}_2^{\mathrm{T}},\ \mathbf{L}=\mathbf{I}_2\mathbf{L}_\zeta,\\&F=\begin{bmatrix}F_z&\mathcal{B}K_0\mathcal{I}_0\\0&F_e\end{bmatrix},\ \mathcal{I}_1=\begin{bmatrix}I_N\\\tilde{\mathcal{I}}_1\end{bmatrix},\ \mathcal{I}=[\tilde{\mathcal{I}}_0,-\mathcal{I}_0],\\&\mathbf{F}(t)=FX(t)+\mathbf{I}_1G(t)+\mathcal{I}_1\mathcal{B}K_0Y_{\tau_u}^{N_0}(t)+\mathbf{I}_2H(t)\\&\qquad+\Lambda Y_r(t)+\mathbf{L}\zeta(t-\tfrac{r}{M}-\tau_y)-\mathbf{L}\mathcal{C}\bar Y_{\bar\tau_y}(t).\end{aligned}$$

We can write (2.34) as

$$\mathrm{d}X(t) = \mathbf{F}(t)\mathrm{d}t + \mathcal{I}_1\sigma(t)\mathrm{d}\mathcal{W}(t), \tag{2.35a}$$

$$\begin{aligned}\mathrm{d}z_n(t) &= [-\lambda_n z_n(t) + g_n(t) - b_n K_0 \mathcal{I} X(t)\\ &\quad + b_n K_0 Y^{N_0}_{\tau_u}(t)]\mathrm{d}t + \sigma_n(t)\mathrm{d}\mathcal{W}(t),\ n > N.\end{aligned} \tag{2.35b}$$

Remark 4. In closed-loop system (2.35), the delay term $\bar{Y}_{\bar{\tau}_y}(t)$ corresponding to $z^j(t-\frac{r}{M})$ is induced by $e^j_M(t) = z^j(t) - \hat{z}^j_M(t-\frac{r}{M})$ with $z^j(t-\frac{r}{M})$ appearing in ODE system for $\hat{z}^{N_0}_M(t-\frac{r}{M})$. To compensate $\bar{Y}_{\bar{\tau}_y}(t)$, it will result in delay terms $z^j(t-\frac{r}{M})$. Besides, both the stochastic term and the delay fraction r/M appear in system e^j_M. Therefore, we need to construct Lyapunov functional (as introduced in [32]) that depends on stochastic term $\mathcal{I}_1\sigma(t)$ for the compensation of $Y_r(t)$ (see V_{Q_r} in (2.38)). Moreover, the delay term $Y^{N_0}_{\tau_u}(t)$ corresponds to $\hat{z}^{N_0}_1(t-r) = \mathcal{I}X(t)$. From (2.30) and (2.35a), it follows

$$\mathrm{d}\mathcal{I}\,X(t) = \mathcal{I}\mathbf{F}(t)\mathrm{d}t + \mathcal{I}\mathcal{I}_1\sigma(t)\mathrm{d}\mathcal{W}(t), \tag{2.36}$$

where $\mathcal{I}\mathcal{I}_1 \neq 0$. This means that we need to construct Lyapunov functional that depends on stochastic term $\mathcal{I}\mathcal{I}_1\sigma(t)$ for the compensation of $Y^{N_0}_{\tau_u}(t)$ (see V_{Q_u} in (2.38)).

For the mean-square L^2 exponential stability of system (2.35), we consider the following Lyapunov functional:

$$V(t) = V_{\text{tail}}(t) + V_P + V_y(t) + V_r(t) + V_u(t) \tag{2.37}$$

with

$$\begin{aligned}
&V_{\text{tail}}(t) = \rho\textstyle\sum_{n=N+1}^{\infty} z_n^2(t), \quad V_P(t) = |X(t)|_P^2,\\
&V_y(t) = V_{S_y}(t) + V_{R_y}(t) + V_{Q_y}(t),\\
&V_{S_y}(t) = \int_{t-\frac{r}{M}-\tau_{M,y}}^{t-\frac{r}{M}} \mathrm{e}^{-2\delta(t-\frac{r}{M}-s)}|\mathbf{I}_1^{\mathrm{T}}X(s)|^2_{S_y}\mathrm{d}s,\\
&V_{R_y}(t) = \tau_{M,y}\int_{-\frac{r}{M}-\tau_{M,y}}^{-\frac{r}{M}}\int_{t+\theta}^{t} \mathrm{e}^{-2\delta(t-\frac{r}{M}-s)}|\mathbf{I}_1^{\mathrm{T}}\mathbf{F}(s)|^2_{R_y}\mathrm{d}s\mathrm{d}\theta,\\
&V_{Q_y}(t) = \int_{-\frac{r}{M}-\tau_{M,y}}^{-\frac{r}{M}}\int_{t+\theta}^{t} \mathrm{e}^{-2\delta(t-\frac{r}{M}-s)}|\sigma(s)|^2_{Q_y}\mathrm{d}s\mathrm{d}\theta,\\
&V_r(t) = V_{S_r}(t) + V_{R_r}(t) + V_{Q_r}(t),\\
&V_{S_r}(t) = \int_{t-\frac{r}{M}}^{t} \mathrm{e}^{-2\delta(t-s)}|X(s)|^2_{S_r}\mathrm{d}s,\\
&V_{R_r}(t) = \frac{r}{M}\int_{-\frac{r}{M}}^{0}\int_{t+\theta}^{t} \mathrm{e}^{-2\delta(t-s)}|\mathbf{F}(s)|^2_{R_r}\mathrm{d}s\mathrm{d}\theta,\\
&V_{Q_r}(t) = \int_{-\frac{r}{M}}^{0}\int_{t+\theta}^{t} \mathrm{e}^{-2\delta(t-s)}|\mathcal{I}_1\sigma(t)|^2_{Q_r}\mathrm{d}s\mathrm{d}\theta,\\
&V_u(t) = V_{S_u}(t) + V_{R_u}(t) + V_{Q_u}(t),\\
&V_{S_u}(t) = \int_{t-\tau_{M,u}}^{t} \mathrm{e}^{-2\delta(t-s)}|\mathcal{I}X(s)|^2_{S_u}\mathrm{d}s,\\
&V_{R_u}(t) = \tau_{M,u}\int_{-\tau_{M,u}}^{0}\int_{t+\theta}^{t} \mathrm{e}^{-2\delta(t-s)}|\mathcal{I}\mathbf{F}(s)|^2_{R_u}\mathrm{d}s\mathrm{d}\theta,\\
&V_{Q_u}(t) = \int_{-\tau_{M,u}}^{0}\int_{t+\theta}^{t} \mathrm{e}^{-2\delta(t-s)}|\mathcal{I}\mathcal{I}_1\sigma(t)|^2_{Q_u}\mathrm{d}s\mathrm{d}\theta,
\end{aligned} \tag{2.38}$$

where P, S_i, R_i, Q_i, $i \in \{y, r, u\}$ are positive matrices of appropriate dimensions and $\rho > 0$ is a scalar. The term V_y is introduced to compensate $\bar{Y}_{\bar{\tau}_y}$. The term V_r is used to compensate Y_r. The term V_u is utilized to compensate $Y^{N_0}_{\tau_u}$. Finally, to compensate $\zeta(t - r/M - \bar{\tau}_y)$, we will use Halanay's inequality with respect to $\mathbb{E}V(t)$.

By Parseval's equality (see [33, Proposition 10.29]) and the change of variables (2.19), we present $V_{\text{tail}}(t)$ in (2.37) as

$$\begin{aligned}&V_{\text{tail}}(t) = -V_1(t) + V_2(w(t), t),\ V_1(t) = \rho|\mathbf{I}_1^{\mathrm{T}}X(t)|^2,\\ &V_2(w(t), t) = \rho\|w(\cdot, t) + \psi(\cdot)u(t-r-\tau_u)\|^2_{L^2}.\end{aligned} \tag{2.39}$$

For function V_1, calculating the generator $\mathcal{L}$ along stochastic ODE (2.35a) (see [34, P. 149]), we have

$$\begin{aligned}\mathcal{L}V_1(t) + 2\delta V_1(t) &= \rho\textstyle\sum_{n=1}^{N} 2(-\lambda_n + \delta)z_n^2(t)\\ &\quad + \rho|\sigma(t)|^2 + \rho\textstyle\sum_{n=1}^{N} 2z_n(t)[g_n(t) - b_n K_0\mathcal{I}X(t)]\\ &\quad + \rho\textstyle\sum_{n=1}^{N} 2z_n(t)b_n K_0 Y^{N_0}_{\tau_u}(t).\end{aligned} \tag{2.40}$$

Note that $w(t)$ is a strong solution to (2.21) (see Section 2.2). For $V_2(w, t)$, calculating the generator $\mathcal{L}$ along (2.21) (see [29, P. 228]) we obtain for continuously differentiable delay τ_u

$$\begin{aligned}\mathcal{L}V_2(w(t), t) &= \tfrac{\partial V_2(w(t),t)}{\partial t} + \langle\tfrac{\partial V_2(w(t),t)}{\partial w}, \mathcal{A}w(t)\rangle\\
&+\langle\tfrac{\partial V_2(w(t),t)}{\partial w}, g(\cdot, w(t) + \psi(\cdot)u(t-r-\tau_u))\rangle\\
&-\langle\tfrac{\partial V_2(w(t),t)}{\partial w}, \psi(\cdot)\rangle[\mu u(t-r-\tau_u) + (1-\tau_u')F_u(t-r-\tau_u)]\\
&+\tfrac{1}{2}\Big\langle\tfrac{\partial^2 V_2(w(t),t)}{\partial w^2}\sigma(\cdot, w(t) + \psi(\cdot)u(t-r-\tau_u)),\\
&\qquad\qquad \sigma(\cdot, w(t) + \psi(\cdot)u(t-r-\tau_u))\Big\rangle,\end{aligned} \tag{2.41}$$

and for sawtooth delays

$$\begin{aligned}\mathcal{L}V_2(w(t), t) &= \langle\tfrac{\partial V_2(w(t),t)}{\partial w}, \mathcal{A}w(t) - \mu\psi(\cdot)u(t-r-\tau_u)\rangle,\\
&+\langle\tfrac{\partial V_2(w(t),t)}{\partial w}, g(\cdot, w(t) + \psi(\cdot)u(t-r-\tau_u))\rangle\\
&+\tfrac{1}{2}\Big\langle\tfrac{\partial^2 V_2(w(t),t)}{\partial w^2}\sigma(\cdot, w(t) + \psi(\cdot)u(t-r-\tau_u)),\\
&\qquad\qquad \sigma(\cdot, w(t) + \psi(\cdot)u(t-r-\tau_u))\Big\rangle.\end{aligned} \tag{2.42}$$

For both cases (2.41) and (2.42), we have

$$\begin{aligned}\mathcal{L}V_2(w(t), t) &= 2\rho\langle z(\cdot, t), \mathcal{A}w(t) + g(\cdot, z(\cdot, t))\rangle\\
&\quad + 2\rho\langle z(\cdot, t), \psi''(\cdot)\rangle u(t-r-\tau_u) + \|\sigma(\cdot, z(\cdot, t))\|^2_{L^2}\\
&\overset{(2.2)}{\leq} 2\rho\langle\mathcal{A}w(t) + g(\cdot, z(\cdot, t)), z(\cdot, t)\rangle + \rho\bar{\sigma}^2\|z(\cdot, t)\|^2_{L^2}\\
&\quad + 2\rho\langle z(\cdot, t), \psi''(\cdot)\rangle u(t-r-\tau_u).\end{aligned} \tag{2.43}$$

By Parseval's equality (see [33, Proposition 10.29]), we have

$$\begin{aligned}&\langle\mathcal{A}w(t) + g(\cdot, z(\cdot, t)), z(\cdot, t)\rangle\\
&= \textstyle\sum_{n=1}^{\infty} z_n(t)\langle\mathcal{A}w(t), \phi_n\rangle + \sum_{n=1}^{\infty} z_n(t)g_n(t),\\
&\langle\psi(\cdot), z(t)\rangle = \textstyle\sum_{n=1}^{\infty} z_n(t)\langle\psi(\cdot), \phi_n\rangle.\end{aligned} \tag{2.44}$$

Using integration by parts, (1.1) and (2.20), we arrive at

$$\begin{aligned}\langle\mathcal{A}w(t), \phi_n\rangle &= -\lambda_n w_n(t)\\
&= -\lambda_n z_n(t) + \lambda_n\langle\psi, \phi_n\rangle u(t-r-\tau_u),\\
\langle\psi''(\cdot), \phi_n\rangle &= b_n - \lambda_n\langle\psi, \phi_n\rangle.\end{aligned} \tag{2.45}$$

Substituting (2.44) and (2.45) into (2.43), and using (2.18), (2.30), we get

$$\begin{aligned}&\mathbb{E}[\mathcal{L}V_2(w(t), t) + 2\delta V_2(w(t), t)]\\
&\leq \rho\mathbb{E}\textstyle\sum_{n=1}^{\infty} 2(-\lambda_n + \delta + \frac{\bar{\sigma}^2}{2})z_n^2(t) + \rho\mathbb{E}\sum_{n=1}^{\infty} 2z_n(t)g_n(t)\\
&\quad + \rho\mathbb{E}\textstyle\sum_{n=1}^{\infty} 2z_n(t)[-b_n K_0\mathcal{I}X(t) + b_n K_0 Y^{N_0}_{\tau_u}(t)].\end{aligned} \tag{2.46}$$

Note that $w(\cdot, t) \in \mathcal{D}(\mathcal{A})$ almost surely and $w \in L^2(\Omega \times [0, \infty)\backslash\mathcal{J}; H^1(0, 1))$, we have $\mathbb{E}\|w(\cdot, t)\|^2_{H^1} < \infty$. By (2.19), we get $\mathbb{E}\|z(\cdot, t)\|^2_{H^1} \leq \mathbb{E}\|w(\cdot, t)\|^2_{H^1} + \|\psi\|^2_{H^1}|u(t-r-\tau_u)|^2$. From (2.18) and $\mathbb{E}|\hat{z}_1(t)|^2 < \infty$ (see Section 2.2), it follows $\mathbb{E}\sum_{n=1}^{\infty}\lambda_n z_n^2(t) \leq \mathbb{E}\|z(\cdot, t)\|^2_{H^1} < \infty$. Let $\alpha_1, \alpha_2, \alpha_3 > 0$. By the Young inequalities we have

$$\begin{aligned}&\textstyle\sum_{n=N+1}^{\infty} 2z_n(t)g_n(t) \leq \sum_{n=N+1}^{\infty}\frac{z_n^2(t)}{\alpha_1} + \alpha_1\sum_{n=N+1}^{\infty} g_n^2(t)\\
&\quad = \textstyle\sum_{n=N+1}^{\infty}\frac{1}{\alpha_1}z_n^2(t) - \alpha_1|G(t)|^2 + \alpha_1\sum_{n=1}^{\infty} g_n^2(t),\\
&\textstyle\sum_{n=N+1}^{\infty} 2z_n(t)[-b_n K_0\mathcal{I}X(t) + b_n K_0 Y^{N_0}_{\tau_u}(t)]\\
&\quad\overset{(2.7)}{\leq} \textstyle\sum_{n=N+1}^{\infty}\frac{\lambda_n}{\alpha_2}z_n^2(t) + \frac{2\alpha_2(N+1)}{N^2\pi^2}|K_0\mathcal{I}X(t)|^2\\
&\qquad + \textstyle\sum_{n=N+1}^{\infty}\frac{\lambda_n}{\alpha_3}z_n^2(t) + \frac{2\alpha_3(N+1)}{N^2\pi^2}|K_0 Y^{N_0}_{\tau_u}(t)|^2.\end{aligned} \tag{2.47}$$

By Parseval's equality we have

$$\begin{aligned}\textstyle\sum_{n=1}^{\infty} g_n^2(t) &\overset{(2.6)}{=} \|g(\cdot, z(\cdot, t))\|^2_{L^2}\\
&\overset{(2.2)}{\leq} \bar{g}^2|\mathbf{I}_1^{\mathrm{T}}X(t)|^2 + \bar{g}^2\textstyle\sum_{n=N+1}^{\infty} z_n^2(t).\end{aligned} \tag{2.48}$$

Combination of (2.40), (2.46), (2.47), and (2.48) gives

$$\begin{aligned}&\mathbb{E}[\mathcal{L}V_{\rm tail}(t)+2\delta V_{\rm tail}(t)]\le\mathbb{E}\textstyle\sum_{n=N+1}^{\infty}\chi_n z_n^2(t)\\&+\tfrac{2\rho(N+1)}{N^2\pi^2}[\alpha_2\mathbb{E}|K_0\mathcal{I}X(t)|^2+\alpha_3\mathbb{E}|K_0Y_{\tau_u}^{N_0}(t)|^2]\\&+\mathbb{E}[\rho(\bar\sigma^2+\alpha_1\bar g^2)|\mathbf{I}_1^{\rm T}X(t)|^2-\rho|\sigma(t)|^2-\rho\alpha_1|G(t)|^2],\end{aligned}\tag{2.49}$$

where $\chi_n=\rho(-2\lambda_n+2\delta+\bar\sigma^2+\alpha_1\bar g^2+\frac{1}{\alpha_1}+\frac{\lambda_n}{\alpha_2}+\frac{\lambda_n}{\alpha_3})$.

For V_P, V_y, V_u, V_r, calculating the generator $\mathcal{L}$ along (2.35a) (see [34, P. 149]), we have

$$\begin{aligned}&\mathcal{L}V_P(t)+2\delta V_P(t)=X^{\rm T}(t)[PF+F^{\rm T}P+2\delta P]X(t)\\&\quad+|\mathcal{I}_1\sigma(t)|_P^2+2X^{\rm T}(t)P[\mathbf{I}_1G(t)+\mathcal{I}_1\mathcal{B}K_0Y_{\tau_u}^{N_0}(t)\\&\quad+\mathbf{I}_2H(t)+\Lambda Y_r(t)+\mathbf{L}\zeta(t-\tfrac{r}{M}-\bar\tau_y)-\mathbf{L}C\bar Y_{\bar\tau_y}(t)],\\&\mathcal{L}V_y(t)+2\delta V_y(t)\le|\mathbf{I}_1^{\rm T}X(t)-\mathbf{I}_1^{\rm T}Y_r(t)|_{S_y}^2\\&\quad-\varepsilon_y|\mathbf{I}_1^{\rm T}X(t)-\mathbf{I}_1^{\rm T}Y_r(t)-\bar Y_{\bar\tau_y}(t)-\bar v_{\bar\tau_y}(t)|_{S_y}^2\\&\quad+\tau_{M,y}^2e^{2\delta r/M}|\mathbf{I}_1^{\rm T}\mathbf{F}(t)|_{R_y}^2-\varepsilon_y\tau_{M,y}\int_{t-\frac{r}{M}-\tau_{M,y}}^{t-\frac{r}{M}}|\mathbf{I}_1^{\rm T}\mathbf{F}(s)|_{R_y}^2ds\\&\quad+\tau_{M,y}e^{2\delta r/M}|\sigma(t)|_{Q_y}^2-\varepsilon_y\int_{t-\frac{r}{M}-\tau_{M,y}}^{t-\frac{r}{M}}|\sigma(s)|_{Q_y}^2ds,\end{aligned}\tag{2.50}$$

$$\begin{aligned}&\mathcal{L}V_u(t)+2\delta V_u(t)\le|\mathcal{I}X(t)|_{S_u}^2-\varepsilon_u|\mathcal{I}X(t)-Y_{\tau_u}^{N_0}(t)-v_{\tau_u}^{N_0}(t)|_{S_u}^2\\&\quad+\tau_{M,u}^2|\mathcal{I}\mathbf{F}(t)|_{R_u}^2-\varepsilon_u\tau_{M,u}\int_{t-\tau_{M,u}}^t|\mathcal{I}\mathbf{F}(s)|_{R_u}^2ds\\&\quad+\tau_{M,u}|\mathcal{I}\mathcal{I}_1\sigma(t)|_{Q_u}^2-\varepsilon_u\int_{t-\tau_{M,u}}^t|\mathcal{I}\mathcal{I}_1\sigma(s)|_{Q_u}^2ds,\\&\mathcal{L}V_r(t)+2\delta V_r(t)\le|X(t)|_{S_r}^2-\varepsilon_M|X(t)-Y_r(t)|_{S_r}^2\\&\quad+\tfrac{r^2}{M^2}|\mathbf{F}(t)|_{R_r}^2-\tfrac{r\varepsilon_M}{M}\int_{t-\frac{r}{M}}^t|\mathbf{F}(s)|_{R_r}^2ds\\&\quad+\tfrac{r}{M}|\mathcal{I}_1\sigma(t)|_{Q_r}^2-\varepsilon_M\int_{t-\frac{r}{M}}^t|\mathcal{I}_1\sigma(s)|_{Q_r}^2ds,\end{aligned}$$

where $\varepsilon_y=e^{-2\delta\tau_{M,y}}$, $\varepsilon_u=e^{-2\delta\tau_{M,u}}$, $\varepsilon_M=e^{-2\delta r/M}$, $\bar v_{\bar\tau_y}(t)=X_z(t-\frac{r}{M}-\bar\tau_y)-X_z(t-\frac{r}{M}-\tau_{M,y})$, $v_{\tau_u}^{N_0}(t)=\hat z_1^{N_0}(t-r-\tau_u)-\hat z_1^{N_0}(t-r-\tau_{M,u})$. By using the Itô integral properties (see [27,34]), we have

$$\begin{aligned}&\mathbb{E}\int_{t-\frac{r}{M}-\tau_{M,y}}^{t-\frac{r}{M}}|\sigma(s)|_{Q_y}^2ds=\mathbb{E}\begin{bmatrix}\xi_{y,1}(t)\\\xi_{y,2}(t)\end{bmatrix}^{\rm T}\begin{bmatrix}Q_y&0\\0&Q_y\end{bmatrix}\begin{bmatrix}\xi_{y,1}(t)\\\xi_{y,2}(t)\end{bmatrix},\\&\mathbb{E}\int_{t-\tau_{M,u}}^t|\mathcal{I}\mathcal{I}_1\sigma(s)|_{Q_u}^2ds=\mathbb{E}\begin{bmatrix}\xi_{u,1}(t)\\\xi_{u,2}(t)\end{bmatrix}^{\rm T}\begin{bmatrix}Q_u&0\\0&Q_u\end{bmatrix}\begin{bmatrix}\xi_{u,1}(t)\\\xi_{u,2}(t)\end{bmatrix},\\&\mathbb{E}\int_{t-\frac{r}{M}}^t|\mathcal{I}_1\sigma(s)|_{Q_r}^2ds=\mathbb{E}|\xi_r(t)|_{Q_r}^2,\ \xi_r(t)=\int_{t-\frac{r}{M}}^t\mathcal{I}_1\sigma(s)d\mathcal{W}(s),\\&\xi_{y,1}(t)=\int_{t-r/M-\tau_y}^{t-r/M}\sigma(s)d\mathcal{W}(s),\quad\xi_{u,1}(t)=\int_{t-\tau_u}^t\mathcal{I}\mathcal{I}_1\sigma(s)d\mathcal{W}(s),\\&\xi_{y,2}(t)=\int_{t-r/M-\tau_{M,u}}^{t-r/M-\tau_u}\sigma(s)d\mathcal{W}(s),\ \xi_{u,2}(t)=\int_{t-\tau_{M,u}}^{t-\tau_u}\mathcal{I}\mathcal{I}_1\sigma(s)d\mathcal{W}(s).\end{aligned}\tag{2.51}$$

Let $G_y\in\mathbb{R}^{N\times N}$, $G_u\in\mathbb{R}^{N_0\times N_0}$ satisfy

$$\begin{bmatrix}R_y&G_y\\ *&R_y\end{bmatrix}\ge0,\quad\begin{bmatrix}R_u&G_u\\ *&R_u\end{bmatrix}\ge0.\tag{2.52}$$

Applying Jensen's and Park's inequalities (see, e.g., Sec. 3.6.3 of [25]), we obtain

$$\tfrac{r}{M}\int_{t-\frac{r}{M}}^t|\mathbf{F}(s)|_{R_r}^2ds\ge|\int_{t-\frac{r}{M}}^t\mathbf{F}(s)ds|_{R_r}^2=|Y_r(t)-\xi_r(t)|_{R_r}^2,$$

$$\begin{aligned}&\tau_{M,y}\int_{t-\frac{r}{M}-\tau_{M,y}}^{t-\frac{r}{M}}|\mathbf{I}_1^{\rm T}\mathbf{F}(s)|_{R_y}^2ds\ge|\int_{t-\frac{r}{M}-\tau_{M,y}}^{t-\frac{r}{M}}\mathbf{I}_1^{\rm T}\mathbf{F}(s)ds|_{R_y}^2\\&\ge\begin{bmatrix}\int_{t-\frac{r}{M}-\bar\tau_y}^{t-\frac{r}{M}}\mathbf{I}_1^{\rm T}\mathbf{F}(s)ds\\\int_{t-\frac{r}{M}-\tau_{M,u}}^{t-\frac{r}{M}-\bar\tau_y}\mathbf{I}_1^{\rm T}\mathbf{F}(s)ds\end{bmatrix}^{\rm T}\begin{bmatrix}R_y&G_y\\ *&R_y\end{bmatrix}\begin{bmatrix}\int_{t-\frac{r}{M}-\bar\tau_y}^{t-\frac{r}{M}}\mathbf{I}_1^{\rm T}\mathbf{F}(s)ds\\\int_{t-\frac{r}{M}-\tau_{M,u}}^{t-\frac{r}{M}-\bar\tau_y}\mathbf{I}_1^{\rm T}\mathbf{F}(s)ds\end{bmatrix}\\&\overset{(2.35a)}{=}\begin{bmatrix}\bar Y_{\bar\tau_y}(t)-\xi_{y,1}(t)\\\bar v_{\bar\tau_y}(t)-\xi_{y,2}(t)\end{bmatrix}^{\rm T}\begin{bmatrix}R_y&G_y\\ *&R_y\end{bmatrix}\begin{bmatrix}\bar Y_{\bar\tau_y}(t)-\xi_{y,1}(t)\\\bar v_{\bar\tau_y}(t)-\xi_{y,2}(t)\end{bmatrix},\end{aligned}$$

$$\begin{aligned}&\tau_{M,u}\int_{t-\tau_{M,u}}^t|\mathcal{I}\mathbf{F}(s)|_{R_u}^2ds\ge|\int_{t-\tau_{M,u}}^t\mathcal{I}\mathbf{F}(s)ds|_{R_u}^2\\&\ge\begin{bmatrix}\int_{t-\tau_u}^t\mathcal{I}\mathbf{F}(s)ds\\\int_{t-\tau_{M,u}}^{t-\tau_u}\mathcal{I}\mathbf{F}(s)ds\end{bmatrix}^{\rm T}\begin{bmatrix}R_u&G_u\\ *&R_u\end{bmatrix}\begin{bmatrix}\int_{t-\tau_u}^t\mathcal{I}\mathbf{F}(s)ds\\\int_{t-\tau_{M,u}}^{t-\tau_u}\mathcal{I}\mathbf{F}(s)ds\end{bmatrix}\\&\overset{(2.36)}{=}\begin{bmatrix}Y_{\tau_u}^{N_0}(t)-\xi_{u,1}(t)\\v_{\tau_u}^{N_0}(t)-\xi_{u,2}(t)\end{bmatrix}^{\rm T}\begin{bmatrix}R_u&G_u\\ *&R_u\end{bmatrix}\begin{bmatrix}Y_{\tau_u}^{N_0}(t)-\xi_{u,1}(t)\\v_{\tau_u}^{N_0}(t)-\xi_{u,2}(t)\end{bmatrix}.\end{aligned}\tag{2.53}$$

To compensate the nonlinear term $H(t)$, following [20] we employ Parseval's equality and use (2.2) to obtain

$$\begin{aligned}&|H_{M+1}^{N_0}(t)|^2+|H_{M+1}^{N-N_0}(t)|^2=\textstyle\sum_{n=1}^N|g_n(t)-\hat g_n^{(M+1)}(t)|^2\\&\le\int_0^1|g(x,z(x,t))-g(x,\Phi_0(x)\hat z_{M+1}^{N_0}(t)+\Phi_1(x)\hat z_{M+1}^{N-N_0}(t))|^2dx\\&\le\bar g^2\int_0^1|z(x,t)-\Phi_0(x)\hat z_{M+1}^{N_0}(t)-\Phi_1(x)\hat z_{M+1}^{N-N_0}(t)|^2dx\\&\le\bar g^2|e_{M+1}^{N_0}(t)|^2+\bar g^2|e_{M+1}^{N-N_0}(t)|^2+\bar g^2\textstyle\sum_{n=N+1}^\infty z_n^2(t),\\&|H_i^{N_0}(t)|^2+|H_i^{N-N_0}(t)|^2\le\bar g^2|e_i^{N_0}(t)|^2+\bar g^2|e_i^{N-N_0}(t)|^2,\end{aligned}$$

where $1\le i\le M$, which implies

$$|H(t)|^2\le\bar g^2|\mathbf{I}_2^{\rm T}X(t)|^2+\bar g^2\textstyle\sum_{n=N+1}^\infty z_n^2(t).\tag{2.54}$$

Besides, from Parseval's equality and (2.2) we have

$$\begin{aligned}&|\sigma(t)|^2=\textstyle\sum_{n=1}^N\sigma_n^2(t)\le\sum_{n=1}^\infty\sigma_n^2(t)=\|\sigma(\cdot,z(\cdot,t))\|_{L^2}^2\\&\le\bar\sigma^2|\mathbf{I}_1^{\rm T}X(t)|^2+\bar\sigma^2\textstyle\sum_{n=N+1}^\infty z_n^2(t).\end{aligned}\tag{2.55}$$

We will compensate $\zeta(t-r/M-\bar\tau_y)$ that appears in $\mathcal{L}V_P$ in (3.12) by employing Halanay's inequality with respect to $\mathbb{E}V(t)$. For this we will use the following bound for $\delta_1\in(0,\delta)$:

$$\begin{aligned}&-2\delta_1\sup_{t-\frac{r}{M}-\tau_{M,y}\le\theta\le t}\mathbb{E}V(\theta)\le-2\delta_1\mathbb{E}V_{\rm tail}(t-\tfrac{r}{M}-\bar\tau_y)\\&\le-2\delta_1\rho\|c\|_N^{-2}\mathbb{E}\zeta^2(t-\tfrac{r}{M}-\bar\tau_y).\end{aligned}\tag{2.56}$$

where the last inequality is obtained by the Cauchy–Schwarz inequality:

$$\zeta^2(t)\le\|c\|_N^2\textstyle\sum_{n=N+1}^\infty z_n^2(t).\tag{2.57}$$

Let $\hat\chi_n=\chi_n+\beta_1\bar g^2+\beta_2\bar\sigma^2$, where $\beta_1,\beta_2>0$, χ_n is defined below (2.49), and define

$$\begin{aligned}\eta(t)=\mathrm{col}\{&X(t),Y_r(t),\xi_r(t),Y_{\tau_u}^{N_0}(t),v_{\tau_u}^{N_0}(t),\xi_{u,1}(t),\xi_{u,2}(t),\\&\bar Y_{\bar\tau_y}(t),\bar v_{\bar\tau_y}(t),\xi_{y,1}(t),\xi_{y,2}(t),\zeta(t-\tfrac{r}{M}-\bar\tau_y),G(t),H(t)\}.\end{aligned}$$

By (2.49)–(2.56) and the S-procedure [25, Sec 3.2.3], we get

$$\begin{aligned}&\mathbb{E}\mathcal{L}V(t)+2\delta\mathbb{E}V(t)-2\delta_1\sup_{t-r/M-\tau_{M,y}\le\theta\le t}\mathbb{E}V(\theta)\\&+\beta_1\mathbb{E}[\bar g^2|\mathbf{I}_2^{\rm T}X(t)|^2+\bar g^2\textstyle\sum_{n=N+1}^\infty z_n^2(t)-|H(t)|^2]\\&+\beta_2\mathbb{E}[\bar\sigma^2|\mathbf{I}_1^{\rm T}X(t)|^2+\bar\sigma^2\textstyle\sum_{n=N+1}^\infty z_n^2(t)-|\sigma(t)|^2]\\&\le\mathbb{E}[\sigma^{\rm T}(t)\Psi_1\sigma(t)+\eta^{\rm T}(t)\Psi_2\eta(t)+\hat\chi_{N+1}\textstyle\sum_{n=N+1}^\infty z_n^2(t)]<0\end{aligned}\tag{2.58}$$

provided $\hat\chi_{N+1}<0$, (2.52) and the following inequalities hold:

$$\begin{aligned}&\Psi_1=\mathcal{I}_1^{\rm T}P\mathcal{I}_1-\rho I-\beta_2I+\tfrac{r}{M}\mathcal{I}_1^{\rm T}Q_r\mathcal{I}_1\\&\quad+\tau_{M,u}\mathcal{I}_1^{\rm T}\mathcal{I}^{\rm T}Q_u\mathcal{I}\mathcal{I}_1+\tau_{M,y}e^{2\delta\frac{r}{M}}Q_y<0,\\&\Psi_2=\left[\begin{array}{c|c|ccc}\Omega_1&\Theta_1&\Theta_2&\Theta_3&\Theta_4\\\hline *&\Omega_2&0&\Theta_5&0\\\hline *&*&\multicolumn{3}{c}{\mathrm{diag}\{\Omega_3,\Omega_4,\Omega_5\}}\end{array}\right]+\Xi^{\rm T}[\tfrac{r^2}{M^2}R_r\\&\quad+\tau_{M,u}^2\mathcal{I}^{\rm T}R_u\mathcal{I}+\tau_{M,y}^2e^{\frac{2\delta r}{M}}\mathbf{I}_1R_y\mathbf{I}_1^{\rm T}]\Xi<0,\end{aligned}\tag{2.59}$$

$$\left[\begin{array}{c|ccc}\rho(-2\lambda_{N+1}+2\delta+\bar{\sigma}^2+\alpha_1\bar{g}^2)+\beta_1\bar{g}^2+\beta_2\bar{\sigma}^2 & 1 & 1 & 1\\ \hline * & \multicolumn{3}{c}{-\frac{1}{\rho}\mathrm{diag}\{\alpha_1, \frac{\alpha_2}{\lambda_{N+1}}, \frac{\alpha_3}{\lambda_{N+1}}\}}\end{array}\right]<0. \tag{2.61}$$

Box I.

where

$$\begin{aligned}
&\Xi=[F,\Lambda,0,\mathcal{I}_1\mathcal{B}K_0,0,0,0,-\mathbf{L}C,0,0,0,\mathbf{L},\mathbf{I}_1,\mathbf{I}_2],\\
&\Omega_1=PF+F^{\mathrm{T}}P+2\delta P+\tfrac{2\alpha_2\rho(N+1)}{N^2\pi^2}\mathcal{I}^{\mathrm{T}}K_0^{\mathrm{T}}K_0\mathcal{I}\\
&\quad+(\rho\bar{\sigma}^2+\rho\alpha_1\bar{g}^2+\beta_2\bar{\sigma}^2)\mathbf{I}_1\mathbf{I}_1^{\mathrm{T}}+\beta_1\bar{g}^2\mathbf{I}_2\mathbf{I}_2^{\mathrm{T}}\\
&\quad+(1-\varepsilon_u)\mathcal{I}^{\mathrm{T}}S_u\mathcal{I}+(1-\varepsilon_y)\mathbf{I}_1S_y\mathbf{I}_1^{\mathrm{T}}+(1-\varepsilon_M)S_r,\\
&\Omega_2=\begin{bmatrix}-\varepsilon_M(S_r+R_r)+(1-\varepsilon_y)\mathbf{I}_1S_y\mathbf{I}_1^{\mathrm{T}} & \varepsilon_MR_r\\ * & -\varepsilon_M(Q_r+R_r)\end{bmatrix},\\
&\Omega_3=\begin{bmatrix}\Omega_3^{(11)} & -\varepsilon_u(G_u+S_u) & \varepsilon_uR_u & \varepsilon_uG_u\\ * & -\varepsilon_u(S_u+R_u) & \varepsilon_uG_u^{\mathrm{T}} & \varepsilon_uR_u\\ * & * & -\varepsilon_u(Q_u+R_u) & 0\\ * & * & * & -\varepsilon_u(Q_u+R_u)\end{bmatrix},\\
&\Omega_3^{(11)}=\tfrac{2\alpha_3\rho(N+1)}{N^2\pi^2}K_0^{\mathrm{T}}K_0-\varepsilon_u(S_u+R_u),\\
&\Omega_4=\begin{bmatrix}-\varepsilon_y(S_y+R_y) & -\varepsilon_y(S_y+G_y) & \varepsilon_yR_y & \varepsilon_yG_y\\ * & -\varepsilon_y(S_y+R_y) & \varepsilon_yG_y^{\mathrm{T}} & \varepsilon_yR_y\\ * & * & -\varepsilon_y(Q_y+R_y) & 0\\ * & * & * & -\varepsilon_y(Q_y+R_y)\end{bmatrix},\\
&\Omega_5=\mathrm{diag}\{-2\delta_1\rho\|c\|_N^{-2},-\alpha_1\rho I,-\beta_1 I\},\\
&\Theta_1=[P\Lambda+\varepsilon_MS_r-(1-\varepsilon_y)\mathbf{I}_1S_y\mathbf{I}_1^{\mathrm{T}},\ 0],\\
&\Theta_2=[P\mathcal{I}_1\mathcal{B}K_0+\varepsilon_u\mathcal{I}^{\mathrm{T}}S_u,\ \varepsilon_u\mathcal{I}^{\mathrm{T}}S_u,\ 0,\ 0],\\
&\Theta_4=[P\mathbf{L},P\mathbf{I}_1,P\mathbf{I}_2],\Theta_5=[-\varepsilon_y\mathbf{I}_1S_y,-\varepsilon_y\mathbf{I}_1S_y,0,0],\\
&\Theta_3=[-P\mathbf{L}C+\varepsilon_y\mathbf{I}_1S_y,\ \varepsilon_y\mathbf{I}_1S_y,\ 0,\ 0].
\end{aligned} \tag{2.60}$$

By Schur complement, we find that $\hat{\chi}_{N+1}<0$ iff (2.61) (see Box I) is feasible. Summarizing, we have:

Theorem 1. *Consider system (2.1) with control law (2.18), measurement (2.3) with $c\in L^2(0,1)$ satisfying (2.11), $z_0\in\mathcal{D}(\mathcal{A})$ almost surely and $z_0\in L^2(\Omega;L^2(0,1))$. Let $N_0\in\mathbb{N}$ satisfy (2.8) and $N\in\mathbb{N}$ satisfy $N\geq N_0$. Assume that L_0 and K_0 are obtained from (2.12) and (2.13), respectively. Given $M\in\mathbb{N}$ and $r,\tau_{M,y},\tau_{M,u},\bar{\sigma},\bar{g},\delta,\delta_1>0$ ($\delta_1<\delta$), let there exist positive definite matrices P, S_y, R_y, Q_y, S_r, R_r, Q_r, S_u, R_u, Q_u, matrices G_y, G_u, positive scalars α_1, α_2, α_3, β_1, β_2 and tuning parameter $\rho>0$ such that LMIs (2.52), (2.59) and (2.61) with Ω_i, Θ_i, $i=1,\dots,5$, Ξ in (2.60) are feasible. Then the following holds:*

- *The solution $z(x,t)$ to (2.1) subject to the control law (2.18) and the corresponding sub-predictor-based observer $\hat{z}(x,t)$ defined by (2.15), (2.17) satisfy*

$$\mathbb{E}[\|z(\cdot,t)\|_{L^2}^2+\|\hat{z}(\cdot,t)\|_{L^2}^2]\leq D\mathrm{e}^{-2\delta_\tau t}\mathbb{E}\|z(\cdot,0)\|_{L^2}^2, \tag{2.62}$$

  for $t\geq 0$ and some $D\geq 1$, where $\delta_\tau>0$ is the unique solution of $\delta_\tau=\delta-\delta_1\mathrm{e}^{2\delta_\tau(\tau_{M,y}+r/M)}$.
- *Given $r>0$, LMIs (2.52), (2.59), and (2.61) are always feasible for M, N being large enough and $\bar{\sigma},\bar{g},\tau_{M,u},\tau_{M,y}>0$ being small enough.*

Proof. First, we prove (2.62). By employing Itô's formula for $V_P(t)$ along stochastic ODE (2.35a) (see [34, Theorem 4.18]), we obtain

$$\begin{aligned}&V_P(t+\Delta)-V_P(t)=\int_t^{t+\Delta}\mathcal{L}V_P(s)\mathrm{d}s\\ &+\int_t^{t+\Delta}\frac{\partial V_P}{\partial X}(s)\mathcal{I}_1\sigma(s)\mathrm{d}\mathcal{W}(s),\ \forall t\geq 0,\Delta>0,\end{aligned} \tag{2.63}$$

where $\frac{\partial V_P}{\partial X}(s)=2X^{\mathrm{T}}(s)P$. Taking expectations on both sides of (2.63) and applying Fubini's theorem (see [34, Theorem 2.39]) we have $\mathbb{E}V_P(t+\Delta)-\mathbb{E}V_P(t)=\int_t^{t+\Delta}\mathbb{E}[\mathcal{L}V_P(s)]\mathrm{d}s$, which implies

$$D^+\mathbb{E}V_P(t)=\limsup_{\Delta\searrow 0}\frac{\mathbb{E}V_P(t+\Delta)-\mathbb{E}V_P(t)}{\Delta}=\mathbb{E}[\mathcal{L}V_P(t)]. \tag{2.64}$$

Note that (2.64) also holds with V_P replaced by V_y, V_r, V_u. Since $w(t)$ is a strong solution to (2.21) (see Section 2.2), applying Itô's formula for $V_2(w(t),t)$ along (2.21) (see [29, Theorem 7.2.1]), we have

$$\begin{aligned}&V_2(w(t+\Delta),t+\Delta)-V_2(w(t),t)=\int_t^{t+\Delta}\mathcal{L}V_2(w(s),s)\mathrm{d}s\\ &+\int_t^{t+\Delta}2\langle w(s),\sigma(w(s)+\psi(\cdot)u(s-r-\tau_u(s)))\rangle\mathrm{d}\mathcal{W}(s)\end{aligned} \tag{2.65}$$

for all $t\geq 0,\Delta>0$. Taking expectation on both sides of (2.65) and applying the infinite dimensional Fubini theorem (see [28, Theorem 4.33]) we arrive at $\mathbb{E}V_2(w(t+\Delta),t+\Delta)-\mathbb{E}V_2(w(t),t)=\int_t^{t+\Delta}\mathbb{E}[\mathcal{L}V_2(w(s),s)]\mathrm{d}s$, which implies

$$\begin{aligned}D^+\mathbb{E}V_2(w(t),t)&=\limsup_{\Delta\searrow 0}\frac{\mathbb{E}V_2(w(t+\Delta),t+\Delta)-\mathbb{E}V_2(w(t),t)}{\Delta}\\ &=\mathbb{E}[\mathcal{L}V_2(w(t),t)].\end{aligned} \tag{2.66}$$

By the definition of $V(t)$ in (2.37), (2.38), and combining (2.64) and (2.66), we arrive at $D^+\mathbb{E}V(t)=\mathbb{E}[\mathcal{L}V(t)]$, which together with (2.58) gives

$$D^+\mathbb{E}V(t)+2\delta_0\mathbb{E}V(t)-2\delta_1\sup_{s_k\leq\theta\leq t}\mathbb{E}V(\theta)\leq 0.$$

From Section 2.2, it follows that $\mathbb{E}\|z(\cdot,t)\|_{L^2}^2$ and $\mathbb{E}|X(t)|^2$ are continuous, which together with the construction of $V(t)$ in (2.37) implies that $\mathbb{E}V(t)$ is continuous. Then employing Halanay's inequality (see [25, P. 138]) we arrive at

$$\mathbb{E}V(t)\leq V(0)\mathrm{e}^{-2\delta_\tau t},\ \ t\geq 0, \tag{2.67}$$

where $\delta_\tau>0$ is the unique solution of $\delta_\tau=\delta-\delta_1\mathrm{e}^{2\delta_\tau(\tau_{M,y}+r/M)}$. From (2.37), it follows

$$\begin{aligned}&V(0)=V_{\mathrm{nom}}(0)\leq\max\{\lambda_{\max}(P),\rho\}\|z_0\|_{L^2}^2,\\ &\mathbb{E}V(t)\geq\mathbb{E}V_{\mathrm{nom}}(t)\geq\min\{\lambda_{\min}(P),\rho\}\mathbb{E}\|z(\cdot,t)\|_{L^2}^2,\end{aligned}$$

which together with (2.67) gives

$$\mathbb{E}[\|z(\cdot,t)\|_{L^2}^2+\|z(\cdot,t)-\hat{z}(\cdot,t)\|_{L^2}^2]\leq D\mathrm{e}^{-2\delta_\tau t}\|z_0\|_{L^2}^2, \tag{2.68}$$

for $t\geq 0$ and some $D\geq 1$.

For any given $r>0$, to prove the feasibility of (2.52), (2.59), and (2.61) for large enough N, M and small enough $\bar{\sigma},\bar{g},\tau_{M,u},\tau_{M,y}>0$, we take $\bar{\sigma},\bar{g},\tau_{M,u},\tau_{M,y}\to 0^+$, $\alpha_1=\alpha_2=\alpha_3=1$, $S_y=0$, $S_u=0$, $Q_y=R_y=r_yI$, $R_u=r_uI$, $P=\mathrm{diag}\{\hat{P}_z,p_1I_{N-N_0},\hat{P}_e,p_2I_{M(N-N_0)}\}$, $S_r=\mathrm{diag}\{s_1I_N,\hat{S}_r,I_{M(N-N_0)}\}$, $R_r=\mathrm{diag}\{r_1I_N,2\hat{R}_r,r_2I_{M(N-N_0)}\}$, $Q_r=R_r$ with $0<\hat{P}_z\in\mathbb{R}^{N_0\times N_0}$, $0<\hat{P}_e\in\mathbb{R}^{MN_0\times MN_0}$, $0<\hat{S}_r,\hat{R}_r\in\mathbb{R}^{MN_0\times MN_0}$ and $s_1,r_1,r_2,r_y,r_u,p_1,p_2>0$. By applying Schur complement repeatedly and letting $p_1,s_1\to 0^+$, $r_u=p_2^2$, $r_1,r_2,p_2,\beta_1,r_y\to\infty$, we find that $\Psi_2<0$ if

$$\begin{aligned}&\left[\begin{array}{cc|cc}\hat{\psi}_{11} & \hat{P}_zB_0K_0\hat{\mathcal{I}}_0 & 0 & \hat{P}_z\\ \hline * & \phi(\hat{P}_e,\hat{S}_r,\hat{R}_r) & \hat{P}_e\hat{\mathbf{L}}_\zeta & 0\\ & & 0 & 0\\ \hline * & * & \multicolumn{2}{c}{\mathrm{diag}\left\{-\frac{2\rho\delta_1}{\|c\|_N^2},-\rho I\right\}}\end{array}\right]+\frac{r^2}{M^2}\hat{\Xi}^{\mathrm{T}}\hat{R}_r\hat{\Xi}\\ &+\frac{2\rho(N+1)}{N^2\pi^2}\mathrm{diag}\{\hat{\mathcal{I}}^{\mathrm{T}}K_0^{\mathrm{T}}K_0\hat{\mathcal{I}},0,0,0\}<0,\end{aligned} \tag{2.69}$$

where

$\hat{\psi}_{11} = \hat{P}_z(A_0 - B_0K_0) + (A_0 - B_0K_0)^{\mathrm{T}}\hat{P}_z + 2\delta\hat{P}_z,$

$$\phi(\hat{P}_e, \hat{S}_e, \hat{R}_e) = \begin{bmatrix} \hat{P}_e F_{e11} + F_{e11}^{\mathrm{T}}\hat{P}_e + 2\delta\hat{P}_e + (1-\varepsilon_M)\hat{S}_r & \hat{P}_e\hat{\Lambda}_e + \varepsilon_M\hat{S}_r \\ * & -\varepsilon_M(\hat{S}_r + \hat{R}_r) \end{bmatrix},$$

$F_{e11} = I_M \otimes (A_0 - L_0C_0) + J_{0,M} \otimes L_0C_0,$

$$\hat{\mathbf{L}}_\zeta = \begin{bmatrix} 0_{(M-2)N_0\times 1} \\ L_0 \\ -L_0 \end{bmatrix}, \ \hat{\mathcal{I}}_0 = [I_{N_0}, \dots, I_{N_0}] \in \mathbb{R}^{N_0\times MN_0},$$

$\hat{\Lambda}_e = I_M \otimes L_0C_0 - J_{0,M} \otimes L_0C_0,$

$\hat{\mathcal{I}} = [-I_{N_0}, \hat{\mathcal{I}}_0], \quad \hat{\Xi} = [0, F_{e11}, \hat{\Lambda}_e, 0, \hat{\mathbf{L}}_\zeta, 0].$

For any given $r > 0$, we first fix very large $M > 0$. From Proposition 1 in [17] we can obtain for suitable positive matrices $\hat{P}_e$, $\hat{S}_r$, $\hat{R}_r$,

$$\phi(\hat{P}_e, \hat{S}_r, \hat{R}_r) + (\tfrac{r}{M})^2 \begin{bmatrix} F_{e11}^{\mathrm{T}} \\ \hat{\Lambda}_e^{\mathrm{T}} \end{bmatrix} \hat{R}_r[F_{e11}, \hat{\Lambda}_e] < 0.$$

Replace $\phi(\hat{P}_e, \hat{S}_r, \hat{R}_r)$ with $\phi(\beta\hat{P}_e, \beta\hat{S}_r, \beta\hat{R}_r) = \beta\phi(\hat{P}_e, \hat{S}_r, \hat{R}_r)$, $\beta > 0$. Let $\hat{P}_z = P_c$, given in (2.13), resulting in $\hat{\psi}_{11} < 0$. Setting $\beta > 0$ to be large enough, then choosing $\rho = \sqrt{N}$ large enough, and applying Schur complement three times in (2.69), we find that (2.69) hold. Fixing such M and N, taking $\beta_2 = \max\{p_2^2, r_y^2\}$, and using continuity, we have that (2.52), (2.59), (2.61) are feasible provided $\bar{\sigma}, \bar{g}, \tau_{M,u}, \tau_{M,y} > 0$ are small enough.

Remark 5. For the case that $g(z) = \bar{g}z$, we have the following closed-loop system

$$\begin{aligned} \mathrm{d}X(t) &= [FX(t) + \Lambda Y_r(t) + \mathcal{I}_1\mathcal{B}K_0Y_{\tau_u}^{N_0}(t) - \mathbf{L}C\bar{Y}_{\bar{\tau}_y}(t) \\ &\quad + \mathbf{L}\zeta(t - \tfrac{r}{M} - \bar{\tau}_y)]\mathrm{d}t + \mathcal{I}_1\sigma(t)\mathrm{d}\mathcal{W}(t), \\ \mathrm{d}z_n(t) &= [-\lambda_n z_n(t) - b_nK_0\mathcal{I}X(t) + b_nK_0Y_{\tau_u}^{N_0}(t)]\mathrm{d}t \\ &\quad + \sigma_n(t)\mathrm{d}\mathcal{W}(t), n > N, \end{aligned} \tag{2.70}$$

where all coefficients are defined in (2.33) with A_0 and A_1 in F replaced by

$$A_0 = \mathrm{diag}\{-\lambda_n + \bar{g}\}_{n=1}^{N_0}, \ A_1 = \mathrm{diag}\{-\lambda_n + \bar{g}\}_{n=N_0+1}^{N}. \tag{2.71}$$

For the stability of system (2.70), we consider Lyapunov functional (2.37). By arguments similar to (2.39)–(2.61), we have

$$\mathbb{E}\mathcal{L}V(t) + 2\delta\mathbb{E}V(t) - 2\delta_1 \sup\nolimits_{t-\frac{r}{M}-\tau_{M,y}\le\theta\le t} \mathbb{E}V(\theta)$$
$$+\beta_2\mathbb{E}[\bar{\sigma}^2|\mathbf{I}_1^{\mathrm{T}}X(t)|^2 + \bar{\sigma}^2\textstyle\sum_{n=N+1}^{\infty} z_n^2(t) - |\sigma(t)|^2] \le 0$$

provide (2.59) and (2.61) hold with Ω_2, Ω_3, Ω_4, Θ_1, Θ_2, Θ_3 in (2.60) and

$$\begin{aligned} \Omega_1 &= PF + F^{\mathrm{T}}P + 2\delta P + \tfrac{2\alpha_2\rho(N+1)}{N^2\pi^2}\mathcal{I}^{\mathrm{T}}K_0^{\mathrm{T}}K_0\mathcal{I} \\ &\quad + (\rho\bar{\sigma}^2 + \beta_2\bar{\sigma}^2)\mathbf{I}_1\mathbf{I}_1^{\mathrm{T}} + (1-\varepsilon_M)S_r \\ &\quad + (1-\varepsilon_u)\mathcal{I}^{\mathrm{T}}S_u\mathcal{I} + (1-\varepsilon_y)\mathbf{I}_1S_y\mathbf{I}_1^{\mathrm{T}}, \\ \Omega_5 &= -2\delta_1\rho\|c\|_N^{-2}, \qquad \Theta_4 = P\mathbf{L}_\zeta, \\ \Xi &= [F, \Lambda, 0, \mathcal{I}_1\mathcal{B}K_0, 0, 0, 0, -\mathbf{L}C, 0, 0, 0, \mathbf{L}]. \end{aligned} \tag{2.72}$$

Summarizing, we obtain that the feasibility of LMIs (2.59) and (2.61) with Ω_2, Ω_3, Ω_4, Θ_1, Θ_2, Θ_3 in (2.60) and Ω_1, Ω_5, Θ_4, Ξ given by (2.72) guarantees the mean-square exponential stability of (2.70).

3. Novel chain of sub-predictors

As stated in Remark 4, for the conventional sub-predictors (2.14), we need to construct three functionals V_{Q_y}, V_{Q_u}, V_{Q_r} that depend on the stochastic terms to compensate the delay terms. In this section, we design a novel chain of sub-predictors (see (3.1) below), where an additional sub-predictor $\hat{z}_{M+1}^j$ ($j \in \{N_0, N-N_0\}$) is introduced compared with (2.14). This splits the stochastic term and the fraction delay r/M into separate systems. In this scenario, delay terms $Y_{i,r}^j$ appear only in system e_i^j, $i = 1, \dots, M$ and stochastic term appears only in system z^j, e_{M+1}^j (cf. (3.6)). This avoids some stochastic-dependent terms in the corresponding Lyapunov functional (see (3.9) below where we construct only one stochastic-dependent term V_{Q_y} due to the measurement delay term Y_{τ_y}).

Consider stochastic systems (2.10). We design a chain of sub-predictors

$$\begin{aligned} &\hat{z}_1^j(t-r) \mapsto \cdots \mapsto z_i^j(t - \tfrac{M-i+1}{M}r) \mapsto \dots \\ &\mapsto \hat{z}_M^j(t - \tfrac{1}{M}r) \mapsto \hat{z}_{M+1}^j(t) \mapsto z^j(t), j \in \{N_0, N-N_0\}. \end{aligned} \tag{3.1}$$

The sub-predictors satisfy

$$\begin{aligned} \mathrm{d}\hat{z}_{M+1}^{N_0}(t) &= [A_0\hat{z}_{M+1}^{N_0}(t) + \hat{G}_{M+1}^{N_0}(t) + B_0u(t-r)]\mathrm{d}t \\ &\quad - L_0[C_0\hat{z}_{M+1}^{N_0}(t) + C_1\hat{z}_{M+1}^{N-N_0}(t) - y(t)]\mathrm{d}t, \\ \mathrm{d}\hat{z}_{M+1}^{N-N_0}(t) &= [A_1\hat{z}_{M+1}^{N-N_0}(t) + \hat{G}_{M+1}^{N-N_0}(t) + B_1u(t-r)]\mathrm{d}t, \\ \mathrm{d}\hat{z}_i^{N_0}(t) &= [A_0\hat{z}_i^{N_0}(t) + \hat{G}_i^{N_0}(t) + B_0u(t - \tfrac{i-1}{M}r)]\mathrm{d}t \\ &\quad - L_0[C_0\hat{z}_i^{N_0}(t - \tfrac{r}{M}) + C_1\hat{z}_i^{N-N_0}(t - \tfrac{r}{M}) \\ &\quad - C_0\hat{z}_{i+1}^{N_0}(t) - C_1\hat{z}_{i+1}^{N-N_0}(t)]\mathrm{d}t, \\ \mathrm{d}\hat{z}_i^{N-N_0}(t) &= [A_1\hat{z}_i^{N-N_0}(t) + \hat{G}_i^{N-N_0}(t) \\ &\quad + B_1u(t - \tfrac{i-1}{M}r)]\mathrm{d}t, \quad 1 \le i \le M, \quad t \ge 0, \end{aligned} \tag{3.2}$$

subject to $\hat{z}_i^{N_0}(t) = 0$, $\hat{z}_i^{N-N_0}(t) = 0$, $t \le 0$, $1 \le i \le M+1$, where $y(t)$ is given by (2.3), $C_1 = [c_{N_0+1}, \dots, c_N]$, $\hat{G}_i^j(t)$ and $\hat{g}_n^{(i)}(t)$ are defined similar to (2.16).

The finite-dimensional observer $\hat{z}(x,t)$ of the state $z(x,t)$, based on (3.2), is given by (2.17). The controller is chosen as (2.18), where $K_0 \in \mathbb{R}^{1\times N_0}$ is determined by (2.13). Define the estimation errors as follows

$$\begin{aligned} e_i^j(t) &= \hat{z}_{i+1}^j(t - \tfrac{M-i}{M}r) - \hat{z}_i^j(t - \tfrac{M-i+1}{M}r), 1 \le i \le M, \\ e_{M+1}^j(t) &= z^j(t) - \hat{z}_{M+1}^j(t), \quad j \in \{N_0, N-N_0\}. \end{aligned} \tag{3.3}$$

Then the last term on the right-hand-side of differential equation for $\hat{z}_{M+1}^{N_0}(t)$ in (3.2) can be presented as

$$\begin{aligned} &C_0\hat{z}_{M+1}^{N_0}(t) + C_1\hat{z}_{M+1}^{N-N_0}(t) - y(t) \\ &\overset{(2.3)}{=} -[C_0e_{M+1}^{N_0}(t) + C_1e_{M+1}^{N-N_0}(t) + \zeta(t-\tau_y)] \\ &\quad + C_0Y_{\tau_y}^{N_0}(t) + C_1Y_{\tau_y}^{N-N_0}(t), \end{aligned} \tag{3.4}$$

where $Y_{\tau_y}^j(t)$, $\zeta(t)$ are defined in (2.29). Furthermore, by (3.3), we get

$$\hat{z}_1^{N_0}(t-r) + \textstyle\sum_{i=1}^{M+1} e_i^{N_0}(t) = z^{N_0}(t). \tag{3.5}$$

In particular, if the errors $e_i^{N_0}(t)$, $1 \le i \le M+1$ converge to zero, from (3.5) we have $\hat{z}_1^{N_0}(t) \to z^{N_0}(t+r)$, meaning that $\hat{z}_1^{N_0}(t)$ predicts the future system state $z^{N_0}(t+r)$.

$$
\begin{aligned}
&X_e=\mathrm{col}\{e_1^{N_0},\ldots,e_{M+1}^{N_0},e_1^{N-N_0},\ldots,e_{M+1}^{N-N_0}\},\\
&X=\mathrm{col}\{X_z,X_e\},\ \mathcal{I}_0=[I_{N_0},\ldots,I_{N_0},0_{N_0\times(M+1)(N-N_0)}],\\
&Y_{\tau_y}=\mathrm{col}\{Y_{\tau_y}^{N_0},Y_{\tau_y}^{N-N_0}\},\ \mathcal{I}=[I_{N_0},0_{N_0\times(N-N_0)},-\mathcal{I}_0],\\
&Y_{e,r}=\mathrm{col}\{Y_{1,r}^{N_0},\ldots,Y_{M,r}^{N_0},Y_{1,r}^{N-N_0},\ldots,Y_{M,r}^{N-N_0}\},\\
&H=\mathrm{col}\{H_1^{N_0},\ldots,H_{M+1}^{N_0},H_1^{N-N_0},\ldots,H_{M+1}^{N-N_0}\},\\
&\mathcal{I}_1=\mathrm{col}\{I_N,0_{MN_0\times N},[I_{N_0}\ 0],0_{M(N-N_0)\times N},[0\ I_{N-N_0}]\},\\
&\mathbf{L}=\mathrm{col}\{0_{(N+(M-1)N_0)\times 1},L_0,-L_0,0_{(M+1)(N-N_0)\times 1}\},\\
&F_e=\left[\begin{array}{c|c} I_{M+1}\otimes(A_0-L_0C_0)+J_{0,M+1}\otimes L_0C_0 & -I_{M+1}\otimes L_0C_1+J_{0,M+1}\otimes L_0C_1\\ 0 & I_{M+1}\otimes A_1\end{array}\right],\\
&F=\begin{bmatrix}F_z & \mathcal{B}K_0\mathcal{I}_0\\ 0 & F_e\end{bmatrix},\ \mathbf{I}_1=\begin{bmatrix}I_N\\ 0_{(M+1)N\times N}\end{bmatrix},\ \mathbf{I}_2=\begin{bmatrix}0_{N\times(M+1)N}\\ I_{(M+1)N}\end{bmatrix},\\
&\Lambda_e=\left[\begin{array}{cc} \multicolumn{2}{c}{0_{N\times MN}}\\ \hline I_M\otimes L_0C_0-J_{0,M}\otimes L_0C_0 & I_M\otimes L_0C_1-J_{0,M}\otimes L_0C_1\\ \hline \multicolumn{2}{c}{0_{((M+1)N-MN_0)\times MN}}\end{array}\right].
\end{aligned}
\tag{3.7}
$$

Box II.

Using (3.2), (3.4), we arrive at

$$
\begin{aligned}
\mathrm{d}e_{M+1}^{N_0}(t)&=[(A_0-L_0C_0)e_{M+1}^{N_0}(t)+H_{M+1}^{N_0}(t)\\
&\quad+B_0K_0Y_{\tau_u}^{N_0}(t)-L_0C_1e_{M+1}^{N-N_0}(t)-L_0\xi(t-\tau_y)\\
&\quad+L_0C_0Y_{\tau_y}^{N_0}(t)+L_0C_1Y_{\tau_y}^{N-N_0}(t)]\mathrm{d}t+\sigma^{N_0}(t)\mathrm{d}\mathcal{W}(t),\\
\mathrm{d}e_{M+1}^{N-N_0}(t)&=[A_1e_{M+1}^{N-N_0}(t)+H_{M+1}^{N-N_0}(t)\\
&\quad+B_1K_0Y_{\tau_u}^{N_0}(t)]\mathrm{d}t+\sigma^{N-N_0}(t)\mathrm{d}\mathcal{W}(t),\\
\mathrm{d}e_{M}^{N_0}(t)&=[(A_0-L_0C_0)e_M^{N_0}(t)+L_0C_0Y_{M,r}^{N_0}(t)\\
&\quad+H_M^{N_0}(t)-L_0C_1e_M^{N-N_0}(t)+L_0C_1Y_{M,r}^{N-N_0}(t)\\
&\quad+L_0C_0e_{M+1}^{N_0}(t)+L_0C_1e_{M+1}^{N-N_0}(t)+L_0\zeta(t-\tau_y)\\
&\quad-L_0C_0Y_{\tau_y}^{N_0}(t)L_0C_1Y_{\tau_y}^{N-N_0}(t)]\mathrm{d}t,\\
\mathrm{d}e_{i}^{N_0}(t)&=[(A_0-L_0C_0)e_i^{N_0}(t)+L_0C_0Y_{i,r}^{N_0}(t)+H_i^{N_0}(t)\\
&\quad+L_0[C_0e_{i+1}^{N_0}(t)+C_1e_{i+1}^{N-N_0}(t)]-L_0C_0Y_{i+1,r}^{N_0}(t)\\
&\quad-L_0C_1Y_{i+1,r}^{N-N_0}(t)-L_0C_1e_i^{N-N_0}(t)\\
&\quad+L_0C_1Y_{i,r}^{N-N_0}(t)]\mathrm{d}t,\ 1\le i\le M-1,\\
\mathrm{d}e_{i}^{N-N_0}(t)&=[A_1e_i^{N-N_0}(t)+H_i^{N-N_0}(t)]\mathrm{d}t, 1\le i\le M,
\end{aligned}
\tag{3.6}
$$

where $Y_{\tau_u}^{N_0}(t)$ and $Y_{i,r}^{j}(t)$, $1\le i\le M, j\in\{N_0,N-N_0\}$ are defined in (2.32), and

$$
\begin{aligned}
&H_i^j(t)=\hat{G}_{i+1}^j(t-\tfrac{M-i}{M}r)-G_i^j(t-\tfrac{M-i+1}{M}r),\ 1\le i\le M,\\
&H_{M+1}^j(t)=G^j(t)-\hat{G}_{M+1}^j(t),\quad j\in\{N_0,N-N_0\}.
\end{aligned}
$$

Consider notations $X_z(t)$, $\mathcal{B}$, $\sigma(t)$, $G(t)$, F_z, C, $Y_{\tau_u}^{N_0}(t)$ defined in (2.33) and redefine the following notations (see Box II). By arguments similar to (2.28)–(2.35), we can obtain from (2.10), (2.18), (3.2)–(3.7) the following closed-loop system for $t\ge 0$,

$$
\mathrm{d}X(t)=\mathbf{F}(t)\mathrm{d}t+\mathcal{I}_1\sigma(t)\mathrm{d}\mathcal{W}(t),\tag{3.8a}
$$

$$
\begin{aligned}
\mathrm{d}z_n(t)&=[-\lambda_nz_n(t)+g_n(t)-b_nK_0\mathcal{I}X(t)\\
&\quad+b_nK_0Y_{\tau_u}^{N_0}(t)]\mathrm{d}t+\sigma_n(t)\mathrm{d}\mathcal{W}(t), n>N,
\end{aligned}
\tag{3.8b}
$$

where

$$
\begin{aligned}
\mathbf{F}(t)&=FX(t)+\mathbf{L}\zeta(t-\tau_y)+\Lambda_eY_{e,r}(t)+\mathbf{I}_1G(t)\\
&\quad+\mathbf{I}_2H(t)-\mathbf{L}CY_{\tau_y}(t)+\mathcal{I}_1\mathcal{B}K_0Y_{\tau_u}^{N_0}(t).
\end{aligned}
$$

For mean-square L^2-stability analysis of (3.8), we consider the Lyapunov functional:

$$
V(t)=V_{\mathrm{tail}}(t)+V_P+V_y(t)+V_r(t)+V_u(t),\tag{3.9}
$$

with $V_{\mathrm{tail}}(t)$ and $V_P(t)$ in (2.38) and

$$
\begin{aligned}
&V_y(t)=V_{S_y}(t)+V_{R_y}(t)+V_{Q_y}(t),\\
&V_{S_y}(t)=\int_{t-\tau_{M,y}}^t \mathrm{e}^{-2\delta(t-s)}|\mathbf{I}_1^{\mathrm{T}}X(s)|_{S_y}^2\,\mathrm{d}s,\\
&V_{R_y}(t)=\tau_{M,y}\int_{-\tau_{M,y}}^0\int_{t+\theta}^t \mathrm{e}^{-2\delta(t-s)}|\mathbf{I}_1^{\mathrm{T}}\mathbf{F}(s)|_{R_y}^2\,\mathrm{d}s\mathrm{d}\theta,\\
&V_{Q_y}(t)=\int_{-\tau_{M,y}}^0\int_{t+\theta}^t \mathrm{e}^{2\delta(s-t)}|\sigma(s)|_{Q_y}^2\,\mathrm{d}s\mathrm{d}\theta,\\
&V_r(t)=V_{S_r}(t)+V_{R_r}(t),\\
&V_{S_r}(t)=\int_{t-\frac{r}{M}}^t \mathrm{e}^{-2\delta(t-s)}|\mathcal{I}_2X(s)|_{S_r}^2\,\mathrm{d}s,\\
&\mathcal{I}_2=\left[\begin{array}{c|cccc} 0_{MN_0\times N} & I_{MN_0} & 0_{MN_0\times N_0} & 0 & 0_{MN_0\times(N-N_0)}\\ 0_{M(N-N_0)\times N} & 0 & 0 & I_{M(N-N_0)} & 0\end{array}\right],\\
&V_{R_r}(t)=\frac{r}{M}\int_{-\frac{r}{M}}^0\int_{t+\theta}^t \mathrm{e}^{-2\delta(t-s)}|\mathcal{I}_2\mathbf{F}(s)|_{R_r}^2\,\mathrm{d}s\mathrm{d}\theta,\\
&V_u(t)=V_{S_u}(t)+V_{R_u}(t),\\
&V_{S_u}(t)=\int_{t-\tau_{M,u}}^t \mathrm{e}^{-2\delta(t-s)}|\mathcal{I}X(s)|_{S_u}^2\,\mathrm{d}s,\\
&V_{R_u}(t)=\tau_{M,u}\int_{-\tau_{M,u}}^0\int_{t+\theta}^t \mathrm{e}^{-2\delta(t-s)}|\mathcal{I}\mathbf{F}(s)|_{R_u}^2\,\mathrm{d}s\mathrm{d}\theta,
\end{aligned}
\tag{3.10}
$$

where P, S_y, R_y, Q_y, S_r, R_r, S_u, R_u are positive matrices of appropriate dimensions and $\rho>0$ is a scalar. The terms V_{S_y}, V_{R_y}, V_{Q_y} are introduced to compensate Y_{τ_y}. The terms V_{S_r}, V_{R_r} are used to compensate Y_r. The terms V_{S_u}, V_{R_u} are utilized to compensate $Y_{\tau_u}^{N_0}$. Finally, to compensate $\zeta(t-\tau_y)$, we will use Halanay's inequality with respect to $\mathbb{E}V(t)$.

Remark 6. Note that delay terms $Y_{i,r}^j(t)$ correspond to $e_i^j(t)$ ($i=1,\ldots,M$, $j\in\{N_0,N-N_0\}$), whereas ODEs for these e_i^j do not contain stochastic term. Also the delay term $Y_{\tau_u}^{N_0}(t)$ corresponds to $\hat{z}_1^{N_0}(t-r)=\mathcal{I}X(t)$. From (3.8a) we have

$$
\mathrm{d}\mathcal{I}X(t)=\mathcal{I}\mathbf{F}(t)\mathrm{d}t+\mathcal{I}\mathcal{I}_1\sigma(t)\mathrm{d}\mathcal{W}(t)=\mathcal{I}\mathbf{F}(t)\mathrm{d}t,\tag{3.11}
$$

which means that ODE for $\mathcal{I}X(t)$ does not contain stochastic term yet. However, the delay term $Y_y^j(t)$ corresponds to $z^j(t)$ which contains stochastic term. Therefore, it is enough to construct the stochastic-term-dependent functional V_{Q_y} to compensate the stochastic part in systems z^j. For the case that there is no output delay τ_y, we do not need V_{Q_y} (see Sec. II-D in [26]).

For $V_{\mathrm{tail}}(t)$, we have (2.49) for $\alpha_1,\alpha_2,\alpha_3>0$. For V_P, V_y, V_u, V_r, by arguments similar to (2.50)–4, we calculate the generator $\mathcal{L}$ along (3.8a) (see [34, P. 149]) and obtain

$$\begin{aligned}
&\mathcal{L}V_P(t)+2\delta V_P(t)=X^{\mathrm T}(t)[PF+F^{\mathrm T}P+2\delta P]X(t)\\
&\quad+|\mathcal{I}_1\sigma(t)|_P^2+2X^{\mathrm T}(t)P[\mathbf{L}\zeta(t-\tau_y)+\Lambda_eY_{e,r}(t)\\
&\quad+\mathbf{I}_1G(t)+\mathbf{I}_2H(t)-\mathbf{L}CY_{\tau_y}(t)+\mathcal{I}_1\mathcal{B}K_0Y_{\tau_u}^{N_0}(t)],\\
&\mathbb{E}[\mathcal{L}V_y(t)+2\delta V_y(t)]\le-\varepsilon_y\mathbb{E}|\mathbf{I}_1^{\mathrm T}X(t)-Y_{\tau_y}(t)-v_{\tau_y}(t)|_{S_y}^2\\
&\quad+\mathbb{E}|\mathbf{I}_1^{\mathrm T}X(t)|_{S_y}^2+\tau_{M,y}^2\mathbb{E}|\mathbf{I}_1^{\mathrm T}\mathbf{F}(t)|_{R_y}^2+\tau_{M,y}\mathbb{E}|\sigma(t)|_{Q_y}^2\\
&\quad-\varepsilon_y\mathbb{E}\begin{bmatrix}Y_{\tau_y}(t)-\xi_1(t)\\ v_{\tau_y}(t)-\xi_2(t)\end{bmatrix}^{\mathrm T}\begin{bmatrix}R_y & G_y\\ * & R_y\end{bmatrix}\begin{bmatrix}Y_{\tau_y}(t)-\xi_1(t)\\ v_{\tau_y}(t)-\xi_2(t)\end{bmatrix}\\
&\quad-\varepsilon_y[\mathbb{E}|\xi_1(t)|_{Q_y}^2+\mathbb{E}|\xi_2(t)|_{Q_y}^2],\\
&\mathcal{L}V_u(t)+2\delta V_u(t)\le-\varepsilon_u|\mathcal{I}X(t)-Y_{\tau_u}^{N_0}(t)-v_{\tau_u}^{N_0}(t)|_{S_u}^2\\
&\quad+|\mathcal{I}X(t)|_{S_u}^2+\tau_{M,u}^2|\mathcal{I}\mathbf{F}(t)|_{R_u}^2\\
&\quad-\varepsilon_u\begin{bmatrix}Y_{\tau_u}^{N_0}(t)\\ v_{\tau_u}^{N_0}(t)\end{bmatrix}^{\mathrm T}\begin{bmatrix}R_u & G_u\\ * & R_u\end{bmatrix}\begin{bmatrix}Y_{\tau_u}^{N_0}(t)\\ v_{\tau_u}^{N_0}(t)\end{bmatrix},\\
&\mathcal{L}V_r(t)+2\delta V_r(t)\le|\mathcal{I}_2X(t)|_{S_r}^2-\varepsilon_M|\mathcal{I}_2X(t)-Y_r(t)|_{S_r}^2\\
&\quad+\tfrac{r^2}{M^2}|\mathcal{I}_2\mathbf{F}(t)|_{R_r}^2-\varepsilon_M|Y_r(t)|_{R_r}^2,
\end{aligned}\tag{3.12}$$

where ε_y, ε_u, ε_M, $v_{\tau_u}^{N_0}(t)$ are defined below (2.50), $G_y\in\mathbb{R}^{N\times N}$ and $G_u\in\mathbb{R}^{N_0\times N_0}$ satisfy (2.52),

$$v_{\tau_y}(t)=\mathbf{I}_1^{\mathrm T}[X(t-\tau_y)-X(t-\tau_{M,y})],$$
$$\xi_1(t)=\textstyle\int_{t-\tau_y}^{t}|\sigma(s)|_{Q_y}^2\mathrm{d}\mathcal{W}(s),\ \ \xi_2(t)=\int_{t-\tau_{M,y}}^{t-\tau_y}|\sigma(s)|_{Q_y}^2\mathrm{d}\mathcal{W}(s).$$

We will compensate $\zeta(t-\tau_y)$ that appears in $\mathcal{L}V_P$ in (3.12) by employing Halanay's inequality with respect to $\mathbb{E}V(t)$. For this, similar to (2.56), we have the following bound for $\delta_1\in(0,\delta)$:

$$\begin{aligned}
&-2\delta_1\sup\nolimits_{t-\tau_{M,y}\le\theta\le t}\mathbb{E}V(\theta)\le-2\delta_1\rho\mathbb{E}[\textstyle\sum_{n=N+1}^{\infty}z_n^2(t-\tau_y)]\\
&\le-2\delta_1\rho\|c\|_N^{-2}\mathbb{E}\zeta^2(t-\tau_y).
\end{aligned}\tag{3.13}$$

Let

$$\begin{aligned}
\eta(t)=\mathrm{col}\{&X(t),Y_{\tau_u}^{N_0}(t),v_{\tau_u}^{N_0}(t),Y_{\tau_y}(t),v_{\tau_y}(t),\\
&\xi_1(t),\xi_2(t),Y_r(t),\zeta(t-\tau_y),G(t),H(t)\}.
\end{aligned}$$

Combination of (2.49), (2.54), (2.55), (3.9), (3.12), and (3.13) gives

$$\begin{aligned}
&\mathbb{E}\mathcal{L}V(t)+2\delta\mathbb{E}V(t)-2\delta_1\sup\nolimits_{t-\tau_{M,y}\le\theta\le t}\mathbb{E}V(\theta)\\
&\quad+\beta_1\mathbb{E}[\bar g^2|\mathbf{I}_2^{\mathrm T}X(t)|^2+\bar g^2\textstyle\sum_{n=N+1}^{\infty}z_n^2(t)-|H(t)|^2]\\
&\quad+\beta_2\mathbb{E}[\bar\sigma^2|\mathbf{I}_1^{\mathrm T}X(t)|^2+\bar\sigma^2\textstyle\sum_{n=N+1}^{\infty}z_n^2(t)-|\sigma(t)|^2]\\
&\le\mathbb{E}\sigma^{\mathrm T}(t)\Psi_1\sigma(t)+\mathbb{E}\eta^{\mathrm T}(t)\Psi_2\eta(t)\\
&\quad+\hat\chi_{N+1}\mathbb{E}\textstyle\sum_{n=N+1}^{\infty}z_n^2(t)<0
\end{aligned}\tag{3.14}$$

provided $\hat\chi_{N+1}<0$ ($\hat\chi_n$ is defined below (2.57)), (2.52) and the following inequalities hold:

$$\begin{aligned}
&\Psi_1=\mathcal{I}_1^{\mathrm T}P\mathcal{I}_1+\tau_{M,y}Q_y-\rho I-\beta_2I<0,\\
&\Psi_2=\left[\begin{array}{c|c}\Omega_1 & \Theta_1\quad\Theta_2\quad\Theta_3\\ \hline * & \mathrm{diag}\{\Omega_2,\Omega_3,\Omega_4\}\end{array}\right]+\Xi^{\mathrm T}[\tau_{M,u}^2\mathcal{I}^{\mathrm T}R_u\mathcal{I}\\
&\quad+\tau_{M,y}^2\mathbf{I}_1R_y\mathbf{I}_1^{\mathrm T}+(r/M)^2\mathcal{I}_2^{\mathrm T}R_r\mathcal{I}_2]\Xi<0,
\end{aligned}\tag{3.15}$$

where

$$\begin{aligned}
\Omega_1&=PF+F^{\mathrm T}P+2\delta P+\tfrac{2\alpha_2\rho(N+1)}{N^2\pi^2}\mathcal{I}^{\mathrm T}K_0^{\mathrm T}K_0\mathcal{I}\\
&\quad+(\rho\bar\sigma^2+\rho\alpha_1\bar g^2+\beta_2\bar\sigma^2)\mathbf{I}_1\mathbf{I}_1^{\mathrm T}+\beta_1\bar g^2\mathbf{I}_2\mathbf{I}_2^{\mathrm T}\\
&\quad+(1-\varepsilon_u)\mathcal{I}^{\mathrm T}S_u\mathcal{I}+(1-\varepsilon_y)\mathbf{I}_1S_y\mathbf{I}_1^{\mathrm T}+(1-\varepsilon_M)\mathcal{I}_2^{\mathrm T}S_r\mathcal{I}_2,\\
\Omega_2&=\begin{bmatrix}\frac{2\alpha_3\rho(N+1)}{N^2\pi^2}K_0^{\mathrm T}K_0-\varepsilon_u(S_u+R_u) & -\varepsilon_u(G_u+S_u)\\ * & -\varepsilon_u(S_u+R_u)\end{bmatrix},\\
\Omega_3&=\begin{bmatrix}-\varepsilon_y(S_y+R_y) & -\varepsilon_y(S_y+G_y) & \varepsilon_yR_y & \varepsilon_yG_y\\ * & -\varepsilon_y(S_y+R_y) & \varepsilon_yG_y^{\mathrm T} & \varepsilon_yR_y\\ * & * & -\varepsilon_y(Q_y+R_y) & 0\\ * & * & * & -\varepsilon_y(Q_y+R_y)\end{bmatrix},\\
\Omega_4&=\begin{bmatrix}-\varepsilon_M(S_r+R_r) & 0 & 0 & 0\\ * & -2\delta_1\rho\|c\|_N^{-2} & 0 & 0\\ * & * & -\alpha_1\rho I & 0\\ * & * & * & -\beta_1I\end{bmatrix},\\
\Theta_1&=[P\mathcal{I}_1\mathcal{B}\mathcal{K}_0+\varepsilon_u\mathcal{I}^{\mathrm T}S_u,\quad\varepsilon_u\mathcal{I}^{\mathrm T}S_u],\\
\Theta_2&=[-P\mathbf{L}_\zeta C+\varepsilon_y\mathbf{I}_1S_y,\quad\varepsilon_y\mathbf{I}_1S_y,\quad 0,\quad 0],\\
\Theta_3&=[P\Lambda_e+\varepsilon_M\mathcal{I}_3^{\mathrm T}S_r,P\mathbf{L}_\zeta,P\mathbf{I}_1,P\mathbf{I}_2],\\
\Xi&=[F,\mathcal{I}_1\mathcal{B}K_0,0,-\mathbf{L}_\zeta C,0,0,0,\Lambda_e,\mathbf{L}_\zeta,\mathbf{I}_1,\mathbf{I}_2].
\end{aligned}\tag{3.16}$$

By Schur complement, $\hat\chi_{N+1}<0$ holds iff (2.61) holds. Summarizing, we obtain:

Theorem 2. *Consider system (2.1) with control law (2.18), measurement (2.3) with $c\in L^2(0,1)$ satisfying (2.11), $z_0\in\mathcal{D}(\mathcal{A})$ almost surely and $z_0\in L^2(\Omega;L^2(0,1))$. Let $N_0\in\mathbb{N}$ satisfy (2.8) and $N\in\mathbb{N}$ satisfy $N\ge N_0$. Assume that L_0 and K_0 are obtained from (2.12)* and *(2.13), respectively. Given $M\in\mathbb{N}$ and $r,\tau_{M,y},\tau_{M,u},\bar\sigma,\bar g,\delta,\delta_1>0$ ($\delta_1<\delta$), let there exist positive definite matrices P, S_y, R_y, Q_y, S_r, R_r, S_u, R_u, matrices G_y, G_u, positive scalars α_1, α_2, α_3, β_1, β_2 and tuning parameter $\rho>0$ such that LMIs (2.52), (3.15) with Ω_1, Ω_2, Ω_3, Ω_4, Θ_1, Θ_2, Θ_3, Ξ in (3.16), and (2.61) are feasible. Then the following holds:*

- *The solution $z(x,t)$ to (2.1) under the control law (2.18) and the corresponding sub-predictor-based observer $\hat z(x,t)$ defined by (2.17), (3.2) satisfy (2.62) for $t\ge0$ and some $D\ge1$, where $\delta_\tau>0$ is the unique solution of $\delta_\tau=\delta-\delta_1e^{2\delta_\tau\tau_{M,y}}$.*
- *Given $r>0$, inequalities (2.52), (2.61), and (3.15) are always feasible for large enough M,N and small enough $\bar\sigma$, $\bar g$, $\tau_{M,u}$, $\tau_{M,y}>0$.*

Remark 7. For $g(z)=\bar gz$, we have the following closed-loop system

$$\begin{aligned}
&\mathrm{d}X(t)=[FX(t)+\mathbf{L}\zeta(t-\tau_y)+\Lambda_eY_{e,r}(t)\\
&\quad-\mathbf{L}CY_{\tau_y}(t)+\mathcal{I}_1\mathcal{B}K_0Y_{\tau_u}^{N_0}(t)]\mathrm{d}t+\mathcal{I}_1\sigma(t)\mathrm{d}\mathcal{W}(t),\\
&\mathrm{d}z_n(t)=[(-\lambda_n+\bar g)z_n(t)-b_nK_0\mathcal{I}X(t)\\
&\quad+b_nK_0Y_{\tau_u}^{N_0}(t)]\mathrm{d}t+\sigma_n(t)\mathrm{d}\mathcal{W}(t),n>N,
\end{aligned}\tag{3.17}$$

where all coefficients are defined in (3.7) with A_0 and A_1 in F replaced by (2.71). By constructing the Lyapunov functional (3.9) and following the arguments similar to (2.50)–4, (3.9)–(3.12), (3.12)–(3.14), we have

$$\mathbb{E}\mathcal{L}V(t)+2\delta\mathbb{E}V(t)-2\delta_1\sup_{t-\tau_{M,y}\le\theta\le t}\mathbb{E}V(\theta)$$
$$+\beta_2\mathbb{E}[\bar{\sigma}^2|\mathbf{I}_1^{\mathrm{T}}X(t)|^2+\bar{\sigma}^2\sum_{n=N+1}^{\infty}z_n^2(t)-|\sigma(t)|^2]\le 0$$

provided

$$\rho(-2\lambda_n+2\delta+\bar{\sigma}^2+\lambda_n/\alpha_2+\lambda_n/\alpha_3)+\beta_2\bar{\sigma}^2<0,\ n>N, \tag{3.18}$$

and (2.52), (3.15) hold with Ω_2, Ω_3, Θ_1, Θ_2 in (3.16) and

$$\begin{aligned}\Omega_1&=PF+F^{\mathrm{T}}P+2\delta P+\tfrac{2\alpha_2\rho(N+1)}{N^2\pi^2}\mathcal{I}^{\mathrm{T}}K_0^{\mathrm{T}}K_0\mathcal{I}\\&\quad+(\rho\bar{\sigma}^2+\beta_2\bar{\sigma}^2)\mathbf{I}_1\mathbf{I}_1^{\mathrm{T}}+(1-\varepsilon_M)\mathcal{I}_3^{\mathrm{T}}S_r\mathcal{I}_3\\&\quad+(1-\varepsilon_u)\mathcal{I}^{\mathrm{T}}S_u\mathcal{I}+(1-\varepsilon_y)\mathbf{I}_1S_y\mathbf{I}_1^{\mathrm{T}},\\\Omega_4&=\begin{bmatrix}-\varepsilon_M(S_r+R_r)&0\\ *&-2\delta_1\rho\|c\|_N^{-2}\end{bmatrix},\\\Theta_3&=[P\Lambda_e+\varepsilon_M\mathcal{I}_3^{\mathrm{T}}S_r,P\mathbf{L}],\\\Xi&=[F,\mathcal{I}_1\mathcal{B}K_0,0,-\mathbf{L}C,0,0,0,\Lambda_e,\mathbf{L}].\end{aligned} \tag{3.19}$$

By the monotonicity of λ_n and Schur complement, we find that (3.18) holds iff

$$\left[\begin{array}{c|cc}\rho(-2\lambda_{N+1}+2\delta+\bar{\sigma}^2)+\beta_2\bar{\sigma}^2&1&1\\ *&\multicolumn{2}{c}{-\frac{1}{\rho}\mathrm{diag}\{\frac{\alpha_2}{\lambda_{N+1}},\frac{\alpha_3}{\lambda_{N+1}}\}}\end{array}\right]<0. \tag{3.20}$$

Summarizing, we obtain that the feasibility of LMIs (2.52), (3.15) with Ω_2, Ω_3, Θ_1, Θ_2 in (3.16) and Ω_1, Ω_4, Θ_3, Ξ in (3.19), and (3.20) guarantees the mean-square L^2 exponential stability of (3.17).

3.1. Observer-based design: delay robustness

For the case of $M=0$ in (3.1), $\hat{z}^j(t)=\hat{z}^j_{M+1}(t)$, $j\in\{N_0,N-N_0\}$ satisfy

$$\begin{aligned}\mathrm{d}\hat{z}^{N_0}(t)&=[A_0\hat{z}^{N_0}(t)+\hat{G}^{N_0}(t)+B_0u(t-r)]\mathrm{d}t\\&\quad-L_0[C_0\hat{z}^{N_0}(t)+C_1\hat{z}^{N-N_0}(t)-y(t)]\mathrm{d}t,\\\mathrm{d}\hat{z}^{N-N_0}(t)&=[A_1\hat{z}^{N-N_0}(t)+\hat{G}^{N-N_0}(t)+B_1u(t-r)]\mathrm{d}t,\end{aligned} \tag{3.21}$$

where $\hat{G}^j(t)=\hat{G}^j_{M+1}(t)$ is defined below (3.2). Then our method degenerates into the observer-based control with the delay robustness as studied in [13] for deterministic PDEs. Differently from [13], here we assume unknown measurement delays. Following [13], we construct a finite-dimensional observer:

$$\hat{z}(x,t)=\Phi_0(x)\hat{z}^{N_0}(t)+\Phi_1(x)\hat{z}^{N-N_0}(t), \tag{3.22}$$

where Φ_0 and Φ_1 are defined below (3.2). We propose the controller

$$u(t)=K_0\hat{z}^{N_0}(t), \tag{3.23}$$

where $K_0\in\mathbb{R}^{1\times N_0}$ is determined by (2.13).

Let $e^j(t)=z^j(t)-\hat{z}^j(t)$, $j\in\{N_0,N-N_0\}$. Using (2.10) and (3.21), we obtain

$$\begin{aligned}\mathrm{d}e^{N_0}(t)&=[(A_0-L_0C_0)e^{N_0}(t)+H^{N_0}(t)+B_0K_0Y^{N_0}_{\tau_u}(t)\\&\quad-L_0C_1e^{N-N_0}(t)-L_0\zeta(t-\tau_y)+L_0C_0Y^{N_0}_{\tau_y}(t)\\&\quad+L_0C_1Y^{N-N_0}_{\tau_y}(t)]\mathrm{d}t+\sigma^{N_0}(t)\mathrm{d}\mathcal{W}(t),\\\mathrm{d}e^{N-N_0}(t)&=[A_1e^{N-N_0}(t)+H^{N-N_0}(t)\\&\quad+B_1K_0Y^{N_0}_{\tau_u}(t)]\mathrm{d}t+\sigma^{N-N_0}(t)\mathrm{d}\mathcal{W}(t),\end{aligned} \tag{3.24}$$

where $H^j(t)=G^j(t)-\hat{G}^j(t)$, $j\in\{N_0,N-N_0\}$. Introduce the following notations:

$$\begin{aligned}&X(t)=\mathrm{col}\{\hat{z}^{N_0}(t),\hat{z}^{N-N_0}(t),e^{N_0}(t),e^{N-N_0}(t)\},\\&F=\begin{bmatrix}A_0-B_0K_0&0&L_0C_0&L_0C_1\\-B_1K_0&A_1&0&0\\0&0&A_0-L_0C_0&-L_0C_1\\0&0&0&A_1\end{bmatrix},\ \hat{G}(t)=\begin{bmatrix}\hat{G}^{N_0}(t)\\\hat{G}^{N-N_0}(t)\end{bmatrix},\\&H(t)=\begin{bmatrix}H^{N_0}(t)\\H^{N-N_0}(t)\end{bmatrix},\ \mathbf{I}_1=\begin{bmatrix}I_N\\0_{N\times N}\end{bmatrix},\ \mathbf{I}_2=\begin{bmatrix}0_{N\times N}\\I_N\end{bmatrix},\\&\mathcal{B}_1=\mathrm{col}\{B_0,B_1,0_{N\times1}\},\ \mathcal{B}_2=\mathrm{col}\{0_{N\times1},B_0,B_1\},\\&Y^{N_0}_r(t)=\hat{z}^{N_0}(t)-\hat{z}^{N_0}(t-r),\quad C=[C_0,C_1],\\&\mathbf{L}_0=\mathrm{col}\{L_0,0,-L_0,0\},\ \mathcal{I}=[I_{N_0},0_{N_0\times(2N-N_0)}],\\&Y^{N_0}_{\tau_u}(t)=\hat{z}^{N_0}(t-r)-\hat{z}^{N_0}(t-r-\tau_u),\\&Y_{\tau_y}(t)=\begin{bmatrix}Y^{N_0}_{\tau_y}(t)\\Y^{N-N_0}_{\tau_y}(t)\end{bmatrix}=\begin{bmatrix}e^{N_0}(t)-e^{N_0}(t-\tau_y)\\e^{N-N_0}(t)-e^{N_0}(t-\tau_y)\end{bmatrix}.\end{aligned} \tag{3.25}$$

From (2.3), (3.21)–(3.24), we have the closed-loop systems:

$$\begin{aligned}\mathrm{d}X(t)&=\mathbf{F}(t)\mathrm{d}t+\mathbf{I}_2\sigma(t)\mathrm{d}\mathcal{W}(t),\\\mathrm{d}z_n(t)&=[-\lambda_nz_n(t)+g_n(t)-b_nK_0\mathcal{I}X(t)\\&\quad+b_nK_0Y^{N_0}_r(t)+b_nK_0Y^{N_0}_{\tau_u}(t)]\mathrm{d}t+\sigma_n(t)\mathrm{d}\mathcal{W}(t),\end{aligned} \tag{3.26}$$

where $\zeta(t)$ is defined in (3.4) and

$$\begin{aligned}\mathbf{F}(t)&=FX(t)+\mathbf{I}_1\hat{G}(t)+\mathbf{I}_2H(t)+\mathcal{B}_1K_0Y^{N_0}_r(t)\\&\quad+\mathcal{B}_2K_0Y^{N_0}_{\tau_u}(t)-\mathbf{L}_0CY_{\tau_y}(t)+\mathbf{L}_0\zeta(t-\tau_y).\end{aligned}$$

Note that in (3.26), the input delays r, τ_u appear only in systems $\hat{z}^{N_0},\hat{z}^{N-N_0}$ and the stochastic term appears only in systems e^{N_0},e^{N-N_0} that contains only the measurement delays τ_y. So there is still a separation between delay terms r, τ_u and stochastic term, and it needs only one stochastic-dependent term in corresponding Lyapunov functional (see V_{Q_y} in (3.9)). For the stability of system (3.26), we consider the Lyapunov functional (3.9) with $\mathcal{I}_2$ in V_{S_r}, V_{R_r} replaced by $\mathcal{I}$ defined in (3.25), $\mathbf{I}_1$ in V_{S_y}, V_{R_y} replaced by $\mathbf{I}_2$ defined in (3.25), and V_{S_u}, V_{R_u} replaced by

$$\begin{aligned}V_{R_u}(t)&=\tau_{M,u}\int_{-r-\tau_{M,u}}^{-r}\int_{t+\theta}^{t}\mathrm{e}^{-2\delta(t-r-s)}|\mathcal{I}_1\mathbf{F}(s)|^2_{R_u}\mathrm{d}s\mathrm{d}\theta,\\V_{S_u}(t)&=\int_{t-r-\tau_{M,u}}^{t-r}\mathrm{e}^{-2\delta(t-r-s)}|\mathcal{I}_1X(s)|^2_{S_u}\mathrm{d}s.\end{aligned} \tag{3.27}$$

Following arguments similar to (2.39)–(2.61), we have

$$\mathbb{E}\mathcal{L}V(t)+2\delta\mathbb{E}V(t)-2\delta_1\sup_{t-\tau_{M,y}\le\theta\le t}\mathbb{E}V(\theta)$$
$$+\beta_1\mathbb{E}[\bar{g}^2|\mathbf{I}_2^{\mathrm{T}}X(t)|^2+\bar{g}^2\sum_{n=N+1}^{\infty}z_n^2(t)-|H(t)|^2]$$
$$+\beta_2\mathbb{E}[\bar{\sigma}^2|\mathbf{I}_3X(t)|^2+\bar{\sigma}^2\sum_{n=N+1}^{\infty}z_n^2(t)-|\sigma(t)|^2]\le 0$$

$$
\begin{aligned}
&\left[\begin{array}{c|cccc} \rho(-2\lambda_{N+1}+2\delta+\bar{\sigma}^2+\alpha_1\bar{g}^2)+\beta_1\bar{g}^2+\beta_2\bar{\sigma}^2 & 1 & 1 & 1 & 1 \\ * & \multicolumn{4}{c}{-\frac{1}{\rho}\mathrm{diag}\{\alpha_1, \frac{\alpha_2}{\lambda_{N+1}}, \frac{\alpha_3}{\lambda_{N+1}}, \frac{\alpha_4}{\lambda_{N+1}}\}} \end{array}\right] < 0, \\
&\Psi_1 = \mathbf{I}_2^{\mathrm{T}} P \mathbf{I}_2 - \rho I + \tau_{M,y} Q_y - \beta_2 I < 0, \\
&\Psi_2 = \left[\begin{array}{c|c|ccc} \Omega_1 & \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 \\ \hline * & \Omega_2 & \Theta_5 & 0 & 0 \\ \hline * & * & \multicolumn{3}{c}{\mathrm{diag}\{\Omega_3, \Omega_4, \Omega_5\}} \end{array}\right] + \tau_{M,y}^2 \Xi^{\mathrm{T}} \mathbf{I}_2 R_y \mathbf{I}_2^{\mathrm{T}} \Xi \\
&\qquad + \Xi^{\mathrm{T}} \mathcal{I}^{\mathrm{T}} [r^2 R_r + \tau_{M,u}^2 \mathrm{e}^{2\delta\tau_{M,u}} R_u] \mathcal{I} \Xi < 0,
\end{aligned}
\tag{3.28}
$$

Box III.

provided (2.52) and inequalities (3.28) (see Box III) hold. where

$$
\begin{aligned}
\Omega_1 &= PF + F^{\mathrm{T}}P + 2\delta P + \tfrac{2\rho\alpha_2(N+1)}{N^2\pi^2}\mathcal{I}^{\mathrm{T}}K_0^{\mathrm{T}}K_0\mathcal{I} \\
&\quad + (\rho\bar{\sigma}^2 + \rho\alpha_1\bar{g}^2 + \beta_2\bar{\sigma}^2)\mathbf{I}_3^{\mathrm{T}}\mathbf{I}_3 + (1-\varepsilon_r)\mathcal{I}^{\mathrm{T}}S_r\mathcal{I} \\
&\quad + \beta_1\bar{g}^2\mathbf{I}_2\mathbf{I}_2^{\mathrm{T}} + (1-\varepsilon_y)\mathbf{I}_2S_y\mathbf{I}_2^{\mathrm{T}} + (1-\varepsilon_u)\mathcal{I}^{\mathrm{T}}S_u\mathcal{I}, \\
\mathbf{I}_3 &= \begin{bmatrix} I_{N_0} & 0 & I_{N_0} & 0 \\ 0 & I_{N-N_0} & 0 & I_{N-N_0} \end{bmatrix}, \\
\Omega_2 &= \tfrac{2\rho\alpha_3(N+1)}{N^2\pi^2}K_0^{\mathrm{T}}K_0 - \varepsilon_r(S_r+R_r) + (1-\varepsilon_u)S_u, \\
\Omega_3 &= \begin{bmatrix} \frac{2\rho\alpha_4(N+1)}{N^2\pi^2}K_0^{\mathrm{T}}K_0 - \varepsilon_u(S_u+R_u) & -\varepsilon_u(S_u+G_u) \\ * & -\varepsilon_u(S_u+R_u) \end{bmatrix}, \\
\Omega_4 &= \\
&\begin{bmatrix} -\varepsilon_y(S_y+R_y) & -\varepsilon_y(S_y+G_y) & \varepsilon_yR_y & \varepsilon_yG_y \\ * & -\varepsilon_y(S_y+R_y) & \varepsilon_yG_y^{\mathrm{T}} & \varepsilon_yR_y \\ * & * & -\varepsilon_y(Q_y+R_y) & 0 \\ * & * & * & -\varepsilon_y(Q_y+R_y) \end{bmatrix}, \\
\Omega_5 &= \begin{bmatrix} -2\delta_1\rho\|c\|_N^{-2} & 0 & 0 \\ * & -\alpha_1\rho I & -\alpha_1\rho I \\ * & * & -(\beta_1+\rho\alpha_1)I \end{bmatrix}, \\
\Theta_1 &= P\mathcal{B}_1K_0 + \varepsilon_r\mathcal{I}^{\mathrm{T}}S_r - (1-\varepsilon_u)\mathcal{I}^{\mathrm{T}}S_u, \\
\Theta_2 &= [P\mathcal{B}_2K_0 + \varepsilon_u\mathcal{I}^{\mathrm{T}}S_u,\ \varepsilon_u\mathcal{I}^{\mathrm{T}}S_u], \\
\Theta_3 &= [\varepsilon_y\mathbf{I}_2S_y - P\mathbf{L}_0C,\ \varepsilon_y\mathbf{I}_2S_y,\ 0,\ 0], \\
\Theta_3 &= [\varepsilon_y\mathbf{I}_2S_y - P\mathbf{L}_0C,\ \varepsilon_y\mathbf{I}_2S_y,\ 0,\ 0], \\
\Theta_4 &= [P\mathbf{L}_0,\ P\mathbf{I}_1,\ P\mathbf{I}_2], \quad \Theta_5 = [-\varepsilon_uS_u,\ -\varepsilon_uS_u], \\
\Xi &= [F, \mathcal{B}_1K_0, \mathcal{B}_2K_0, 0, -\mathbf{L}_0C, 0, 0, 0, \mathbf{L}_0, \mathbf{I}_1, \mathbf{I}_2].
\end{aligned}
\tag{3.29}
$$

Summarizing, we have

Proposition 2. *Consider system (2.1) with control law (3.23), measurement (2.3) with $c \in L^2(0,1)$ satisfying (2.11), $z_0 \in \mathcal{D}(\mathcal{A})$ almost surely and $z_0 \in L^2(\Omega; L^2(0,1))$. Let $N_0 \in \mathbb{N}$ satisfy (2.8) and $N \in \mathbb{N}$ satisfy $N \geq N_0$. Assume that L_0 and K_0 are obtained from (2.12) and (2.13), respectively. Given $r, \tau_{M,y}, \tau_{M,u}, \bar{\sigma}, \bar{g}, \delta, \delta_1 > 0$ ($\delta_1 < \delta$), let there exist positive definite matrices P, S_y, R_y, Q_y, S_r, R_r, S_u, R_u, positive scalars α_1, α_2, α_3, α_4, β_1, β_2 and tuning parameter $\rho > 0$ such that LMIs (2.52) and (3.28) are feasible with Ω_i, Θ_i, $i = 1,\ldots,5$, Ξ in (3.29). Then the following holds:*

- *The solution $z(x,t)$ to (2.1) subject to the control law (3.23) and the corresponding observer $\hat{z}(x,t)$ given by (3.21), (3.22) satisfy (2.62) for $t \geq 0$ and some $D \geq 1$, where $\delta_\tau > 0$ is the unique solution of $\delta_\tau = \delta - \delta_1\mathrm{e}^{2\delta_\tau\tau_{M,y}}$.*
- *Inequalities (2.52) and (3.28) are always feasible for N large enough and $r, \bar{\sigma}, \bar{g}, \tau_{M,u}, \tau_{M,y} > 0$ small enough.*

Remark 8. For $g(z) = \bar{g}z$, we have the closed-loop systems:

$$
\begin{aligned}
\mathrm{d}X(t) &= [FX(t) + \mathcal{B}_1K_0Y_r^{N_0}(t) + \mathcal{B}_2K_0Y_{\tau_u}^{N_0}(t) \\
&\quad - \mathbf{L}_0CY_{\tau_y}(t) + \mathbf{L}_0\zeta(t-\tau_y)]\mathrm{d}t + \mathbf{I}_2\sigma(t)\mathrm{d}\mathcal{W}(t), \\
\mathrm{d}z_n(t) &= [-\lambda_nz_n(t) - b_nK_0\mathcal{I}_1X(t) + b_nK_0Y_r^{N_0}(t) \\
&\quad + b_nK_0Y_{\tau_u}^{N_0}(t)]\mathrm{d}t + \sigma_n(t)\mathrm{d}\mathcal{W}(t),
\end{aligned}
\tag{3.30}
$$

where all coefficients are defined in (3.25) with A_0 and A_1 in F replaced by (2.71). Construct Lyapunov functional (3.9) with $\mathcal{I}_2 = \mathcal{I}$, and $V_{S_y}(t)$, $V_{R_y}(t)$ replaced by (3.27). Following the arguments similar to (2.39)–(2.61), we have

$$
\begin{aligned}
&\mathbb{E}\mathcal{L}V(t) + 2\delta\mathbb{E}V(t) - 2\delta_1\sup_{t-\tau_{M,y}\leq\theta\leq t}\mathbb{E}V(\theta) \\
&+\beta_2\mathbb{E}[\bar{\sigma}^2|\mathbf{I}_3X(t)|^2 + \bar{\sigma}^2\textstyle\sum_{n=N+1}^{\infty}z_n^2(t) - |\sigma(t)|^2] \leq 0
\end{aligned}
$$

provided (2.52),

$$
\left[\begin{array}{c|ccc} \rho(-2\lambda_{N+1}+2\delta+\bar{\sigma}^2)+\beta_2\bar{\sigma}^2 & 1 & 1 & 1 \\ * & \multicolumn{3}{c}{-\frac{1}{\rho}\mathrm{diag}\{\frac{\alpha_2}{\lambda_{N+1}}, \frac{\alpha_3}{\lambda_{N+1}}, \frac{\alpha_4}{\lambda_{N+1}}\}} \end{array}\right] < 0,
\tag{3.31}
$$

and (3.28) hold, where Ω_2, Ω_3, Ω_4, Θ_1, Θ_2, Θ_3, Θ_5 are given in (3.29) and

$$
\begin{aligned}
\Omega_1 &= PF + F^{\mathrm{T}}P + 2\delta P + \tfrac{2\rho\alpha_2(N+1)}{N^2\pi^2}\mathcal{I}^{\mathrm{T}}K_0^{\mathrm{T}}K_0\mathcal{I} \\
&\quad + (\rho\bar{\sigma}^2 + \beta_2\bar{\sigma}^2)\mathbf{I}_3^{\mathrm{T}}\mathbf{I}_3 + (1-\varepsilon_r)\mathcal{I}^{\mathrm{T}}S_r\mathcal{I} \\
&\quad + (1-\varepsilon_y)\mathbf{I}_2S_y\mathbf{I}_2^{\mathrm{T}} + (1-\varepsilon_u)\mathcal{I}^{\mathrm{T}}S_u\mathcal{I}, \\
\Theta_4 &= P\mathbf{L}_0, \quad \Omega_5 = -2\delta_1\rho\|c\|_N^{-2}, \\
\Xi &= [F, \mathcal{B}_1K_0, \mathcal{B}_2K_0, 0, \mathbf{L}_0C, 0, 0, \mathbf{L}_0].
\end{aligned}
\tag{3.32}
$$

Summarizing, we obtain that the feasibility of LMIs (2.52), (3.28) and (3.31) with Ω_2, Ω_3, Ω_4, Θ_1, Θ_2, Θ_3, Θ_5 given in (3.29), and Ω_1, Ω_5, Θ_4, Ξ given in (3.32) guarantees the mean-square exponential stability of (3.30).

4. Classical predictor for stochastic heat equation

In this section, we aim to study the classical predictor for stochastic heat equation (2.1) with $g(z) = \bar{g}z$ under delayed non-local measurement (2.3). Differently from [17–19], here we assume unknown measurement delays. Due to the multiplicative noise in our system, we have the coupling between all modes of the solution. Therefore, the reduced-order method in [17,18] is ineffective for our system.

Presenting the solution to (2.1) with $g(z) = \bar{g}z$ as (2.4), we get

$$
\begin{aligned}
\mathrm{d}z_n(t) &= [(-\lambda_n + \bar{g})z_n(t) + b_nu(t-r-\tau_u)]\mathrm{d}t \\
&\quad + \sigma_n(t)\mathrm{d}\mathcal{W}(t), \quad n \geq 1,
\end{aligned}
\tag{4.1}
$$

where $\sigma_n(t)$ is defined in (2.6). By notations z^{N_0}, z^{N-N_0}, σ^{N_0}, σ^{N-N_0}, B_0, B_1 defined in (2.9) and A_0, A_1 defined in (2.71), we have that $z^{N_0}(t)$ and $z^{N-N_0}(t)$ satisfy

$$
\begin{aligned}
\mathrm{d}z^{N_0}(t) &= [A_0z^{N_0}(t) + B_0u(t-r-\tau_u)]\mathrm{d}t \\
&\quad + \sigma^{N_0}(t)\mathrm{d}\mathcal{W}(t), \\
\mathrm{d}z^{N-N_0}(t) &= [A_1z^{N-N_0}(t) + B_1u(t-r-\tau_u)]\mathrm{d}t \\
&\quad + \sigma^{N-N_0}(t)\mathrm{d}\mathcal{W}(t).
\end{aligned}
\tag{4.2}
$$

Following [17], we consider a N dimensional observer of the form

$$\hat{z}(x,t) = \Phi_0(x)\hat{z}^{N_0}(t) + \Phi_1(x)\hat{z}^{N-N_0}(t), \tag{4.3}$$

where Φ_0 and Φ_1 are defined below (3.2), $\hat{z}^{N_0}(t)$ and $\hat{z}^{N-N_0}(t)$ satisfy

$$\begin{aligned} d\hat{z}^{N_0}(t) &= [A_0\hat{z}^{N_0}(t) + B_0u(t-r)]dt \\ &\quad - L_0[C_0\hat{z}^{N_0}(t) + C_1\hat{z}^{N-N_0}(t) - y(t)]dt, \\ d\hat{z}^{N-N_0}(t) &= [A_1\hat{z}^{N-N_0}(t) + B_1u(t-r)]dt, \\ \hat{z}^{N_0}(t) &= 0, \quad \hat{z}^{N-N_0}(t) = 0, \quad t \le 0, \end{aligned} \tag{4.4}$$

with $y(t)$ given by (2.3). Following [17,18], we propose the predictor-based control law

$$\begin{aligned} \bar{z}^{N_0}(t) &= e^{A_0r}\hat{z}^{N_0}(t) + \int_{t-r}^{t} e^{A_0(t-s)}B_0u(s)ds, \\ u(t) &= -K_0\bar{z}^{N_0}(t). \end{aligned} \tag{4.5}$$

Taking the time derivative of $\bar{z}^{N_0}(t)$ in (4.5) along (4.4), we arrive at

$$\begin{aligned} d\bar{z}^{N_0}(t) &= (A_0 - B_0K_0)\bar{z}^{N_0}(t)dt \\ &\quad - e^{A_0r}L_0[C_0\hat{z}^{N_0}(t) + C_1\hat{z}^{N-N_0}(t) - y(t)]dt, \\ \bar{z}^{N_0}(0) &= 0, \quad t = 0. \end{aligned} \tag{4.6}$$

Remark 9. In [19], the transformation for the estimation of $\hat{z}^{N-N_0}(t)$ was introduced to eliminate the input delay that appears in system $\hat{z}^{N-N_0}(t)$. Differently from [19], we construct predictor only for $\hat{z}^{N_0}(t)$. If we consider also the transformation for $\hat{z}^{N-N_0}(t)$:

$$\bar{z}^{N-N_0}(t) = e^{A_1r}\hat{z}^{N-N_0}(t) + \int_{t-r}^{t} e^{A_1(t-s)}B_1u(s)ds, \tag{4.7}$$

we will have the differential equation for $e^{N-N_0}(t) = z^{N-N_0}(t) - \hat{z}^{N-N_0}(t)$ (assume $\sigma(z) = \bar{\sigma}z$ for simplicity):

$$\begin{aligned} & de^{N-N_0}(t) = [A_1e^{N-N_0}(t) + B_1K_0\bar{z}^{N_0}(t-r) \\ & -B_1K_0\bar{z}^{N_0}(t-r-\tau_u)]dt + \bar{\sigma}z^{N-N_0}(t)d\mathcal{W}(t) \\ & \overset{(4.7)}{=} [A_1e^{N-N_0}(t) - B_1K_0\bar{z}^{N_0}(t-r-\tau_u) \\ & +B_1K_0\bar{z}^{N_0}(t-r)]dt + \bar{\sigma}[e^{N-N_0}(t) + e^{-A_1r}\bar{z}^{N-N_0}(t) \\ & +e^{-A_1r}\int_{t-r}^{t} e^{A_1(t-s)}B_1K_0\bar{z}^{N_0}(s)ds]d\mathcal{W}(t). \end{aligned} \tag{4.8}$$

For any $r > 0$ and $\bar{\sigma} > 0$, the last stochastic term in (4.8) with $-A_1 > 0$ will blow up for $N \to \infty$.

For well-posedness, we introduce the change of variables (2.19) to obtain the equivalent stochastic heat equation

$$\begin{aligned} & dw(t) = [\mathcal{A}w(t) + \bar{g}[w(t) + \psi(\cdot)u(t-r-\tau_u)] \\ & -\psi(\cdot)[\mu u(t-r-\tau_u) + (1-\tau_u')F_u(t-r-\tau_u)]]dt \\ & +\sigma(w(t) + \psi(\cdot)u(t-r-\tau_u))d\mathcal{W}(t), \ t \ge 0, \end{aligned} \tag{4.9}$$

where $F_u(t) = -K_0(A_0 - B_0K_0)\bar{z}^{N_0}(t) + K_0e^{A_0r}L_0[C_0\hat{z}^{N_0}(t) + C_1\hat{z}^{N-N_0}(t) - \langle c, z(t-\tau_y)\rangle]$ and $u(t)$ is given by (4.5). Similar to Section 2.2 we first consider $t \in [0, t_u^*]$ and obtain that (2.21) admits a unique solution $w \in L^2(\Omega; C([0,t_u^*]; L^2(0,1)) \cap L^2(\Omega\times[0,t_u^*]; H^1(0,1))$ and $w(\cdot,t) \in \mathcal{D}(\mathcal{A})$, $t \in [0, t_u^*]$, almost surely. Therefore, there exists a constant $\kappa_1 > 0$ such that (2.25) holds. Next, we consider

$$\begin{aligned} d\bar{z}^{N_0}(t) &= (A_0 - B_0K_0 - e^{A_0r}L_0C_0e^{-A_0r})\bar{z}^{N_0}(t) \\ &\quad - e^{A_0r}L_0[C_0G(t) + C_1\hat{z}^{N-N_0}(t) - y(t)]dt, \\ G(t) &= e^{-A_0r}\int_{t-r}^{t} e^{A_0(t-s)}B_0K_0\bar{z}^{N_0}(s)ds, \\ d\hat{z}^{N-N_0}(t) &= [A_1\hat{z}^{N-N_0}(t) - B_1K_0\bar{z}^{N_0}(t-r)]dt. \end{aligned} \tag{4.10}$$

Differently from Section 2.2, due to the distributed delay term $G(t)$, here we treat only $y(t)$ as the non-homogeneous term and consider (4.10) as a functional differential equation in $C([-r,0], \mathbb{R}^N)$. From Theorem 5.2.2 in [27], we have (4.10) has a unique solution $\hat{Z} = \mathrm{col}\{\bar{z}^{N_0}, \hat{z}^{N-N_0}\}$ on $[0, t_u^*]$ that satisfies (2.26). Then following the arguments similar to Section 2.2, we use the step method on $[t_u^*+\tau_{m,y}, t_u^*+2\tau_{m,y}]$, $[t_u^*+2\tau_{m,y}, t_u^*+3\tau_{m,y}]$, … and obtain the existence of a unique strong solution to (4.9) under control law (4.5) satisfying $w \in L^2(\Omega; C([0,\infty); L^2(0,1))) \cap L^2(\Omega\times[0,\infty)\backslash\mathcal{J}; H^1(0,1))$ such that $w(\cdot,t) \in \mathcal{D}(\mathcal{A})$, $t \ge 0$, almost surely, where $\mathcal{J} = \{t_u^* + j\tau_{m,y}\}_{j=0}^{\infty}$, and a unique solution to (4.10) satisfying $\hat{Z} \in L^2(\Omega\times[0,\infty); \mathbb{R}^{(M+1)N})$.

Define $e^j(t) = z^j(t) - \hat{z}^j(t)$, $j \in \{N_0, N-N_0\}$. The last term on the right hand side of the differential system for $\hat{z}^{N_0}(t)$ given in (4.4) can be represented as

$$\begin{aligned} & C_0\hat{z}^{N_0}(t) + C_1\hat{z}^{N-N_0}(t) - y(t) \\ & = -C_0e^{N_0}(t) - C_1e^{N-N_0}(t) - \zeta(t-\tau_y) \\ & \quad + C_0Y_{\tau_y}^{N_0}(t) + C_1Y_{\tau_y}^{N-N_0}(t), \\ & Y_{\tau_y}^{j}(t) = z^j(t) - z^j(t-\tau_y), \ j \in \{N_0, N-N_0\}, \end{aligned} \tag{4.11}$$

where $\zeta(t)$ is defined in (3.4). From (4.2), (4.4), and (4.11), we have

$$\begin{aligned} de^{N_0}(t) &= [(A_0 - L_0C_0)e^{N_0}(t) - L_0C_1e^{N-N_0}(t) \\ &\quad - L_0\zeta(t-\tau_y) + B_0K_0\bar{Y}_{\tau_u}^{N_0}(t) + L_0C_0Y_{\tau_y}^{N_0}(t) \\ &\quad + L_0C_1Y_{\tau_y}^{N-N_0}(t)]dt + \sigma^{N_0}(t)d\mathcal{W}(t), \\ de^{N-N_0}(t) &= [A_1e^{N-N_0}(t) + B_1K_0\bar{Y}_{\tau_u}^{N_0}(t)]dt \\ &\quad + \sigma^{N-N_0}(t)d\mathcal{W}(t), \\ \bar{Y}_{\tau_u}^{N_0}(t) &= \bar{z}^{N_0}(t-r) - \bar{z}^{N_0}(t-r-\tau_u). \end{aligned} \tag{4.12}$$

Denote

$$\begin{aligned} & X(t) = \mathrm{col}\{\bar{z}^{N_0}(t), \hat{z}^{N-N_0}(t), e^{N_0}(t), e^{N-N_0}(t)\}, \\ & \mathbf{L}_0 = \mathrm{col}\{e^{A_0r}L_0, 0_{(N-N_0)\times 1}, -L_0, 0_{(N-N_0)\times 1}\}, \\ & F = \begin{bmatrix} A_0 - B_0K_0 & 0 & e^{A_0r}L_0C_0 & e^{A_0r}L_0C_1 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_0 - L_0C_0 & -L_0C_1 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \\ & \mathcal{C} = [C_0, C_1], \quad \mathcal{B}_0 = \mathrm{col}\{0_{N\times 1}, B_0, B_1\}, \\ & \mathcal{I}_1 = \mathrm{col}\{0_{N\times N}, I_N\}, \ \mathcal{B}_1 = \mathrm{col}\{0_{N_0\times 1}, B_1, 0_{N\times 1}\}, \\ & Y_{\tau_y}(t) = \mathrm{col}\{Y_{\tau_y}^{N_0}(t), Y_{\tau_y}^{N-N_0}(t)\}. \end{aligned}$$

By (4.4), (4.6) and (4.12), we have the closed-loop system:

$$dX(t) = \mathbf{F}(t)dt + \mathcal{I}_1\sigma(t)d\mathcal{W}(t), \tag{4.13a}$$

$$\begin{aligned} dz_n(t) &= [(-\lambda_n + a)z_n(t) - b_nK_0\bar{z}^{N_0}(t-r) \\ &\quad + b_nK_0\bar{Y}_{\tau_u}^{N_0}(t)]dt + \sigma_n(t)d\mathcal{W}(t), \quad n > N, \end{aligned} \tag{4.13b}$$

where

$$\begin{aligned} \mathbf{F}(t) &= FX(t) + \mathbf{L}_0\zeta(t-\tau_y) - \mathbf{L}_0\mathcal{C}Y_{\tau_y}(t) \\ &\quad - \mathcal{B}_1K_0\bar{z}^{N_0}(t-r) + \mathcal{B}_0K_0\bar{Y}_{\tau_u}^{N_0}(t). \end{aligned}$$

In the following Lyapunov-based stability analysis of closed-loop system (4.13), to compensate the unknown delay τ_y which appears in $z^{N_0}(t), z^{N-N_0}(t)$ (that are not in the state of closed-loop system (4.13)), we consider

$$X_z(t) = \mathrm{col}\{z^{N_0}(t), z^{N-N_0}(t)\} = \Sigma X(t) + \mathcal{I}_2G(t), \tag{4.14}$$

where $G(t)$ is defined in (4.10). From (4.13a) and (4.14), we have

$$\begin{aligned} & dX_z(t) = \mathbf{F}_z(t)dt + \sigma(t)d\mathcal{W}(t), \\ & \mathbf{F}_z(t) = \Sigma\mathbf{F}(t) + \mathcal{I}_2[e^{-A_0r}B_0K_0\mathcal{I}X(t) - B_0K_0\bar{z}^{N_0}(t-r)], \\ & \Sigma = \begin{bmatrix} e^{-A_0r} & 0 & I_{N_0} & 0 \\ 0 & I_{N-N_0} & 0 & I_{N-N_0} \end{bmatrix}, \mathcal{I}_2 = \begin{bmatrix} I_{N_0} \\ 0_{(N-N_0)\times N_0} \end{bmatrix}, \\ & \mathcal{I} = [I_{N_0}, 0_{N_0\times(2N-N_0)}]. \end{aligned} \tag{4.15}$$

For mean-square L^2-stability of (4.13), we consider the following Lyapunov functional:

$$\begin{aligned} V(t) &= V_{\mathrm{tail}}(t) + V_P(t) + V_r(t) + V_u(t) + V_y(t) + V_q(t), \\ V_q(t) &= q\int_{-r}^{0}\int_{t+\theta}^{t} e^{-2\delta(t-s)}|e^{-A_0(r+\theta)}B_0K_0\bar{z}^{N_0}(s)|^2 ds d\theta, \end{aligned} \tag{4.16}$$

where $V_{\rm tail}$, V_P, V_r, V_u, V_y are defined in (3.10) with

$$\begin{aligned}
&V_{S_r}(t) = \int_{t-r}^{t} e^{-2\delta(t-s)}|\mathcal{I}X(s)|^2_{S_r}ds,\\
&V_{R_r}(t) = r\int_{-r}^{0}\int_{t+\theta}^{t} e^{-2\delta(t-s)}|\mathcal{I}\mathbf{F}(s)|^2_{R_r}dsd\theta,\\
&V_{S_u}(t) = \int_{t-r-\tau_{M,u}}^{t-r} e^{-2\delta(t-r-s)}|\mathcal{I}X(s)|^2_{S_r}ds,\\
&V_{R_u}(t) = \tau_{M,u}\int_{-r-\tau_{M,u}}^{-r}\int_{t+\theta}^{t} e^{-2\delta(t-r-s)}|\mathcal{I}\mathbf{F}(s)|^2_{R_u}dsd\theta,\\
&V_{S_y}(t) = \int_{t-\tau_{M,y}}^{t} e^{-2\delta(t-s)}|X_z(s)|^2_{S_y}ds,\\
&V_{R_y}(t) = \tau_{M,y}\int_{-\tau_{M,y}}^{0}\int_{t+\theta}^{t} e^{-2\delta(t-s)}|\mathbf{F}_z(s)|^2_{R_y}dsd\theta,\\
&V_{Q_y}(t) = \int_{-\tau_{M,y}}^{0}\int_{t+\theta}^{t} e^{-2\delta(t-s)}|\sigma(s)|^2_{Q_y}dsd\theta,
\end{aligned} \tag{4.17}$$

where P, S_r, R_r, S_u, R_u, S_y, R_y, Q_y are positive-definite matrices of appropriate dimensions and q, ρ are positive scalars. Note that V_{S_r}, V_{R_r} are employed to address $\bar{z}^{N_0}(t-r)$ in (4.13), V_{S_u}, V_{R_u} are utilized to compensate $\bar{Y}^{N_0}_{\tau_u}$, V_{S_y}, V_{R_y}, V_{Q_y} are employed to compensate Y_{τ_y}. The term V_q which stems from [22] is used to compensate $G(t)$.

For $V_q(t)$, we have

$$\begin{aligned}
&\mathcal{L}V_q(t) + 2\delta V_q(t) = -\Gamma(t)\\
&\qquad + q\int_{-r}^{0}\left|e^{-A_0(r+\theta)}B_0K_0\mathcal{I}_0X(t)\right|^2 d\theta,\\
&\Gamma(t) = q\int_{-r}^{0} e^{2\delta\theta}\left|e^{-A_0(r+\theta)}B_0K_0\bar{z}^{N_0}(t+\theta)\right|^2 d\theta.
\end{aligned} \tag{4.18}$$

By Jensen's inequality, we obtain

$$-\Gamma(t) \le -\tfrac{q}{r}\varepsilon_r|G(t)|^2. \tag{4.19}$$

Let $\eta(t) = \mathrm{col}\{X(t), \bar{z}^{N_0}(t-r), G(t), \zeta(t-\tau_y), \bar{Y}^{N_0}_{\tau_u}(t), \bar{v}^{j}_{\tau_u}(t), Y_{\tau_y}(t), v^{N_0}_{\tau_y}(t), \xi_1(t), \xi_2(t)\}$, where $v_{\tau_y}(t) = X_z(t-\tau_y) - X_z(t-\tau_{M,y})$, $\bar{v}^{j}_{\tau_u}(t) = \bar{z}^{j}(t-r-\tau_u) - \bar{z}^{j}(t-r-\tau_{M,u})$, $\xi_1(t)$ and $\xi_2(t)$ are defined in (2.51). By using (4.18), (4.19) and following arguments similar to (2.39)–(2.61), we arrive at

$$\begin{aligned}
&\mathbb{E}\mathcal{L}V(t) + 2\delta\mathbb{E}V(t) - 2\delta_1\sup_{t-\tau_{M,y}\le\theta\le t}\mathbb{E}V(\theta)\\
&+\beta\mathbb{E}[\bar{\sigma}^2|\Sigma X(t) + \mathcal{I}_2G(t)|^2 + \bar{\sigma}^2\textstyle\sum_{n=N+1}^{\infty} z_n^2(t) - |\sigma(t)|^2]\\
&\le \mathbb{E}\sigma^{\rm T}(t)\Psi_1\sigma(t) + \mathbb{E}\eta^{\rm T}(t)\Psi_2\eta(t) + \mathbb{E}\textstyle\sum_{n=N+1}^{\infty}\bar{\chi}_n z_n^2(t) < 0
\end{aligned}$$

provided

$$\begin{aligned}
\bar{\chi}_n &= \rho(-2\lambda_n + 2\bar{g} + 2\delta + \bar{\sigma}^2 + \tfrac{\lambda_n}{\alpha_1} + \tfrac{\lambda_n}{\alpha_2})\\
&\quad + \beta\bar{\sigma}^2 < 0, \quad \forall n > N
\end{aligned} \tag{4.20}$$

and the following inequalities hold:

$$\begin{aligned}
&\Psi_1 = \mathcal{I}_1^{\rm T}P\mathcal{I}_1 + \tau_{M,y}Q_y - \rho I - \beta I < 0,\\
&\Psi_2 = \left[\begin{array}{c|c|cc}
\Omega_1 & \begin{matrix} P\mathbf{L}_0 \\ 0 \\ 0\end{matrix} & \Theta_1 & \Theta_2\\ \hline
* & -2\delta_1\rho\|c\|_N^{-2} & 0 & 0\\ \hline
* & * & \multicolumn{2}{c}{\mathrm{diag}\{\Omega_2,\Omega_3\}}
\end{array}\right] + \tau_{M,y}^2\Xi_2^{\rm T}R_y\Xi_2\\
&\qquad + \Xi_1^{\rm T}[\tau_{M,u}^2e^{2\delta r}R_u + r^2R_r]\Xi_1 < 0,
\end{aligned} \tag{4.21}$$

where

$$\begin{aligned}
&\Omega_1 =\\
&\begin{bmatrix}
\Omega_1^{(11)} & \varepsilon_r\mathcal{I}_0^{\rm T}R_r - P\mathcal{B}_1K_0 & (\beta+\rho)\bar{\sigma}^2\Sigma^{\rm T}\mathcal{I}_2 + (1-\varepsilon_y)\Sigma^{\rm T}S_y\mathcal{I}_2\\
* & \Omega_1^{(22)} & 0\\
* & * & \Omega_1^{(33)}
\end{bmatrix},\\
&\Omega_1^{(11)} = PF + F^{\rm T}P + 2\delta P + (\rho+\beta)\bar{\sigma}^2\Sigma^{\rm T}\Sigma\\
&\qquad + \mathcal{I}_0^{\rm T}(S_r + q\Phi - \varepsilon_rR_r)\mathcal{I}_0 + (1-\varepsilon_y)\Sigma^{\rm T}S_y\Sigma,\\
&\Phi = K_0^{\rm T}B_0^{\rm T}\int_{-r}^{0}e^{-2A_0(r+\theta)}d\theta B_0K_0,\\
&\Omega_1^{(22)} = \tfrac{2\alpha_1\rho(N+1)}{N^2\pi^2}K_0^{\rm T}K_0 - \varepsilon_r(S_r+R_r) + (1-\varepsilon_u)S_u,\\
&\Omega_1^{(33)} = (\beta+\rho)\bar{\sigma}^2\mathcal{I}_2^{\rm T}\mathcal{I}_2 + (1-\varepsilon_y)\mathcal{I}_2^{\rm T}S_y\mathcal{I}_2 - \tfrac{q}{r}\varepsilon_rI,\\
&\Omega_2 = \begin{bmatrix}\tfrac{2\alpha_2\rho(N+1)}{N^2\pi^2}K_0^{\rm T}K_0 - \varepsilon_u(S_u+R_u) & -\varepsilon_u(G_u+S_u)\\ * & -\varepsilon_u(S_u+R_u)\end{bmatrix},\\
&\Omega_3 =\\
&\begin{bmatrix}
-\varepsilon_y(S_y+R_y) & -\varepsilon_y(S_y+G_y) & \varepsilon_yR_y & \varepsilon_yG_y\\
* & -\varepsilon_y(S_y+R_y) & \varepsilon_yG_y^{\rm T} & \varepsilon_yR_y\\
* & * & -\varepsilon_y(Q_y+R_y) & 0\\
* & * & * & -\varepsilon_y(Q_y+R_y)
\end{bmatrix},\\
&\Theta_1 = \begin{bmatrix}P\mathcal{B}_0K_0 & 0\\ \varepsilon_uS_u & \varepsilon_uS_u\\ 0 & 0\end{bmatrix},\\
&\Theta_2 = \begin{bmatrix}-P\mathbf{L}_0\mathcal{C} - \varepsilon_y\Sigma^{\rm T}S_y & -\varepsilon_y\Sigma^{\rm T}S_y & 0 & 0\\ 0 & 0 & 0 & 0\\ -\varepsilon_y\mathcal{I}_2^{\rm T}S_y & -\varepsilon_y\mathcal{I}_2^{\rm T}S_y & 0 & 0\end{bmatrix},\\
&\Xi_1 = \mathcal{I}_0[F, -\mathcal{B}_1K_0, \mathbf{L}_0, 0, \mathcal{B}_0K_0, 0, -\mathbf{L}_0\mathcal{C}, 0, 0, 0],\\
&\Xi_2 = [\Sigma F + \mathcal{I}_2e^{-A_0r}B_0K_0\mathcal{I}_0, -\Sigma\mathcal{B}_1K_0 - \mathcal{I}_2B_0K_0,\\
&\qquad \Sigma\mathbf{L}_0, 0, \Sigma\mathcal{B}_0K_0, 0, -\Sigma\mathbf{L}_0\mathcal{C},\ 0,\ 0,\ 0].
\end{aligned} \tag{4.22}$$

By monotonicity of λ_n and Schur complement, we find that (4.20) holds iff

$$\left[\begin{array}{c|cc}
\rho(-2\lambda_{N+1} + 2\bar{g} + 2\delta + \bar{\sigma}^2) + \beta\bar{\sigma}^2 & 1 & 1\\ \hline
* & \multicolumn{2}{c}{-\tfrac{1}{\rho}\mathrm{diag}\{\tfrac{\alpha_1}{\lambda_{N+1}}, \tfrac{\alpha_2}{\lambda_{N+1}}\}}
\end{array}\right] < 0. \tag{4.23}$$

Summarizing, we have

Theorem 3. *Consider system (2.1) with $g(z) = \bar{g}z$, measurement (2.3) with $c \in L^2(0,1)$ satisfying (2.11), control law (4.5), $z_0 \in \mathcal{D}(\mathcal{A})$ almost surely and $z_0 \in L^2(\Omega; L^2(0,1))$. Let $N_0 \in \mathbb{N}$ satisfy (2.8) and $N \in \mathbb{N}$ satisfy $N \ge N_0$. Assume that L_0 and K_0 are obtained from (2.12) and (2.13), respectively. Given $r, \tau_{M,y}, \tau_{M,u}, \bar{\sigma}, \delta, \delta_1 > 0$ ($\delta_1 < \delta$), let there exist positive definite matrices P, S_r, R_r, S_u, R_u, S_y, R_y, Q_y, matrices G_y, G_u, positive scalars q, α_1, α_2, β and tuning parameter ρ such that LMIs (2.52), (4.21), (4.23) are feasible with Ω_1, Ω_2, Ω_3, Θ_1, Θ_2, Ξ_1, Ξ_2 in (4.22). Then the following holds:*

- *The solution $z(x,t)$ to (2.1) with $g(z) = \bar{g}z$ subject to the control law (4.5), and the corresponding observer $\hat{z}(x,t)$ given by (1) satisfy (2.62) for $t \ge 0$ and some $D \ge 1$, where $\delta_\tau > 0$ is the unique solution of $\delta_\tau = \delta - \delta_1e^{2\delta_\tau\tau_{M,y}}$.*
- *Given $r > 0$, LMIs (2.52), (4.21), (4.23) are always feasible for large enough N and small enough $\tau_{M,y}, \tau_{M,u}, \bar{\sigma} > 0$.*

Proof. For any given $r > 0$, to show the feasibility of inequality (2.52), (4.21), (4.23) for large enough N and small enough $\bar{\sigma}, \tau_{M,y}, \tau_{M,u} > 0$, we take $\bar{\sigma}, \tau_{M,y}, \tau_{M,u} \to 0^+$, $\alpha_1 = \alpha_2 = 1$, $R_r = 0$, $S_u = G_u = 0$, $S_y = G_y = 0$.

Take $P = \mathrm{diag}\{P_z, p_1I_{N-N_0}, P_e, p_2I_{N-N_0}\}$ with $P_z, P_e \in \mathbb{R}^{2N_0}$ and $p_1, p_2 > 0$. By Schur complement and letting $R_u = p_2^2I$, $Q_y = R_y = p_2I$,

Table 1
Maximal r for feasibility of LMIs with $\tau_{M,y} = 0.02$, $\tau_{M,u} = 0.01$: Proposition 2 with delay robustness VS. Theorem 1 with $M = 1$.

$\bar{\sigma}$	$N = 4$			$N = 6$			$N = 8$		
	0.2	0.3	0.4	0.2	0.3	0.4	0.2	0.3	0.4
Proposition 2	0.151	0.146	0.142	0.171	0.165	0.159	0.181	0.174	0.167
Remark 8 ($g(z) = \bar{g}z$)	0.162	0.157	0.152	0.183	0.178	0.171	0.194	0.187	0.179
Theorem 1	0.165	0.155	0.141	0.172	0.162	0.145	0.173	0.162	0.147
Remark 5 ($g(z) = \bar{g}z$)	0.173	0.163	0.150	0.182	0.170	0.153	0.183	0.171	0.156

Table 2
Number of decision variables: Theorem 2 (sub-predictors with an additional sub-predictor) VS. Theorem 1 (conventional sub-predictors).

No of sub-predictors	Theorem 2	Theorem 1
2	$\frac{15N^2+9N+3N_0^2+3N_0}{2} + 5$	$20N^2 + 8N + 2N_0^2 + 2N_0 + 5$
3	$12N^2 + 6N + \frac{3N_0^2+3N_0}{2} + 5$	$34N^2 + 10N + 2N_0^2 + 2N_0 + 5$
4	$\frac{47N^2+15N+3N_0^2+3N_0}{2} + 5$	$52N^2 + 12N + 2N_0^2 + 2N_0 + 5$

$p_1 \to 0^+$, $p_2 \to \infty$, we find that $\Psi_2 < 0$ in (4.21) holds if

$$\varepsilon_r S_r - \frac{2\rho(N+1)}{N^2\pi^2} K_0^{\mathrm{T}} K_0 > 0, \quad \begin{bmatrix} \Omega_z & P_z e^{A_0 r} L_0 C_0 \\ * & \Omega_e \end{bmatrix} < 0,$$
$$\begin{aligned} \Omega_z &= P_z(A_0 - B_0K_0) + (A_0 - B_0K_0)^{\mathrm{T}} P_z \\ &\quad + 2\delta P_z + S_r + \frac{\|c\|_N^2}{2\delta_1\rho} P_z e^{A_0 r} L_0 L_0^{\mathrm{T}} e^{A_0 r} P_z, \\ \Omega_e &= P_e(A_0 - L_0C_0) + (A_0 - L_0C_0)^{\mathrm{T}} P_e \\ &\quad + 2\delta P_e + \frac{\|c\|_N^2}{2\delta_1\rho} P_e L_0 L_0^{\mathrm{T}} P_e. \end{aligned} \tag{4.24}$$

Let $P_z = P_c$, given in (2.13), resulting in $P_z(A_0-B_0K_0)+(A_0-B_0K_0)^{\mathrm{T}}P_z+2\delta P_z < 0$. Let P_e solves the following Lyapunov equation

$$P_e(A_0 - L_0C_0 + \delta I) + (A_0 - L_0C_0 + \delta I)^{\mathrm{T}} P_e = -N^{\frac{1}{4}} I.$$

We have $\|P\| = \mathcal{O}(N^{\frac{1}{4}})$, $N \to \infty$. Then choosing $\rho = N^{\frac{1}{3}}$, $S_r = N^{-\frac{1}{2}} I$, $\beta = p_2^2$, we find that (2.52), $\Psi_1 < 0$ in (4.21), and (4.23) hold for large enough N. Fixing such N, by continuity, we have that (2.52), (4.21), and (4.23) are feasible provided $\bar{\sigma}, \tau_{M,u}, \tau_{M,y} > 0$ are small enough.

5. A numerical example

In this section, we consider the stochastic semilinear heat equation (2.1) where g satisfies (2.2) with $\bar{g} = 0.5$ (or the linear case $g(z) = \bar{g}z$), which results in an unstable open-loop system for $\sigma(z) \equiv 0$. Let $N_0 = 1$ and $c(x) = \chi_{[0,0.9]}(x)$ (an indicator function). Take $\delta = 4$. The observer and controller gains L_0 and K_0 are found from (2.12) and (2.13) and are given by $L_0 = 5$, $K_0 = 4.5$.

For verification of LMIs feasibility by using MatLab toolbox, we take $\delta = 0.55$, $\delta_1 = 0.55$, $\rho = 1$, $\tau_{M,y} = 0.02$, $\tau_{M,u} = 0.01$, $\bar{\sigma} \in \{0.2, 0.3, 0.4\}$, and $N \in \{4, 6, 8\}$. First, we consider the observer-based control for delay robustness (i.e., Proposition 2 with delay robustness and Theorem 1 with 1 sub-predictor). The LMIs in Theorem 1 and Proposition 2 as well as their counterparts for $g(z) = \bar{g}z$ (Remarks 5 and 8) were verified, respectively, for maximal values of r that preserve the feasibility of LMIs. The results are given in Table 1, which shows that Proposition 2 with simpler LMIs (only one stochastic-dependent term in Lyapunov functional) allows slightly larger delays than Theorem 1 with 1 sub-predictor for larger N and upper bound of noise intensity.

Next, we consider the predictor methods. We choose the number of sub-predictors as $2, 3, 4$. For the semilinear case, we present the number of scalar decision variables of LMIs in Theorem 2 (sub-predictors with an additional sub-predictor $\hat{z}^j_{M+1}(t)$) and Theorem 1 (conventional sub-predictors) in Table 2, which shows that the introduction of additional sub-predictor has less computational complexity for the same number of sub-predictors compared with the conventional sub-predictors. Then we verify the feasibility of LMIs in Theorems 2 and 1, respectively, to obtain the maximal values of r. The results are given in Table 3. For the case that $g(z) = \bar{g}z$, the LMIs in Remark 7 (sub-predictors with an additional sub-predictor), Remark 5 (conventional sub-predictors), and Theorem 3 (classical predictor) were verified, respectively, to obtain the maximal values of r which preserve the feasibility of LMIs. The results are given in Table 4. From Tables 3 and 4, we see that Theorems 2 and 1 lead to complementary results, whereas Theorem 2 (with an additional sub-predictor) leads to a larger delay for comparatively large M and upper bound of noise intensity, whereas the classical predictor always allows larger delays.

Similar to the deterministic case in [17,20] for large number of sub-predictors, we need much larger N to guarantee the feasibility of LMIs in (3.15) and (2.59) due to the term $\frac{2\rho\alpha_2(N+1)}{N^2\pi^2} \mathcal{I}_0^{\mathrm{T}} K_0^{\mathrm{T}} K_0 \mathcal{I}_0$ therein. Note that for larger M, we require smaller upper bounds $\bar{\sigma}$ on the noise intensity to guarantee the feasibility of LMIs.

For simulations of the closed-loop system (2.1) under the control law (2.18) and sub-predictors (2.15), (2.17) (with an additional sub-predictor), we fix $N = 6$ and choose the time-varying delays τ_u, τ_y and nonlinear functions g, σ as

$$\begin{aligned} &\tau_u = 0.01 \sin^2 t, \quad \tau_y = 0.001 + 0.019 \cos^2 t, \\ &g(z) = 0.5 \sin z, \quad \sigma(z) = 0.2 \sin z, \end{aligned} \tag{5.1}$$

which satisfy (2.2) with $\bar{g} = 0.5$ and $\bar{\sigma} = 0.2$. From Table 3, we have maximal $r = 0.3$. Take the initial condition $z_0(x) = 3x^2 - 2x^3$. The simulations were carried out by using the FTCS (Forward Time Centered Space) method and the Euler–Maruyama method (see [35]) with time step 0.001 and space step 0.05. The evolutions of $\mathbb{E}[\|z(\cdot,t)\|^2_{L^2} + \|\hat{z}(\cdot,t)\|^2_{L^2}]$ and a surface plot of the solution $\mathbb{E}z(x,t)$ are presented in Fig. 1. Similarly, for simulations of the closed-loop system (2.1) with $g(z) = 0.5z$ subject to the control law (4.5) and observer given by (1) (classical predictors), we fix $N = 6$ and choose $\tau_u, \tau_y, g, \sigma$ as (5.1). From Table 4, we have maximal $r = 0.64$. The simulations are presented in Fig. 2. The numerical simulations validate the theoretical results. The stability of the closed-loop systems in simulations was preserved for $r = 0.57$ for the sub-predictors and $r = 1.14$ for the classical predictor, which may indicate that our approach is somewhat conservative in this example.

6. Conclusions

In this paper, we considered output-feedback control of 1D stochastic semilinear heat equation with constant input delay and nonlinear noise under Neumann actuation and nonlocal measurement. To compensate delay we constructed a nonlinear sequential sub-predictor, whereas in the case of linear deterministic part we suggested a classical predictor. Improvements and extension of predictor-based control to higher dimensional stochastic PDEs may be topics for future research.

Table 3
Maximal r for feasibility of LMIs with $\tau_{M,y} = 0.02$, $\tau_{M,u} = 0.01$: Theorem 2 (sub-predictors with an additional sub-predictor) VS. Theorem 1 (conventional sub-predictors).

	No of sub-predictors\\$\bar{\sigma}$	$N = 4$			$N = 6$			$N = 8$		
		0.2	0.3	0.4	0.2	0.3	0.4	0.2	0.3	0.4
Theorem 2	2	0.18	0.17	0.16	0.18	0.17	0.16	0.19	0.18	0.17
	3	0.28	0.26	0.22	0.30	0.27	0.23	0.30	0.28	0.23
	4	0.34	0.22	–	0.37	0.26	–	0.37	0.28	–
Theorem 1	2	0.25	0.23	0.20	0.26	0.24	0.21	0.26	0.24	0.22
	3	0.31	0.24	0.12	0.33	0.27	0.15	0.33	0.28	0.17
	4	0.27	0.08	–	0.33	0.13	–	0.34	0.15	–

Table 4
Maximal r for feasibility of LMIs with $\tau_{M,y} = 0.02$, $\tau_{M,u} = 0.01$: Remark 7 (sub-predictors with an additional sub-predictor) VS. Remark 5 (conventional sub-predictors) VS. Theorem 3 (classical predictor).

	No of sub-predictors\\$\bar{\sigma}$	$N = 4$			$N = 6$			$N = 8$		
		0.2	0.3	0.4	0.2	0.3	0.4	0.2	0.3	0.4
Remark 7	2	0.19	0.18	0.17	0.19	0.18	0.17	0.20	0.19	0.17
	3	0.31	0.30	0.25	0.33	0.31	0.27	0.34	0.31	0.27
	4	0.41	0.33	–	0.43	0.33	0.09	0.44	0.37	0.11
Remark 5	2	0.27	0.25	0.25	0.29	0.26	0.23	0.29	0.27	0.23
	3	0.36	0.30	0.15	0.37	0.32	0.18	0.38	0.33	0.19
	4	0.39	0.16	–	0.44	0.21	0.01	0.45	0.24	0.02
Theorem 3	–	0.59	0.52	0.44	0.64	0.56	0.47	0.66	0.58	0.48

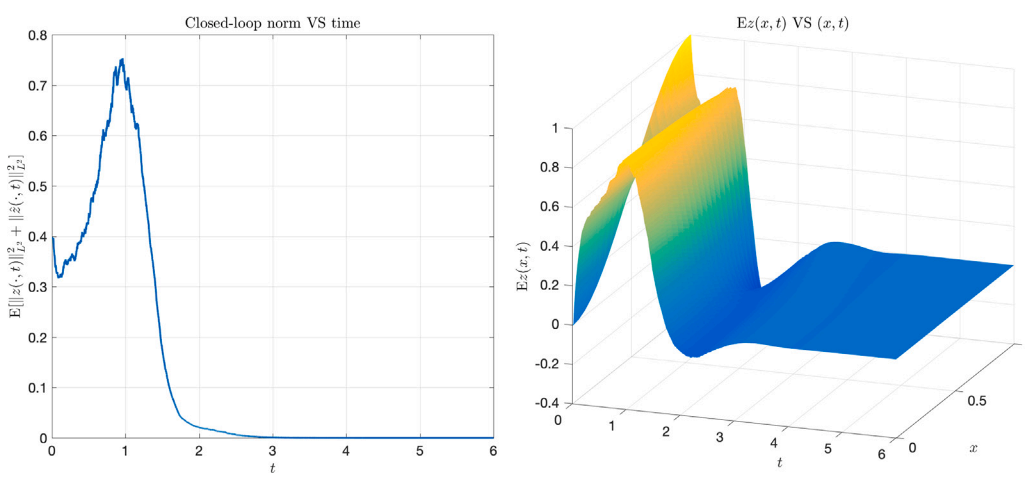

Fig. 1. Sub-predictors: $\mathbb{E}[\|z(\cdot,t)\|^2_{L^2} + \|\hat{z}(\cdot,t)\|^2_{L^2}]$ VS. t and $\mathbb{E}z(x,t)$ VS. (x,t) ($\mathbb{E}$ means taking average over 50 sample trajectories).

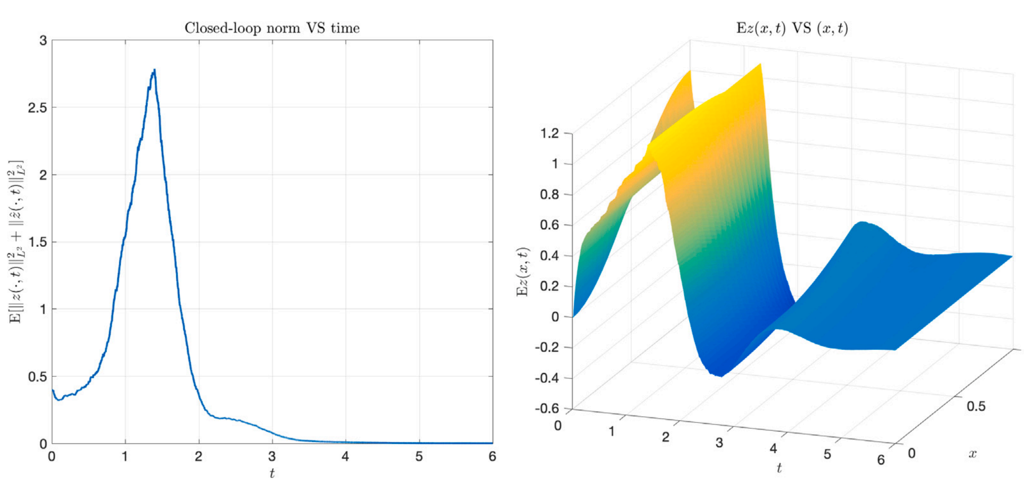

Fig. 2. Classical predictor: $\mathbb{E}[\|z(\cdot,t)\|^2_{L^2} + \|\hat{z}(\cdot,t)\|^2_{L^2}]$ VS. t and $\mathbb{E}z(x,t)$ VS. (x,t) ($\mathbb{E}$ means taking average over 50 sample trajectories).

CRediT authorship contribution statement

Pengfei Wang: Writing – original draft, Writing – review & editing, Methodology, Validation, Investigation. **Emilia Fridman:** Supervision, Investigation, Methodology.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article

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