This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2021.3056364, IEEE Transactions on Automatic Control

Digital implementation of derivative-dependent control by using delays for stochastic multi-agents

Jin Zhang and Emilia Fridman

***Abstract*—In this paper, we study digital implementation of derivative-dependent control for consensus of stochastic multi-agent systems. The consensus controllers that depend on the output and its derivatives are approximated as delayed sampled-data controllers. First, we consider the nth-order stochastic multi-agent systems. Second, we consider PID control of the second-order stochastic multi-agent systems. For the consensus analysis, we propose novel Lyapunov functionals to derive linear matrix inequalities (LMIs) that allow to find admissible sampling period. The efficiency of the presented approach is illustrated by numerical examples.**

***Index Terms*—Sampled-data control, stochastic multi-agent systems, consensus, LMIs.**

I. Introduction

During the last decade, consensus of multi-agent systems has received much attention due to its wide applications [1]. Consensus requires all agents to achieve a desired objective via neighbors' information. For example, the second-order consensus problem was studied by the position and velocity information [2], [3]. If the velocity (i.e. the derivative of the position) is not available, it can be approximated by finite differences leading to a delayed feedback (see e.g. [4], [5], [6], [7], [8], [9], [10] and reference therein). This idea has been employed for the second-order deterministic multi-agent systems in [11], [12], [13].

In networked control systems (NCSs), the asynchronous and aperiodic sampling may emerge. An estimate on the bound of the coupling strength preserving multiconsensus of single integrators under the asynchronous and aperiodic sampling was provided in [14]. Consensus of second-order multi-agent systems with arbitrary asynchronous and aperiodic sampling periods was studied in [15] by designing an observer-based controller that uses sampled position information only. Leader-following consensus problem of nonlinear high-order systems subject to additive bounded disturbances and asynchronously sampled outputs was studied in [16].

In many areas of applications, e.g. aircraft engineering, process control, population dynamics, multiplicative noises that occur due to the parameter uncertainties and nonlinearities cannot be avoided [17], [18]. Consensus of stochastic multi-agent systems was studied in [19], [20], [21]. However, the idea of using the delayed position information has not been studied yet for the nth-order deterministic multi-agents ($n \geq 3$) or stochastic multi-agents ($n \geq 2$).

In this paper, we first study digital implementation of derivative-dependent controller by using delays for the nth-order stochastic multi-agent systems. Following the improved approximation method by using consecutive sampled outputs [9], [10], we approximate the consensus controllers that depend on the output and its derivatives up to the order $n-1$ as delayed sampled-data controllers. Note that extension to multi-agent case of appropriate Lyapunov-Krasovskii (L–K) method is not straightforward since we have to compensate an additional error due to the sampling that appears in multi-agent models only. To compensate this error, we construct additional terms for the corresponding Lyapunov functionals that lead to LMI conditions.

It is well known that L–K method allows to cope with H_∞ performance analysis. As a next step, we consider H_∞ PID control of the second-order stochastic multi-agent systems. Note that sampled-data PID control of the second-order deterministic single-agent systems has been studied in [22], [23]. If we apply the transformation of [22], [23] to the stochastic case, we will have an additional non-zero term that has to be compensated by additional stochastic extension of Lyapunov functional. Our novel transformation significantly simplifies the analysis in the stochastic case. Then we propose appropriate Lyapunov functionals that depend on the deterministic and stochastic parts of the stochastic system. Finally, we present numerical examples to illustrate the efficiency of the presented approach. A conference version of the results of Section II was presented in [24].

Notations and graph theory:

Throughout this paper, $\mathbf{1}_n = [1, \dots, 1]^T \in \mathbb{R}^n$, $\mathbf{0}_n = [0, \dots, 0]^T \in \mathbb{R}^n$, I_n is the identity $n \times n$ matrix, $\otimes$ stands for the Kronecker product, the superscript T stands for matrix transposition. $\mathbb{R}^n$ denotes the n dimensional Euclidean space with Euclidean norm $|\cdot|$, $\mathbb{R}^{n\times m}$ denotes the set of all $n\times m$ real matrices. Denote by $\mathrm{diag}\{\dots\}$ and $\mathrm{col}\{\dots\}$ block-diagonal matrix and block-column vector, respectively. $X > 0$ implies that X is a positive definite symmetric matrix, $|X|_S^2$ denotes X^TSX with matrix S and vector X of appropriate dimensions, $\mathbf{E}x$ denotes the mathematical expectation of stochastic variable x, and the space of the square integrable functions on $[0, \infty)$ with the norm $\|\cdot\|_{L_2}$ is denoted by $L_2[0, \infty)$.

The communication topology among N agents is represented by a directed weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix with $a_{ij} \geq 0$, $\forall i, j \in \mathcal{V}$. It is assumed that $a_{ii} = 0$, $\forall i \in \mathcal{V}$. Notice

This work was supported by Israel Science Foundation under Grant 673/19, by C. and H. Manderman Chair on System Control at Tel Aviv University and by the Planning and Budgeting Committee (PBC) Fellowship from the Council for Higher Education, Israel. The authors are with the School of Electrical Engineering, Tel Aviv University, Tel Aviv 69978, Israel (e-mail addresses: zhangjin1116@126.com, emilia@tauex.tau.ac.il). Corresponding author: Jin Zhang.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2021.3056364, IEEE Transactions on Automatic Control

that $(i,j)\in\mathcal{E}$ when $a_{ij}>0$. An edge $(i,j)\in\mathcal{E}$ implies that node i can receive information from node j. Correspondingly, the Laplacian matrix $L=[L_{ij}]\in\mathbb{R}^{N\times N}$ of graph $\mathcal{G}$ is defined by $L_{ii}=\sum_{j=1}^N a_{ij}$ and $L_{ij}=-a_{ij}$ when $i\neq j$. Graph $\mathcal{G}$ is said to have a spanning tree if there exists node $i\in\mathcal{V}$ such that node i is reachable from any other nodes.

II. Derivative-dependent control

Consider the nth-order stochastic dynamic for each agent $i\in\mathcal{V}$ as follows

$$y_i^{(n)}(t)=\sum_{j=0}^{n-1}\big(a_j+c_j\dot w(t)\big)y_i^{(j)}(t)+bu_i(t), \tag{1}$$

where $y_i(t)=y_i^{(0)}(t)\in\mathbb{R}^p$ is the output, $y_i^{(j)}(t)$ is the jth derivative of $y_i(t)$, $u_i(t)\in\mathbb{R}^q$ is the control input, $w(t)$ is the scalar standard Wiener process [17], [18], and $a_j,\ c_j\in\mathbb{R}^{p\times p}$, $b\in\mathbb{R}^{p\times q}$ are constant matrices. Denoting

$$\begin{aligned}
&x_i(t)=\mathrm{col}\{y_i^{(0)}(t),\dots,y_i^{(n-1)}(t)\}\\
&\quad\ =\mathrm{col}\{x_{i,0}(t),\dots,x_{i,n-1}(t)\}\in\mathbb{R}^{np},\\
&A=\begin{bmatrix}0 & I_p & 0 & \dots & 0\\ 0 & 0 & I_p & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & I_p\\ a_0 & a_1 & a_2 & \dots & a_{n-1}\end{bmatrix}\in\mathbb{R}^{np\times np},\\
&B=\mathrm{col}\{0,b\}\in\mathbb{R}^{np\times q},\quad C=\mathrm{col}\{0,\bar C\}\in\mathbb{R}^{np\times np},\\
&\bar C=[c_0,\dots,c_{n-1}]\in\mathbb{R}^{p\times np},
\end{aligned}$$

we present (1) as

$$dx_i(t)=\big(Ax_i(t)+Bu_i(t)\big)dt+Cx_i(t)dw(t), \tag{2}$$

where the initial condition is given by $x_i(0)=x_i^0$.

Given a Laplacian matrix $L=[L_{ij}]$ of graph $\mathcal{G}$ that contains at least one spanning tree, for the stochastic multi-agent system (2), it is common to look for a consensus controller of the form [25]

$$u_i(t)=\bar K\sum_{j=1}^N L_{ij}x_j(t)=\sum_{j=1}^N\sum_{l=0}^{n-1}L_{ij}\bar K_l x_{j,l}(t) \tag{3}$$

with $\bar K=[\bar K_0,\dots,\bar K_{n-1}]\in\mathbb{R}^{q\times np}$, such that for any initial conditions, consensus of the system (2), (3) is exponentially mean-square achieved with a decay rate $\alpha>0$, i.e.

$$\mathbf{E}|x_i(t)-x_j(t)|\le ce^{-\alpha t}\mathbf{E}|x_i(0)-x_j(0)|\quad \forall i,\ j\in\mathcal{V},$$

where $c>0$ is some constant.

Differently from the state-feedback case with the full knowledge of the agents' state, we consider the output-feedback control where the derivatives $x_{j,l}(t)$ in (3) are not available. As in [9], we employ their finite-difference approximations:

$$\begin{aligned}
&x_{j,0}(t)=\bar x_{j,0}(t),\\
&x_{j,l}(t)\approx\bar x_{j,l}(t)=\tfrac{\bar x_{j,l-1}(t)-\bar x_{j,l-1}(t-h)}{h}\\
&\qquad\qquad=\tfrac{1}{h^l}\sum_{m=0}^l(-1)^m\binom{l}{m}x_{j,0}(t-mh),\\
&\qquad j=1,\dots,N,\quad l=1,\dots,n-1
\end{aligned} \tag{4}$$

with a constant delay $h>0$ and the binomial coefficients $\binom{l}{m}=\frac{l!}{m!(l-m)!}$. By replacing $x_{j,l}(t)$ in (3) with their approximations, we have the following delay-dependent controller

$$\begin{aligned}
u_i(t)&=\sum_{j=1}^N\sum_{l=0}^{n-1}L_{ij}\bar K_l\bar x_{j,l}(t)\\
&=\sum_{j=1}^N\sum_{l=0}^{n-1}L_{ij}K_l x_{j,0}(t-lh),
\end{aligned} \tag{5}$$

where $x_{j,0}(t)=x_{j,0}(0)$ for $t<0$ and

$$K_l=(-1)^l\sum_{m=l}^{n-1}\binom{m}{l}\tfrac{1}{h^m}\bar K_m,\quad l=0,\dots,n-1. \tag{6}$$

Suppose that $x_{j,0}(t)$ is only available at the time instants $t_k=kh$, $k\in\mathbb{N}_0$ where $h>0$ is the sampling period. Then the consensus controller (3) is approximated by the sampled-data controller

$$\begin{aligned}
u_i(t)&=\sum_{j=1}^N\sum_{l=0}^{n-1}L_{ij}\bar K_l\bar x_{j,l}(t_k)\\
&=\sum_{j=1}^N\sum_{l=0}^{n-1}L_{ij}K_l x_{j,0}(t_{k-l}),\\
&t\in[t_k,t_{k+1}),\quad k\in\mathbb{N}_0,
\end{aligned} \tag{7}$$

where $\bar x_{j,l}(t)$ and K_l are from (4) and (6), respectively. For the sampled-data controller (7), we introduce the errors due to sampling

$$\begin{aligned}
&\bar x_{j,0}(t_k)=x_{j,0}(t)-\int_{t_k}^t\dot x_{j,0}(s)ds,\\
&\bar x_{j,l}(t_k)=\bar x_{j,l}(t)-\int_{t_k}^t\dot{\bar x}_{j,l}(s)ds,\ \ l=1,\dots,n-1.
\end{aligned} \tag{8}$$

Then we follow the idea of [9] to present the approximation errors $x_{j,l}(t)-\bar x_{j,l}(t)$ $(l=1,\dots,n-1)$ as

$$\bar x_{j,l}(t)=x_{j,l}(t)-\int_{t-lh}^t\varphi_l(t-s)\dot x_{j,l}(s)ds, \tag{9}$$

where $\varphi_1(v)=\frac{h-v}{h}$, $v\in[0,h]$ and for $l=1,\dots,n-2$

$$\varphi_{l+1}(v)=\begin{cases}\frac1h\int_0^v\varphi_l(\lambda)d\lambda+\frac{h-v}{h}, & v\in[0,h]\\ \frac1h\int_{v-h}^v\varphi_l(\lambda)d\lambda, & v\in(h,lh).\\ \frac1h\int_{v-h}^{lh}\varphi_l(\lambda)d\lambda, & v\in[lh,lh+h].\end{cases}$$

The functions $\varphi_l(s)$ $(l=1,\dots,n-1)$ have the following properties [26], [9]:

$$\begin{cases}0\le\varphi_l(v)\le1,\ \ v\in[0,lh],\\ \varphi_l(0)=1,\ \ \varphi_l(lh)=0,\\ \int_0^{lh}\varphi_l(v)dv=\frac{lh}{2},\\ \frac{d}{dv}\varphi_l(v)\in[-\frac1h,0),\ \ v\in[0,lh].\end{cases} \tag{10}$$

Based on the properties (10), it follows from (9) that

$$\begin{aligned}
\dot{\bar x}_{j,l}(t)&=\dot x_{j,l}(t)-\varphi_l(0)\dot x_{j,l}(t)+\varphi_l(lh)\dot x_{j,l}(t-lh)\\
&\quad-\int_{t-lh}^t\tfrac{d}{dt}\varphi_l(t-s)\dot x_{j,l}(s)ds\\
&=\int_{t-lh}^t\psi_l(t-s)\dot x_{j,l}(s)ds,\\
\psi_l(t-s)&=-\tfrac{d}{dt}\varphi_l(t-s)\in(0,\tfrac1h],\quad l=1,\dots,n-1.
\end{aligned} \tag{11}$$

For the modeling, we denote

$$\begin{aligned}
&x(t)=[x_1^T(t),\dots,x_N^T(t)]^T,\ \ \chi(t)=[\chi_2^T(t),\dots,\chi_N^T(t)]^T,\\
&\chi_j(t)=x_1(t)-x_j(t)=[\chi_{j,0}^T(t),\dots,\chi_{j,n-1}^T(t)]^T,\\
&\bar x(t)=[\bar x_1^T(t),\dots,\bar x_N^T(t)]^T,\ \ \bar\chi(t)=[\bar\chi_2^T(t),\dots,\bar\chi_N^T(t)]^T,\\
&\bar\chi_j(t)=\bar x_1(t)-\bar x_j(t)=[\bar\chi_{j,0}^T(t),\dots,\bar\chi_{j,n-1}^T(t)]^T.
\end{aligned} \tag{12}$$

According to [25], we have

$$\begin{aligned}
&\chi(t)=(E_1\otimes I_{np})x(t),\quad \bar\chi(t)=(E_1\otimes I_{np})\bar x(t),\\
&x(t)=(E_2\otimes I_{np})\chi(t)+(\mathbf{1}_N\otimes I_{np})x_1(t),\\
&\bar x(t)=(E_2\otimes I_{np})\bar\chi(t)+(\mathbf{1}_N\otimes I_{np})\bar x_1(t),\\
&E_1=[\mathbf{1}_{N-1},-I_{N-1}],\quad E_2=[\mathbf{0}_{N-1},-I_{N-1}]^T.
\end{aligned}$$

Then the system (2), (3) takes the form

$$d\chi(t)=D\chi(t)dt+g(t)dw(t), \tag{13}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2021.3056364, IEEE Transactions on Automatic Control

where

$$\begin{array}{l} D = I_{N-1} \otimes A + \mathcal{L} \otimes B\bar{K}, \quad \mathcal{L} = E_1 L E_2, \\ g(t) = (I_{N-1} \otimes C)\chi(t). \end{array} \tag{14}$$

With the same D, $\mathcal{L}$ and $g(t)$, using (8) and (9) the system (2), (7) takes the form

$$d\chi(t) = f_1(t)dt + g(t)dw(t), \tag{15}$$

where

$$\begin{array}{l} f_1(t) = D\chi(t) + \sum_{i=1}^{n-1} (\mathcal{L} \otimes B\bar{K}_i)\kappa_i(t) + (\mathcal{L} \otimes B\bar{K})\delta(t), \\ \kappa_i(t) = -\int_{t-ih}^{t} \varphi_i(t-s) H_i \dot{\chi}(s) ds, \quad \delta(t) = -\int_{t_k}^{t} \dot{\chi}(s) ds, \\ H_i = I_{N-1} \otimes \varepsilon_i, \quad \varepsilon_i = [0_{p\times ip}, I_p, 0_{p\times(n-i-1)p}]. \end{array} \tag{16}$$

Remark 1: Comparatively to the single-agent system in [10], the multi-agent system (15) contains novel term $\delta(t)$ due to the sampling. This term will be further compensated by novel Lyapunov functionals (see e.g. (31), (33) and (36)).

As in [10] we will show that for small enough stochastic perturbations (i.e. small enough $|C|$), if the system (13) is exponentially mean-square stable with a decay rate $\bar{\alpha} > 0$, then for any $\alpha \in (0, \bar{\alpha})$ the system (15) is exponentially mean-square stable with a decay rate and small enough $h > 0$.

Following arguments of [25], we have the following result:

Proposition 1: Assume that directed graph $\mathcal{G}$ has a spanning tree. Consensus of multi-agent system (2) under sampled-data controller (7) with controller gains (6) can be exponentially mean-square achieved if and only if system (15) is exponentially mean-square stable.

It is clear from Proposition 1 that consensus of multi-agent system (2) under sampled-data controller (7) with controller gains (6) is converted into the stability problem of system (15). We now present the following LMI conditions:

Theorem 1: Given $\bar{K} = [\bar{K}_0, \ldots, \bar{K}_{n-1}]$ such that system (13) with $C = 0$ is exponentially mean-square stable with a decay rate $\bar{\alpha} > 0$.

(i) Given tuning parameters $h > 0$ and $\alpha \in (0, \bar{\alpha})$, if there exist $(N-1)np \times (N-1)np$ matrix $P > 0$, $(N-1)p \times (N-1)p$ matrices $R_i > 0$ $(i = 1, \ldots, n-1)$, $Q > 0$, $F_1 > 0$, $F_2 > 0$ and $p \times p$ matrices $W_i > 0$ $(i = 0, \ldots, n-1)$ that satisfy

$$\Phi < 0, \quad \Omega < 0, \tag{17}$$

where Φ and Ω are, respectively, the symmetric matrices composed from

$$\begin{array}{l} \Phi_{11} = PD + D^T P + 2\alpha P + \sum_{i=1}^{n-2} \frac{(ih)^2}{4} |H_{i+1}|^2_{R_i} \\ \quad + |I_{N-1} \otimes C|^2_P + \frac{(n-1)h}{2} |H_{n-1}(I_{N-1} \otimes C)|^2_{F_1+F_2} \\ \quad + \sum_{i=0}^{n-2} h^2 e^{2\alpha ih} |H_{i+1}|^2_{I_{N-1}\otimes W_i}, \\ \Phi_{12} = P[\mathcal{L} \otimes B\bar{K}_1, \ldots, \mathcal{L} \otimes B\bar{K}_{n-1}], \\ \Phi_{14} = P(\mathcal{L} \otimes B\bar{K}), \\ \Phi_{15} = D^T H^T_{n-1}(R_{n-1} + Q), \\ \Phi_{22} = -\mathrm{diag}\{e^{-2\alpha h} R_1, \ldots, e^{-2\alpha(n-1)h} R_{n-1}\}, \\ \Phi_{23} = -[0_{(N-1)p\times(n-2)(N-1)p}, e^{-2\alpha(n-1)h} R_{n-1}]^T, \\ \Phi_{25} = [\mathcal{L} \otimes B\bar{K}_1, \ldots, \mathcal{L} \otimes B\bar{K}_{n-1}]^T H^T_{n-1}(R_{n-1} + Q), \\ \Phi_{33} = -e^{-2\alpha(n-1)h}(R_{n-1} + F_1), \\ \Phi_{44} = -\frac{\pi^2}{4} e^{-2\alpha h} I_{N-1} \otimes \mathrm{diag}\{W_0, \ldots, W_{n-1}\}, \\ \Phi_{45} = (\mathcal{L} \otimes B\bar{K})^T H^T_{n-1}(R_{n-1} + Q), \\ \Phi_{55} = -\frac{4}{(nh-h)^2}(R_{n-1} + Q), \end{array}$$

$$\begin{array}{l} \Omega_{11} = I_{N-1} \otimes W_{n-1} - \frac{(n-1)^2}{4} e^{-2\alpha(n-1)h} Q, \\ \Omega_{12} = I_{N-1} \otimes W_{n-1}, \\ \Omega_{22} = I_{N-1} \otimes W_{n-1} - \frac{n-1}{2} e^{-2\alpha(n-1)h} F_2, \end{array} \tag{18}$$

and other blocks are zero matrices with D and $\mathcal{L}$ given by (14) and H_i $(i = 1, \ldots, n-1)$ given by (16), then consensus of multi-agent system (2) under sampled-data controller (7) with controller gains (6) is exponentially mean-square achieved with a decay rate α.

(ii) Given any $\alpha \in (0, \bar{\alpha})$, LMI $\Phi < 0$ is always feasible for small enough stochastic perturbations and $h > 0$ (meaning that consensus of multi-agent system (2) under sampled-data controller (7) with controller gains (6) is exponentially mean-square achieved with a decay rate $\alpha > 0$).

Proof: (i) Let $\mathfrak{L}$ be the generator of the system (15) [17], [27]. For the standard term

$$V_P = |\chi(t)|^2_P, \quad P > 0, \tag{19}$$

along (15) we have

$$\mathfrak{L}V_P + 2\alpha V_P = 2\chi^T(t) P f_1(t) + 2\alpha |\chi(t)|^2_P + |g(t)|^2_P. \tag{20}$$

To compensate the terms $\kappa_i(t)$ $(i = 1, \ldots, n-2)$, we consider

$$\begin{array}{l} V_{R_i} = \frac{ih}{2} \int_{t-ih}^{t} \int_{t-s}^{ih} e^{-2\alpha(t-s)} \varphi_i(v) |H_{i+1}\chi(s)|^2_{R_i} dv ds, \\ R_i > 0, \quad i = 1, \ldots, n-2. \end{array} \tag{21}$$

Taking into account the relation $H_{i+1}\chi(t) = H_i\dot{\chi}(t)$ $(i = 1, \ldots, n-2)$ and using Jensen's inequality [28], via (10) we have that for $i = 1, \ldots, n-2$

$$\begin{array}{l} \mathfrak{L}V_{R_i} + 2\alpha V_{R_i} = \frac{(ih)^2}{4} |H_{i+1}\chi(t)|^2_{R_i} \\ \quad - \frac{ih}{2} \int_{t-ih}^{t} e^{-2\alpha(t-s)} \varphi_i(t-s) |H_i \dot{\chi}(s)|^2_{R_i} ds \\ \leq \frac{(ih)^2}{4} |H_{i+1}\chi(t)|^2_{R_i} - e^{-2\alpha ih} |\kappa_i(t)|^2_{R_i}. \end{array} \tag{22}$$

For the term $\kappa_{n-1}(t)$, we consider

$$\begin{array}{l} V_{R_{n-1}} = \frac{(n-1)h}{2} \int_{t-(n-1)h}^{t} \int_{t-s}^{(n-1)h} e^{-2\alpha(t-s)} \varphi_{n-1}(v) \\ \quad \times |H_{n-1} f_1(s)|^2_{R_{n-1}} dv ds, \quad R_{n-1} > 0. \end{array} \tag{23}$$

Then we have

$$\begin{array}{l} \mathfrak{L}V_{R_{n-1}} + 2\alpha V_{R_{n-1}} \leq \frac{(nh-h)^2}{4} |H_{n-1} f_1(t)|^2_{R_{n-1}} \\ \quad - e^{-2\alpha(n-1)h} |\kappa_{n-1}(t) + \rho_1(t)|^2_{R_{n-1}}, \end{array} \tag{24}$$

where

$$\rho_1(t) = \int_{t-(n-1)h}^{t} \varphi_{n-1}(t-s) H_{n-1} g(s) dw(s).$$

Using Itô integral properties (see e.g. [17], [27]), via (10) we have for any matrix $F_1 > 0$

$$\begin{array}{l} \mathbf{E} e^{-2\alpha(n-1)h} |\rho_1(t)|^2_{F_1} \\ = \mathbf{E} e^{-2\alpha(n-1)h} \int_{t-(n-1)h}^{t} \varphi^2_{n-1}(t-s) |H_{n-1} g(s)|^2_{F_1} ds \\ \leq \mathbf{E} \int_{t-(n-1)h}^{t} e^{-2\alpha(t-s)} \varphi_{n-1}(t-s) |H_{n-1} g(s)|^2_{F_1} ds. \end{array} \tag{25}$$

Then for the term

$$\begin{array}{l} V_{F_1} = \int_{t-(n-1)h}^{t} \int_{t-s}^{(n-1)h} e^{-2\alpha(t-s)} \varphi_{n-1}(v) \\ \quad \times |H_{n-1} g(s)|^2_{F_1} dv ds, \quad F_1 > 0, \end{array} \tag{26}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2021.3056364, IEEE Transactions on Automatic Control

we have

$$\begin{array}{l}\mathbf{E}\mathfrak{L}V_{F_1}+\mathbf{E}2\alpha V_{F_1}\le \mathbf{E}\frac{(n-1)h}{2}|H_{n-1}g(t)|^2_{F_1}\\ -\mathbf{E}e^{-2\alpha(n-1)h}|\rho_1(t)|^2_{F_1}.\end{array} \tag{27}$$

Denote $W=I_{N-1}\otimes \text{diag}\{W_0,\ldots,W_{n-1}\}>0$. We consider

$$\begin{array}{l}V_W=h^2\int_{t_k}^t e^{-2\alpha(t-s)}|\dot{\chi}(s)|^2_W ds\\ -\frac{\pi^2}{4}e^{-2\alpha h}\int_{t_k}^t e^{-2\alpha(t-s)}|\delta(s)|^2_W ds\end{array} \tag{28}$$

to compensate $\delta(t)$. The exponential Wirtinger's inequality [29] implies $V_W\ge 0$ since $\dot{\delta}(t)=-\dot{\chi}(t)$, $\delta(t_k)=0$ and $W>0$. One can easily arrive at

$$\mathfrak{L}V_W+2\alpha V_W=h^2|\dot{\chi}(t)|^2_W-\frac{\pi^2}{4}e^{-2\alpha h}|\delta(t)|^2_W. \tag{29}$$

From (11) and (12), it follows that

$$\begin{array}{l}|\dot{\chi}(t)|^2_W=|H_1\chi(t)|^2_{I_{N-1}\otimes W_0}+|(\rho_2(t)+\rho_3(t))|^2_{I_{N-1}\otimes W_{n-1}}\\ +\sum_{i=1}^{n-2}|\int_{t-ih}^t\psi_i(t-s)H_{i+1}\chi(s)ds|^2_{I_{N-1}\otimes W_i},\end{array} \tag{30}$$

where

$$\begin{array}{l}\rho_2(t)=\int_{t-(n-1)h}^t\psi_{n-1}(t-s)H_{n-1}f_1(s)ds,\\ \rho_3(t)=\int_{t-(n-1)h}^t\psi_{n-1}(t-s)H_{n-1}g(s)dw(s).\end{array}$$

with H_i $(i=1,\ldots,n-1)$ given by (16). To compensate the term $\rho_2(t)$, we consider

$$\begin{array}{l}V_Q=\frac{(nh-h)^2}{4}\int_{t-(n-1)h}^t e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)\\ \times|H_{n-1}f_1(s)|^2_Q ds,\quad Q>0.\end{array} \tag{31}$$

Using Jensen's inequality [28], via (10) we have

$$\begin{array}{l}\mathfrak{L}V_Q+2\alpha V_Q\le\frac{(nh-h)^2}{4}|H_{n-1}f_1(t)|^2_Q\\ -\frac{(nh-h)^2}{4}e^{-2\alpha(n-1)h}|\rho_2(t)|^2_Q.\end{array} \tag{32}$$

Similarly for the term $\rho_3(t)$, we consider

$$\begin{array}{l}V_{F_2}=\frac{(n-1)h}{2}\int_{t-(n-1)h}^t e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)\\ \times|H_{n-1}g(s)|^2_{F_2}ds,\quad F_2>0.\end{array} \tag{33}$$

Via

$$\begin{array}{l}\mathbf{E}he^{-2\alpha(n-1)h}|\rho_3(t)|^2_{F_2}\\ =\mathbf{E}he^{-2\alpha(n-1)h}\int_{t-(n-1)h}^t\psi^2_{n-1}(t-s)|H_{n-1}g(s)|^2_{F_2}ds\\ \le\mathbf{E}\int_{t-(n-1)h}^t e^{-2\alpha(t-s)}\psi_{n-1}(t-s)|H_{n-1}g(s)|^2_{F_2}ds,\end{array} \tag{34}$$

we have

$$\begin{array}{l}\mathbf{E}\mathfrak{L}V_{F_2}+\mathbf{E}2\alpha V_{F_2}\le\mathbf{E}\frac{(n-1)h}{2}|H_{n-1}g(t)|^2_{F_2}\\ -\mathbf{E}\frac{(n-1)h^2}{2}e^{-2\alpha(n-1)h}|\rho_3(t)|^2_{F_2}.\end{array} \tag{35}$$

To cancel the last term on the right-hand side of (30), we additionally consider

$$\begin{array}{l}V_{W_i}=h^2e^{2\alpha ih}\int_{t-ih}^t e^{-2\alpha(t-s)}\varphi_i(t-s)\\ \times|H_{i+1}\chi(s)|^2_{I_{N-1}\otimes W_i}ds,\quad i=1,\ldots,n-2,\end{array} \tag{36}$$

Using Jensen's inequality [28], via (10) we have

$$\begin{array}{l}\mathfrak{L}V_{W_i}+2\alpha V_{W_i}\le h^2e^{2\alpha ih}|H_{i+1}\chi(t)|^2_{I_{N-1}\otimes W_i}\\ -h^2|\int_{t-ih}^t\psi_i(t-s)H_{i+1}\chi(s)ds|^2_{I_{N-1}\otimes W_i}.\end{array} \tag{37}$$

We now consider the functional

$$V_1=V_P+\sum_{i=1}^{n-1}V_{R_i}+V_{F_1}+V_W+\sum_{i=1}^{n-2}V_{W_i}+V_Q+V_{F_2}. \tag{38}$$

Then in view of (19)-(35), we have

$$\begin{array}{l}\mathbf{E}\mathfrak{L}V_1+\mathbf{E}2\alpha V_1\le\mathbf{E}\xi^T(t)\bar{\Phi}\xi(t)+\mathbf{E}h^2\zeta^T(t)\Omega\zeta(t)\\ +\mathbf{E}\frac{(nh-h)^2}{4}|H_{n-1}f_1(t)|^2_{R_{n-1}+Q},\end{array} \tag{39}$$

where $\xi(t)=[\chi^T(t),\kappa_1^T(t),\ldots,\kappa_{n-1}^T(t),\rho_1^T(t),\delta^T(t)]^T$, $\zeta(t)=[\rho_2^T(t),\rho_3^T(t)]^T$, and $\bar{\Phi}$ is obtained from Φ by taking away the last block-column and block-row. By substituting $f_1(t)$ given by (16) into (39) and applying Schur's complement, it follows from $\Phi<0$ and $\Omega<0$ that $\mathbf{E}\mathfrak{L}V_1+\mathbf{E}2\alpha V_1\le 0$ implying the exponential mean-square stability of system (15) with a decay rate α. By using Proposition 1, consensus of multi-agent system (2) under sampled-data controller (7) with controller gains (6) is thus exponentially mean-square achieved with a decay rate α.

(ii) If system (13) with $C=0$ is exponentially mean-square stable with a decay rate $\bar{\alpha}>0$, then for any $\alpha\in(0,\bar{\alpha})$ there exists matrix $P>0$ of appropriate dimension such that $PD+D^TP+2\alpha P<0$. Thus

$$PD+D^TP+2\alpha P+|I_{N-1}\otimes C|^2_P<0 \tag{40}$$

for small enough $|C|$. We choose R_i $(i=1,\ldots,n)$, Q, F_1, F_2 as $\frac{1}{\sqrt{h}}I_{(N-1)p}$ and W_i $(0=1,\ldots,n)$ as $\frac{1}{\sqrt{h}}I_p$. By using Schur's complement, $\bar{\Phi}<0$ defined below (39) is equivalent to

$$PD+D^TP+2\alpha P+|I_{N-1}\otimes C|^2_P+\sqrt{h}(G_1+hG_2)<0, \tag{41}$$

where

$$\begin{array}{l}G_1=\frac{n-1}{2}|H_{n-1}(I_{N-1}\otimes C)|^2+\frac{4}{\pi^2}e^{2\alpha h}|P(\mathcal{L}\otimes B\bar{K})|\\ +\sum_{i=1}^{n-2}e^{2\alpha ih}|P(\mathcal{L}\otimes B\bar{K}_i)|\\ +2e^{2\alpha(n-1)h}|P(\mathcal{L}\otimes B\bar{K}_{n-1})|,\\ G_2=\sum_{i=1}^{n-2}(\frac{i^2}{4}+e^{2\alpha ih})|H_{i+1}|^2.\end{array}$$

It is clear that (40) implies (41) for small enough $h>0$ since $\sqrt{h}(G_1+hG_2)\to 0$ for $h\to 0$, implying the feasibility of $\bar{\Phi}<0$ for small enough $h>0$. Finally, applying Schur's complement to the last block-column and block-row of Φ given by (17), we find that $\Phi<0$ for small enough $h>0$ if $\bar{\Phi}<0$ is feasible. Therefore, LMI $\Phi<0$ is always feasible for small enough $h>0$ and $|C|$.

It follows from (39) that for small enough $h>0$, $\mathbf{E}\mathfrak{L}V_1+\mathbf{E}2\alpha V_1\le 0$ always holds provided (40) holds. This implies that for small enough $h>0$ and $|C|$, consensus of multi-agent system (2) under sampled-data controller (7) with controller gains (6) is exponentially mean-square achieved with a decay rate α. □

Remark 2: As in [9], we consider the consensus problem of stochastic multi-agent system (2) under continuous-time control (i.e. delay-dependent controller (5)) that leads to the system (15) with $\delta(t)=0$. By taking $Q=F_2=0$ and $W_i=0$ $(i=0,\ldots,n-1)$ in LMIs of Theorem 1, one can obtain some LMIs guaranteeing that consensus of multi-agent system (2) under delay-dependent controller (5) with controller gains (6) is exponentially mean-square achieved with a decay rate α.

Remark 3: In the present paper, we study the consensus problem for stochastic multi-agents in the leaderless case.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2021.3056364, IEEE Transactions on Automatic Control

Our results can be extended to the leader-following consensus problem where the leader takes the form

$$y_0^{(n)}(t) = \sum_{j=0}^{n-1} \big(a_j + c_j \dot{w}(t)\big) y_0^{(j)}(t),$$

and the followers take the form of (1). Denote by $M = \mathrm{diag}\{m_1, \ldots, m_N\}$ the leader adjacency matrix. Thus, the consensus controller (3) becomes

$$u_i(t) = \bar{K} \sum_{j=1}^N L_{ij} x_j(t) + \bar{K} m_i (x_i(t) - x_0(t)).$$

Following the approximations (4) with $j = 0, \ldots, N$, we arrive at the system (15) with $\chi(t)$, $\bar{\chi}(t)$, $\mathcal{L}$ and I_{N-1} respectively changed by $[x_1^T(t) - x_0^T(t), \ldots, x_1^T(t) - x_0^T(t)]^T$, $[\bar{x}_1^T(t) - \bar{x}_0^T(t), \ldots, \bar{x}_1^T(t) - \bar{x}_0^T(t)]^T$, $L + M$ and I_N. Then by using similar L-K functionals, the leader-following consensus problem can be solved. However, this extension is not in the scope of the present paper.

Remark 4: As in [6], [7] one can consider the case with a small input delay h_0, where $u_i(t)$ in system (1) is changed by $u_i(t - h_0)$. Note that this delay can be constant or piecewise-continuous in time satisfying $h_0 = h_0(t) \in [0, h_{0M}]$. For the case of time-varying delay, system (15) includes additionally the error $-\int_{t-h_0(t)}^t \dot{\bar{\chi}}(s)ds$. This would require to add the term $\int_{t-h_{0M}}^t (s - t + h_{0M}) \|\dot{\bar{\chi}}(s)\|^2_{I_{N-1}\otimes S} ds$ with S of appropriate dimension to L-K functional V_1 that leads to more complicated LMIs with one additional block-column and block-row. We study the non-delay case for simplicity. Similarly, by using non-consecutive measurements [23], the consensus problem subject to the asynchronous and aperiodic sampling can be tackled.

III. H_∞ PID CONTROL

In this section, we consider the second-order stochastic dynamic for each agent $i \in \mathcal{V}$

$$\begin{aligned} \ddot{y}_i(t) &= (a_0 + c_0 \dot{w}(t)) y_i(t) + (a_1 + c_1 \dot{w}(t)) \dot{y}_i(t) \\ &\quad + b u_i(t) + b_v v_i(t), \\ z_i(t) &= c_z y_i(t) + d_z v_i(t), \end{aligned} \tag{42}$$

under the PID consensus control

$$u_i(t) = \sum_{j=1}^N L_{ij} \big(\bar{K}_P y_j(t) + \bar{K}_I \int_0^t y_j(s)ds + \bar{K}_D \dot{y}_j(t)\big), \tag{43}$$

where $y_i(t) \in \mathbb{R}^p$ is the output, $u_i(t) \in \mathbb{R}^q$ is the control input, $w(t)$ is the scalar standard Wiener process [17], [18], $z_i(t) \in \mathbb{R}^\ell$ is the controlled output vector and $v_i(t) \in \mathbb{R}^m$ is the exogenous disturbance, a_j, $c_j \in \mathbb{R}^{p\times p}$ $(j = 0, 1)$, $b \in \mathbb{R}^{p\times q}$, $b_v \in \mathbb{R}^{p\times m}$, $c_z \in \mathbb{R}^{\ell\times p}$, $d_z \in \mathbb{R}^{\ell\times m}$ are constant matrices, and $\bar{K}_P$, $\bar{K}_I$ and $\bar{K}_D \in \mathbb{R}^{q\times p}$ are controller gains.

Let us present the sampled-data implementation of (43). For $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}_0$ where $t_k = kh$ and h is the sampling period, we have the following approximations

$$\begin{aligned} &\int_0^t y_j(s)ds \approx \int_0^{t_k} y_j(s)ds \approx h \sum_{l=0}^{k-1} y_j(t_l), \\ &\dot{y}_j(t) \approx \dot{y}_j(t_k) \approx \dot{\bar{y}}_j(t_k), \quad \dot{\bar{y}}_j(t) = \frac{y_j(t) - y_j(t-h)}{h}, \end{aligned} \tag{44}$$

where $y_j(t) = y_j(0)$ for $t < 0$. Associating with the approximations (44), we have the following sampled-data controller

$$\begin{aligned} u_i(t) = \sum_{j=1}^N L_{ij} \big(&\bar{K}_P y_j(t_k) + h\bar{K}_I \sum_{l=0}^{k-1} y_j(t_l) \\ &+ \bar{K}_D \dot{\bar{y}}_j(t_k)\big), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0. \end{aligned} \tag{45}$$

Denoting

$$\begin{aligned} x_i(t) &= [x_{i,0}^T(t), x_{i,1}^T(t), x_{i,2}^T(t)]^T \in \mathbb{R}^{3p} \\ &= [y_i^T(t), \dot{\bar{y}}_i^T(t), (t - t_k) y_i^T(t_k) + h \textstyle\sum_{l=0}^{k-1} y_i^T(t_l)]^T, \\ \mathcal{K} &= [\bar{K}_P, \bar{K}_D, \bar{K}_I] \in \mathbb{R}^{q\times 3p}, \quad \epsilon_2 = [I_p, 0_{p\times p}], \\ \epsilon_i &= [0_{p\times ip}, I_p, 0_{p\times(2-i)p}], \quad i = 0, 1, \end{aligned} \tag{46}$$

via (9) the sampled-data controller (45) is rewritten as

$$\begin{aligned} u_i(t) = \sum_{j=1}^N L_{ij} \big(&\mathcal{K} x_j(t) + \bar{K}_P (x_{j,0}(t_k) - x_{j,0}(t)) \\ &+ \bar{K}_I (x_{j,2}(t_k) - x_{j,2}(t)) - \bar{K}_D \int_{t_k}^t \ddot{\bar{y}}_j(s)ds \\ &- \bar{K}_D \int_{t-h}^t \varphi_1(t-s) \epsilon_1 \dot{x}_j(s)ds\big). \end{aligned} \tag{47}$$

Note that the first integral on the left-hand side of (47) is the error due to the sampling whereas the second is the approximation error $\dot{y}_j(t) - \dot{\bar{y}}_j(t)$. We further denote

$$\begin{aligned} &\chi(t) = [\chi_2^T(t), \ldots, \chi_N^T(t)]^T, \quad \theta(t) = [\theta_2^T(t), \ldots, \theta_N^T(t)]^T, \\ &\chi_i(t) = x_1(t) - x_i(t) = [\chi_{i,0}^T(t), \chi_{i,1}^T(t), \chi_{i,2}^T(t)]^T, \\ &\theta_i(t) = [\chi_{i,0}^T(t) - \chi_{i,0}^T(t_k), \chi_{i,2}^T(t) - \chi_{i,2}^T(t_k)]^T, \\ &\tilde{y}(t) = [\tilde{y}_2^T(t), \ldots, \tilde{y}_N^T(t)]^T, \quad \tilde{y}_i(t) = \bar{y}_1(t) - \bar{y}_i(t), \\ &\bar{z}(t) = [\bar{z}_2^T(t), \ldots, \bar{z}_N^T(t)]^T, \quad \bar{z}_i(t) = z_1(t) - z_i(t), \\ &\bar{v}(t) = [\bar{v}_2^T(t), \ldots, \bar{v}_N^T(t)]^T, \quad \bar{v}_i(t) = v_1(t) - v_i(t). \end{aligned} \tag{48}$$

From (44) and (48), it follows that

$$\begin{aligned} \ddot{\tilde{y}}_i(t) &= \ddot{\bar{y}}_1(t) - \ddot{\bar{y}}_i(t) \\ &= \tfrac{1}{h}[\dot{y}_1(t) - \dot{y}_i(t) - (\dot{y}_1(t-h) - \dot{y}_i(t-h))] \\ &= \tfrac{1}{h}\epsilon_1(\chi_i(t) - \chi_i(t-h)) \\ &= \tfrac{1}{h}\int_{t-h}^t \epsilon_1 \dot{\chi}_i(s)ds, \\ \dot{\theta}_i(t) &= [\dot{\chi}_{i,0}^T(t), \dot{\chi}_{i,2}^T(t)]^T, \\ &= [\chi_i^T(t)\epsilon_1^T, \chi_i^T(t)\epsilon_0^T + \theta_i^T(t)\epsilon_2^T]^T, \end{aligned}$$

where ϵ_i $(i = 0, 1, 2)$ are given by (46). Then we have

$$\begin{aligned} \ddot{\tilde{y}}(t) &= \tfrac{1}{h}\int_{t-h}^t \mathcal{H}_1 \dot{\chi}(s)ds, \\ \dot{\theta}(t) &= [\chi^T(t)\mathcal{H}_1^T, \chi^T(t)\mathcal{H}_0^T + \theta^T(t)\mathcal{H}_2^T]^T, \\ \mathcal{H}_i &= I_{N-1} \otimes \epsilon_i, \quad i = 0, 1, 2. \end{aligned} \tag{49}$$

Using (48), the system (42), (47) takes the form

$$\begin{aligned} d\chi(t) &= f_2(t)dt + g(t)dw(t), \\ \bar{z}(t) &= (I_{N-1} \otimes C_z)\chi(t) + (I_{N-1} \otimes d_z)\bar{v}(t), \end{aligned} \tag{50}$$

where

$$\begin{aligned} f_2(t) &= \mathcal{D}\chi(t) + \mathcal{B}_\theta \theta(t) + (\mathcal{L} \otimes B\bar{K}_D)(\beta(t) + \mu(t)) \\ &\quad + (I_{N-1} \otimes B_v)\bar{v}(t), \quad \mathcal{D} = I_{N-1} \otimes A + \mathcal{L} \otimes B\mathcal{K}, \\ \mathcal{B}_\theta &= -[I_{N-1} \otimes A_1 + \mathcal{L} \otimes B\bar{K}_P, \mathcal{L} \otimes B\bar{K}_I], \end{aligned} \tag{51}$$

and $g(t)$ and $\mathcal{L}$ are given by (14) with

$$A = \begin{bmatrix} 0 & I_p & 0 \\ a_0 & a_1 & 0 \\ I_p & 0 & 0 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 \\ 0 \\ I_p \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}, \ B_v = \begin{bmatrix} 0 \\ b_v \\ 0 \end{bmatrix},$$

$$C = [0, \bar{C}^T, 0]^T, \ \bar{C} = [c_0, c_1, 0], \ C_z = [c_z, 0, 0],$$

$$\beta(t) = -\int_{t_k}^t \ddot{\tilde{y}}(s)ds, \ \mu(t) = -\int_{t-h}^t \varphi_1(t-s)\mathcal{H}_1 \dot{\chi}(s)ds.$$

Remark 5: If we apply the transformation of [22], [23] (modified to the multi-agent case), we will arrive at $\theta_i(t)$ that contains non-zero term $\chi_{i,1}(t) - \chi_{i,1}(t_k) = \int_{t_k}^t \mathcal{H}_1 f_2(s)ds + \Pi$ with $\Pi = \int_{t_k}^t \mathcal{H}_1 g(s)dw(s)$. In the stochastic case, term Π has to be compensated by adding additional terms to Lyapunov functional. Our novel transformation significantly simplifies the analysis in the stochastic case.

We now consider the following L-K functional

$$V_2 = V_P + \hat{V}_R + \hat{V}_{F_1} + \hat{V}_Q + \hat{V}_{F_2} + V_{W_0} + V_{W_1}, \quad (52)$$

where V_P is from (19), $\hat{V}_R$, $\hat{V}_{F_1}$, $\hat{V}_Q$ and $\hat{V}_{F_2}$ are obtained from $V_{R_{n-1}}$, V_{F_1}, V_Q and V_{F_2} (that are defined by (23), (26), (31) and (33), respectively) by setting $n = 2$ and replacing H_1, $f_1(t)$ with $\mathcal{H}_1$, $f_2(t)$, and

$$\begin{aligned} V_{W_0} &= h^2 \int_{t_k}^{t} e^{-2\alpha(t-s)} |\dot{\theta}(s)|^2_{I_{N-1}\otimes W_0} ds \\ &\quad - \frac{\pi^2}{4} e^{-2\alpha h} \int_{t_k}^{t} e^{-2\alpha(t-s)} |\theta(s)|^2_{I_{N-1}\otimes W_0} ds, \\ V_{W_1} &= h^2 \int_{t_k}^{t} e^{-2\alpha(t-s)} |\ddot{y}(s)|^2_{I_{N-1}\otimes W_1} ds \\ &\quad - \frac{\pi^2}{4} e^{-2\alpha h} \int_{t_k}^{t} e^{-2\alpha(t-s)} |\beta(s)|^2_{I_{N-1}\otimes W_1} ds, \\ W_0 &> 0, \quad W_1 > 0, \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0. \end{aligned}$$

The exponential Wirtinger's inequality [29] implies $V_{W_0} \geq 0$ and $V_{W_1} \geq 0$ that respectively compensate $\theta(t)$ and $\beta(t)$.

For system (50), we consider the following performance index

$$J = \mathbf{E}\|\bar{z}\|^2_{L_2} - \mathbf{E}\gamma^2\|\bar{v}\|^2_{L_2},$$

where $\gamma > 0$ is the performance level and $\bar{v} \in L_2[0,\infty)$. Then the problem of H_∞ PID control using sampled outputs can be formulated as follows:

1) when $\bar{v}(t) = 0$, the system (50) is exponentially mean-square stable;
2) under the zero initial condition $\chi(t) = 0$, $J \leq 0$ holds for any non-zero $\bar{v} \in L_2[0,\infty)$.

It is well known that the following condition holds [30]:

$$\mathbf{E}\mathfrak{L}V_2 + \mathbf{E}2\alpha V_2 + \mathbf{E}\bar{z}^T(t)\bar{z}(t) - \mathbf{E}\gamma^2\bar{v}^T(t)\bar{v}(t) \leq 0, \quad (53)$$

then the problem of H_∞ PID control using sampled outputs is solved. Then LMI conditions are derived as follows:

Theorem 2: (i) Given tuning parameters $h > 0$, $\alpha > 0$ and $\gamma > 0$, and controller gains $\bar{K}_P$, $\bar{K}_I$ and $\bar{K}_D$, if there exist $3(N-1)p \times 3(N-1)p$ matrix $P > 0$, $(N-1)p \times (N-1)p$ matrices $R > 0$, $Q > 0$, $F_1 > 0$, $F_2 > 0$, $2p \times 2p$ matrix $W_0 > 0$ and $p \times p$ matrix $W_1 > 0$ that satisfy

$$\Xi < 0, \quad \hat{\Omega} < 0, \quad (54)$$

where Ξ is the symmetric matrix composed from

$$\begin{aligned} \Xi_{11} &= P\mathcal{D} + \mathcal{D}^T P + 2\alpha P + |I_{N-1}\otimes C|^2_P \\ &\quad + \frac{h}{2}|\mathcal{H}_1(I_{N-1}\otimes C)|^2_{F_1+F_2} + h^2|\mathbb{H}_0|^2_{I_{N-1}\otimes W_0}, \\ \Xi_{12} &= P\mathcal{B}_\theta + h^2\mathbb{H}_0^T(I_{N-1}\otimes W_0)\mathbb{H}_1, \\ \Xi_{13} &= \Xi_{14} = P(\mathcal{L}\otimes B\bar{K}_D), \quad \Xi_{16} = P(I_{N-1}\otimes B_v), \\ \Xi_{17} &= \mathcal{D}^T\mathcal{H}_1^T(R+Q), \quad \Xi_{18} = I_{N-1}\otimes C_z^T, \\ \Xi_{22} &= -\frac{\pi^2}{4}e^{-2\alpha h}(I_{N-1}\otimes W_0) + h^2|\mathbb{H}_1|^2_{I_{N-1}\otimes W_0}, \\ \Xi_{27} &= \mathcal{B}_\theta^T\mathcal{H}_1^T(R+Q), \quad \Xi_{33} = -\frac{\pi^2}{4}e^{-2\alpha h}(I_{N-1}\otimes W_1), \\ \Xi_{37} &= \Xi_{47} = (\mathcal{L}\otimes B\bar{K}_D)^T\mathcal{H}_1^T(R+Q), \\ \Xi_{44} &= \Xi_{45} = -e^{-2\alpha h}R, \quad \Xi_{55} = -e^{-2\alpha h}(R+F_1), \\ \Xi_{66} &= -\gamma^2 I_{(N-1)m}, \quad \Xi_{67} = (I_{N-1}\otimes B_v)^T\mathcal{H}_1^T(R+Q), \\ \Xi_{68} &= I_{N-1}\otimes d_z^T, \quad \Xi_{77} = -\frac{4}{h^2}(R+Q), \\ \Xi_{88} &= -I_{(N-1)\ell}, \quad \mathbb{H}_0 = [\mathcal{H}_1, \mathcal{H}_0], \quad \mathbb{H}_1 = [0, \mathcal{H}_2], \end{aligned}$$

and other blocks are zero matrices with $\mathcal{L}$, $\mathcal{H}_i$ $(i = 0,1,2)$ and $\mathcal{D}$ respectively given by (14), (49) and (51), and $\hat{\Omega}$ is obtained

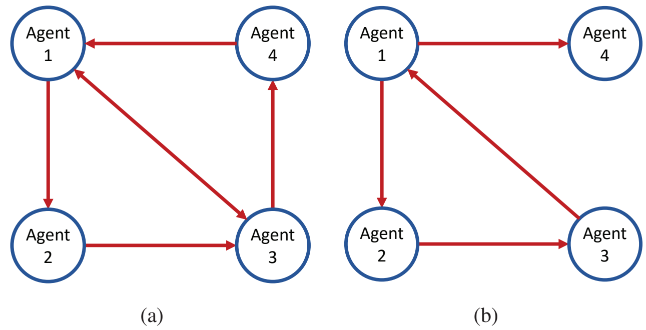

Fig. 1. Directed graphs (Example 1).

from Ω (that is composed from (18)) by setting $n = 2$, then consensus of multi-agent system (42) under sampled-data controller (45) is exponentially mean-square achieved with a decay rate α and a H_∞ performance γ.

(ii) Let there exist controller gains $\bar{K}_P$, $\bar{K}_I$ and $\bar{K}_D$ such that consensus of multi-agent system (42) with $c_0 = c_1 = 0$ and $b_v = 0$ under the PID controller (43) is exponentially achieved with a decay rate $\bar{\alpha}$. Then given any $\alpha \in (0, \bar{\alpha})$, LMI $\Xi < 0$ is always feasible for small enough stochastic perturbations and $h > 0$ and large enough γ (meaning that consensus of multi-agent system (42) under sampled-data controller (45) is exponentially mean-square achieved with a decay rate α).

Proof: (i) Following the proof of Theorem 1 and using V_2 given by (52), via (49) we easily arrive at

$$\begin{aligned} &\mathbf{E}\mathfrak{L}V_2 + \mathbf{E}2\alpha V_2 + \mathbf{E}\bar{z}^T(t)\bar{z}(t) - \mathbf{E}\gamma^2\bar{v}^T(t)\bar{v}(t) \\ &\leq \mathbf{E}\hat{\xi}^T(t)\bar{\Xi}\hat{\xi}(t) + \mathbf{E}\hat{\zeta}^T(t)\hat{\Omega}\hat{\zeta}(t) + \mathbf{E}\bar{z}^T(t)\bar{z}(t) \\ &\quad + \mathbf{E}\frac{h^2}{4}|\mathcal{H}_1 f_2(t)|^2_{R+Q}, \end{aligned} \quad (55)$$

where $\hat{\xi}(t) = [\chi^T(t), \theta^T(t), \beta^T(t), \mu^T(t), \hat{\rho}_1^T(t), \bar{v}(t)]^T$, $\hat{\zeta}(t) = [\hat{\rho}_2^T(t), \hat{\rho}_3^T(t)]^T$, $\bar{\Xi}$ is obtained from Ξ by taking away the last two block-columns and block-rows, and

$$\begin{aligned} &\hat{\rho}_1(t) = \int_{t-h}^{t} \varphi_1(t-s)\mathcal{H}_1 g(s)dw(s), \\ &\hat{\rho}_2(t) = \frac{1}{h}\int_{t-h}^{t} \mathcal{H}_1 f_2(s)ds, \quad \hat{\rho}_3(t) = \frac{1}{h}\int_{t-h}^{t} \mathcal{H}_1 g(s)dw(s). \end{aligned}$$

Further by substituting $\bar{z}(t)$ and $f_2(t)$ respectively given by (50) and (16) into (55) and applying Schur's complement, it follows from $\Xi < 0$ and $\hat{\Omega} < 0$ that (53) holds implying the exponential mean-square stability of system (50) with a decay rate α and a H_∞ performance γ. Thus, consensus of multi-agent system (42) under sampled-data controller (45) is exponentially mean-square achieved with a decay rate α and a H_∞ performance γ.

(ii) The proof of (ii) is similar to (ii) of Theorem 1. □

IV. Numerical examples

Example 1: We consider each agents described by (1) with

$$a_j = 0, \quad b = 1, \quad c_j = \sigma \in \mathbb{R}, \quad j = 0, \ldots, n-1. \quad (56)$$

Two communication topologies are given as directed graphs with a spanning tree in Fig. 1. Without loss of generality, all the weights are assumed to be 1.

Case I: $n = 2$, $\sigma = 0$. First, we consider the multi-agent systems (1), (56) under continuous-time controller (5) with

TABLE I
MAXIMUM VALUES OF h FOR DIFFERENT σ AND $\alpha = 0.1$ (EXAMPLE 1)

σ	0.02	0.1	0.2	0.5	1
Continuous-time control	0.145	0.141	0.135	0.085	0.012
Sampled-data control	0.079	0.076	0.070	0.037	0.004

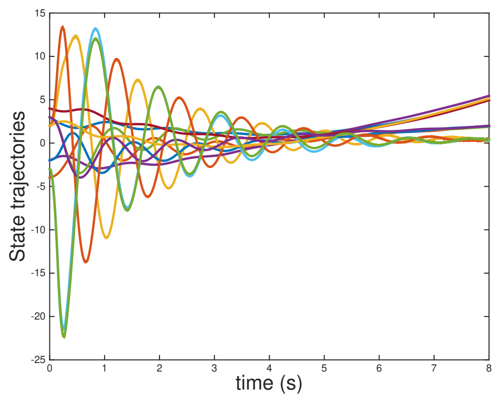

Fig. 2. State trajectories under continuous-time controller (5) with $h = 0.145$ (Example 1).

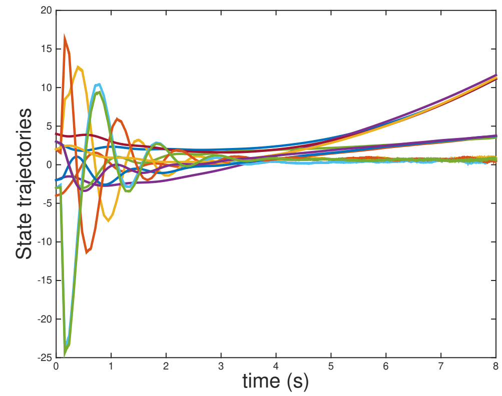

Fig. 3. State trajectories under sampled-data controller (7) with $h = 0.079$ (Example 1).

TABLE II
MAXIMUM VALUES OF h FOR DIFFERENT σ AND $\alpha = 5$ (EXAMPLE 2)

σ	0	0.2	0.5	1	2	3
[23]	0.0047	–	–	–	–	–
[22]	0.019	–	–	–	–	–
Theorem 2	0.019	0.017	0.015	0.011	0.005	0.001

$K_0 = -1$ and $K_1 = 0.5$, and the communication topology shown in Fig. 1(a). Via the frequency-domain approach in [11], the maximum value of h is obtained as 1.8137. From (6), one obtains $\bar{K}_0 = -0.5$, $\bar{K}_1 = -0.9069$. For $\alpha = 0$, via LMIs in Remark 2 corresponding to the continuous-time control, we obtain the maximum value of h as 1.1025. Second, as in [12], we consider the multi-agent systems (1), (56) under sampled-data controller (7) with $\bar{K}_0 = -2.5$ and $\bar{K}_1 = -2$. Under the communication topology Fig. 1(b), the direct discretization approach in [12] leads to the maximum value of h as 0.2, whereas LMIs in Theorem 1 corresponding to the sampled-data control with $\alpha = 0$ give the maximum value of h as 0.13. It should be pointed out that the approaches in [11], [12] are only applicable to the second-order deterministic multi-agent systems (i.e. $n = 2$, $\sigma = 0$). Instead, our method allows to cope with high-order stochastic multi-agent systems (i.e. $n \geq 3$, $\sigma \neq 0$).

Case II: $n = 3$, $\sigma \neq 0$. Choose $\bar{K}_0 = -1$, $\bar{K}_1 = -2$ and $\bar{K}_2 = -3$ such that D defined by (14) is Hurwitz, and consider the communication topology shown in Fig. 1(b). For different values of σ and $\alpha = 0.1$, Table I presents the maximum values of h that preserve consensus of multi-agent respectively under the continuous-time control and sampled-data control. For further simulations, we choose $\sigma = 0.02$ and the initial conditions of the four agents as $x_1(0) = [3, -4, 2]^T$, $x_2(0) = [-2, 3, -3]^T$, $x_3(0) = [4, -2, 2]^T$ and $x_4(0) = [2, 3, -3]^T$. Figs. 2 and 3 depict the state trajectories under delay-dependent controller (5) with $h = 0.145$ and sampled-data controller (7) with $h = 0.079$, respectively. It is clear that consensus is achieved in the presence of stochastic perturbations.

Example 2: We will show that LMIs in Theorem 2 are applicable to the single-agent stochastic systems. Following [22], [23], we consider (42) with

$$a_0 = 0, \quad a_1 = -35.71, \quad b = 1, \quad c_0 = c_1 = \sigma \in \mathbb{R}, \quad (57)$$

and others being 0, and choose $\bar{K}_P = -10$, $\bar{K}_I = -40$, $\bar{K}_D = -0.65$. For different values of σ and $\alpha = 5$, Table II presents the maximum values of h via LMIs of Theorem 2 with $N = 2$, $\mathcal{L} = 1$, [22] and [23]. In the deterministic case ($\sigma = 0$), LMIs in Theorem 2 and [22] give the same result which is better than [23]. In the stochastic case ($\sigma \neq 0$), LMIs of Theorem 2 lead to efficient results whereas [22], [23] fail.

Example 3: We consider each agents described by (42) with

$$\begin{aligned} a_0 = a_1 = 0, \quad b = 1, \quad b_v = 0.2, \\ c_0 = c_1 = 0.01, \quad c_z = 0.05, \quad d_z = 0.1 \end{aligned} \quad (58)$$

under communication topology shown in Fig. 4. Choose $\bar{K}_P = -10$, $\bar{K}_I = -15$ and $\bar{K}_D = -20$ such that $\mathcal{D}$ defined by (51) is Hurwitz. For $\gamma = 2$ and $\alpha = 0.01$, LMIs of Theorem 2 lead to the maximum value of h as 0.013. Then we depict the state trajectories under sampled-data controller (45) with $h = 0.013$ in Fig. 5. where $v_i(t) = 0.1e^{-2t}$ and the initial conditions $[y_i(0), \dot{y}_i(0)]^T$, $i = 1, \ldots, 6$ are given as $[0, 1]^T$, $[0.5, 1.5]^T$, $[-1, 0]^T$, $[0.5, -0.5]^T$, $[-0.5, 0.4]^T$ and $[0.2, -0.2]^T$. Clearly, all the six agents' states indeed achieve consensus.

V. CONCLUSION

In this paper, the digital implementation of derivative-dependent control by using delays has been investigated for consensus of stochastic multi-agent systems. Simple LMIs that allow to find admissible sampling period have been presented by using appropriate Lyapunov functionals. The efficiency of the presented approach has been illustrated by numerical examples. Future work may involve consideration of the asynchronous and aperiodic sampling (as initiated in [15], [16]).

REFERENCES

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2021.3056364, IEEE Transactions on Automatic Control

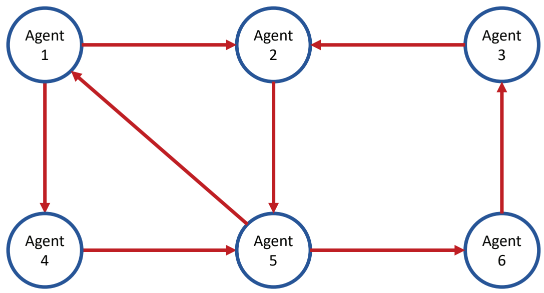

Fig. 4. Directed graph (Example 3).

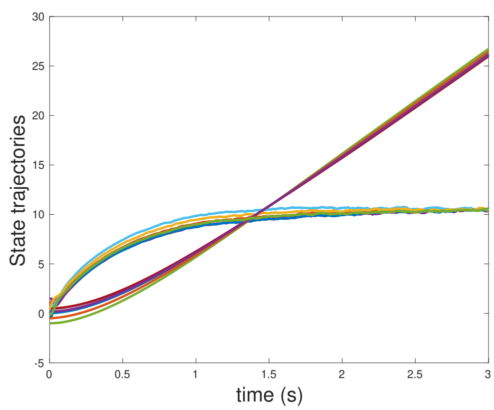

Fig. 5. State trajectories under sampled-data controller (45) with $h = 0.013$ (Example 3).

[2] W. Yu, G. Chen, and M. Cao, "Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems," *Automatica*, vol. 46, no. 6, pp. 1089–1095, 2010.

[3] Y. Gao and L. Wang, "Sampled-data based consensus of continuous-time multi-agent systems with time-varying topology," *IEEE Transactions on Automatic Control*, vol. 56, no. 5, pp. 1226–1231, 2011.

[4] S.-I. Niculescu and W. Michiels, "Stabilizing a chain of integrators using multiple delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 5, pp. 802–807, 2004.

[5] V. L. Kharitonov, S.-I. Niculescu, J. Moreno, and W. Michiels, "Static output feedback stabilization: Necessary conditions for multiple delay controllers," *IEEE Transactions on Automatic Control*, vol. 50, no. 1, pp. 82–86, 2005.

[6] E. Fridman and L. Shaikhet, "Delay-induced stability of vector second-order systems via simple Lyapunov functionals," *Automatica*, vol. 74, pp. 288–296, 2016.

[7] E. Fridman and L. Shaikhet, "Stabilization by using artificial delays: An LMI approach," *Automatica*, vol. 81, pp. 429–437, 2017.

[8] A. Ramírez, S. Mondié, R. Garrido, and R. Sipahi, "Design of proportional-integral-retarded (PIR) controllers for second-order LTI systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 6, pp. 1688–1693, 2016.

[9] A. Selivanov and E. Fridman, "An improved time-delay implementation of derivative-dependent feedback," *Automatica*, vol. 98, pp. 269–276, 2018.

[10] J. Zhang and E. Fridman, "Improved derivative-dependent control of stochastic systems via delayed feedback implementation," *Automatica*, vol. 119, p. 109101, 2020.

[11] W. Yu, G. Chen, M. Cao, and W. Ren, "Delay-induced consensus and quasi-consensus in multi-agent dynamical systems," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 60, no. 10, pp. 2679–2687, 2013.

[12] Q. Ma, S. Xu, and F. L. Lewis, "Second-order consensus for directed multi-agent systems with sampled data," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 16, pp. 2560–2573, 2014.

[13] B. Cui, Y. Xia, K. Liu, and J. Zhang, "Cooperative tracking control for general linear multiagent systems with directed intermittent communications: An artificial delay approach," *International Journal of Robust and Nonlinear Control*, vol. 29, pp. 3063–3077, 2019.

[14] M. Mattioni, "On multiconsensus of multi-agent systems under aperiodic and asynchronous sampling," *IEEE Control Systems Letters*, vol. 4, no. 4, pp. 839–844, 2020.

[15] T. Ménard, E. Moulay, P. Coirault, and M. Defoort, "Observer-based consensus for second-order multi-agent systems with arbitrary asynchronous and aperiodic sampling periods," *Automatica*, vol. 99, pp. 237–245, 2019.

[16] T. Ménard, S. A. Ajwad, E. Moulay, P. Coirault, and M. Defoort, "Leader-following consensus for multi-agent systems with nonlinear dynamics subject to additive bounded disturbances and asynchronously sampled outputs," *Automatica*, vol. 121, p. 109176, 2020.

[17] X. Mao, *Stochastic differential equations and applications*. Chichester: Horwood, 2007.

[18] L. Shaikhet, *Lyapunov functionals and stability of stochastic functional differential equations*. Springer Science & Business Media, 2013.

[19] T. Li and J.-F. Zhang, "Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises," *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2043–2057, 2010.

[20] D. Ding, Z. Wang, B. Shen, and G. Wei, "Event-triggered consensus control for discrete-time stochastic multi-agent systems: The input-to-state stability in probability," *Automatica*, vol. 62, pp. 284–291, 2015.

[21] L. Ma, Z. Wang, Q.-L. Han, and Y. Liu, "Consensus control of stochastic multi-agent systems: a survey," *Science China Information Sciences*, vol. 60, no. 12, p. 120201, 2017.

[22] A. Selivanov and E. Fridman, "Robust sampled-data implementation of PID controller," in *IEEE Conference on Decision and Control*, pp. 932–936, 2018.

[23] A. Selivanov and E. Fridman, "Sampled-data implementation of derivative-dependent control using artificial delays," *IEEE Transactions on Automatic Control*, vol. 63, no. 10, pp. 3594–3600, 2018.

[24] J. Zhang and E. Fridman, "Using delays for digital implementation of derivative-dependent control of stochastic multi-agents," in *IFAC World Congress*, IFAC, 2020.

[25] Y. G. Sun and L. Wang, "Consensus of multi-agent systems in directed networks with nonuniform time-varying delays," *IEEE Transactions on Automatic Control*, vol. 54, no. 7, pp. 1607–1613, 2009.

[26] J. Zhang and E. Fridman, "Derivative-dependent control of stochastic systems via delayed feedback implementation," in *IEEE Conference on Decision and Control*, pp. 66–71, IEEE, 2019.

[27] E. Fridman and L. Shaikhet, "Simple LMIs for stability of stochastic systems with delay term given by Stieltjes integral or with stabilizing delay," *Systems & Control Letters*, vol. 124, pp. 83–91, 2019.

[28] O. Solomon and E. Fridman, "New stability conditions for systems with distributed delays," *Automatica*, vol. 49, no. 11, pp. 3467–3475, 2013.

[29] A. Selivanov and E. Fridman, "Observer-based input-to-state stabilization of networked control systems with large uncertain delays," *Automatica*, vol. 74, pp. 63–70, 2016.

[30] E. Gershon, E. Fridman, and U. Shaked, "Predictor-based control of systems with state multiplicative noise," *IEEE Transactions on Automatic Control*, vol. 62, no. 2, pp. 914–920, 2017.