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Brief paper

Delayed finite-dimensional observer-based control of 1D parabolic PDEs via reduced-order LMIs[☆]

Rami Katz, Emilia Fridman [*]

School of Electrical Engineering, Tel-Aviv University, Tel-Aviv, Israel

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ABSTRACT

Recently a constructive method was introduced for finite-dimensional observer-based control of 1D parabolic PDEs. In this paper we present an improved method in terms of the reduced-order LMIs (that significantly reduce the computation time) and introduce predictors to manage with larger delays. We treat the case of a 1D heat equation under Neumann actuation and non-local measurement, that has not been studied yet. We apply modal decomposition and prove L^2 exponential stability by a direct Lyapunov method. We provide reduced-order LMI conditions for finding the observer dimension N and resulting decay rate. The LMI dimension does not grow with N. The LMI is always feasible for large N, and feasibility for N implies feasibility for $N+1$. For the first time we manage with delayed implementation of the controller in the presence of fast-varying (without any constraints on the delay-derivative) input and output delays. To manage with larger delays, we construct classical observer-based predictors. For the known input delay, the LMIs' dimension does not grow with N, whereas for unknown one the LMIs dimension grows, but it is essentially smaller than in the existing results. A numerical example demonstrates the efficiency of our method.

1. Introduction

Observer-based controllers for PDEs with observers in the form of PDEs have been constructed in Curtain (1982), Krstic and Smyshlyaev (2008), Lasiecka and Triggiani (2000) (to name a few). Very attractive for practical applications finite-dimensional observer-based controllers for parabolic systems were studied by using the modal decomposition approach in Balas (1988), Christofides (2001), Curtain (1982), Ghantasala and El-Farra (2012), Harkort and Deutscher (2011). The recent papers (Katz & Fridman, 2020a, 2020b, 2021) on constructive LMI-based finite-dimensional observer-based control have introduced N-dimensional observers, where the observer gains (and the controller gains) are based only on the $N_0 \leq N$ unstable modes. However, the stability analysis was based on the full-order closed-loop systems. The latter led to higher-order LMIs whose dimension grows with N and complicated proofs of their feasibility.

Delayed and/or sampled-data finite-dimensional controllers were designed in Fridman and Blighovsky (2012) for distributed static output-feedback control and in Espitia, Karafyllis, and Krstic (2021), Karafyllis and Krstic (2018) for boundary state-feedback control. Delayed implementation of finite-dimensional observer-based controllers for the 1D heat equation was presented in Katz and Fridman (2021a). In the case of Dirichlet actuation considered in Katz and Fridman (2021a), the results were not applicable to the case where both input and output delays are fast-varying (without any constraints on the time-derivative that correspond e.g. to sampled-data and network-based control). For boundary control in the presence of fast-varying input and output delays only infinite-dimensional PDE observers have been suggested till now (Katz, Fridman, & Selivanov, 2021).

Large input delays for PDEs can be compensated by classical predictors (Krstic, 2009). Predictor-based controllers for ODEs that compensated an arbitrary large constant part of a delay were suggested in Karafyllis and Krstic (2017), Mazenc and Normand-Cyrot (2013), Selivanov and Fridman (2016) and extended to state-feedback boundary control of parabolic PDEs in Lhachemi, Prieur, and Shorten (2019), Prieur and Trélat (2018). For coupled systems of ODEs, predictors may enlarge the constant part of the delay which preserves stability, but cannot manage with arbitrary large constant delays due to coupling (Liu, Sun, & Krstic, 2018). However, the finite-dimensional observer-based predictors have not been constructed yet for PDEs.

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* Corresponding author.
E-mail addresses: rami@benis.co.il (R. Katz), emilia@tauex.tau.ac.il (E. Fridman).

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In the present paper, we introduce finite-dimensional observer-based controllers for the 1D heat equation under Neumann actuation and non-local measurement. We apply modal decomposition to the original system (without dynamic extension) and prove L^2 exponential stability of the closed-loop system by a direct Lyapunov method. The paper contribution to challenging finite-dimensional observer-based control can be summarized as follows:

(1) The paper introduces *reduced-order closed-loop* system that reveals the *singularly perturbed structure* of the system, leads to *reduced-order LMIs*, trivializes the LMIs feasibility proof and the fact that their feasibility for the observer dimension N implies feasibility for $N+1$. In the example, the feasibility of the reduced-order LMIs for the delayed case can be easily verified for $N=30$, whereas in Katz and Fridman (2021a) the corresponding conditions could not be verified for $N=9$. Note that larger N enlarges delays that preserve the stability.
(2) For the first time in the case of boundary control, the results are applicable to *fast-varying input and output delays*. This is because the proportional controller under Neumann actuation and non-local measurement leads to L^2 convergence. For briefness, our results are presented for differentiable delays. However, via the time-delay approach to networked control (Fridman, 2014), the same LMI conditions are applicable to networked control implementation via a zero-order-hold device, under sampled-data delayed measurements.
(3) The first *finite-dimensional observer-based predictor* is constructed to compensate the constant part of input fast-varying delay, and this is in the presence of the small output fast-varying delay. We present the classical predictors using the reduction approach (Artstein, 1982). We predict the future state of the observer, whereas the infinite-dimensional part depends on the uncompensated large delay. We consider the case of either known or unknown input delay. For the known input delay, the LMIs dimension does not grow with N, whereas for the unknown one it grows, but is essentially smaller than in Katz and Fridman (2021a). An example demonstrates the efficiency of the method and shows that predictors allow for larger delays which preserve the stability.

Our new method can be applied to other classes of parabolic PDEs (see Remark 2.1). In the conference version of the paper (Katz, Basre, & Fridman, 2021a) predictors were not considered.

Notations and preliminaries: $L^2(0,1)$ is the Hilbert space of Lebesgue measurable and square integrable functions $f:[0,1]\to\mathbb{R}$ with inner product $\langle f,g\rangle := \int_0^1 f(x)g(x)dx$ and norm $\|f\|^2 := \langle f,f\rangle$. $H^k(0,1)$ is the Sobolev space of functions $f:[0,1]\to\mathbb{R}$ having k square integrable weak derivative, with norm $\|f\|^2_{H^k} := \sum_{j=0}^k \left\|\frac{d^jf}{dx^j}\right\|^2$. The Euclidean norm on $\mathbb{R}^n$ will be denoted by $|\cdot|$. For $P\in\mathbb{R}^{n\times n}$, the notation $P>0$ means that P is symmetric and positive definite. The sub-diagonal elements of a symmetric matrix are denoted by $*$. For $U\in\mathbb{R}^{n\times n}$, $U>0$ and $X\in\mathbb{R}^n$ we denote $|X|^2_U = X^TUX$. We denote by $\mathbb{Z}_+$ the set of nonnegative integers.

Recall that the Sturm–Liouville eigenvalue problem

$$\phi''+\lambda\phi=0,\quad x\in[0,1]\quad;\quad \phi'(0)=\phi'(1)=0, \tag{1.1}$$

induces a sequence of eigenvalues $\lambda_n = n^2\pi^2, n\geq 0$ with corresponding eigenfunctions

$$\phi_0(x)=1,\quad \phi_n(x)=\sqrt{2}\cos\left(\sqrt{\lambda_n}x\right),\ n\geq 1. \tag{1.2}$$

Moreover, the eigenfunctions form a complete orthonormal system in $L^2(0,1)$. Given $N\in\mathbb{Z}_+$ and $h\in L^2(0,1)$ satisfying $h\overset{L^2}{=}\sum_{n=0}^\infty h_n\phi_n$ we will use the notation $\|h\|_N^2 = \|h\|^2-\sum_{n=0}^N h_n^2 = \sum_{n=N+1}^\infty h_n^2$.

2. Non-delayed L^2-stabilization

Consider the reaction–diffusion system

$$z_t(x,t)=z_{xx}(x,t)+qz(x,t),\ z_x(0,t)=0,\ z_x(1,t)=u(t) \tag{2.1}$$

where $t\geq 0$, $x\in[0,1]$, $z(x,t)\in\mathbb{R}$ and $q\in\mathbb{R}$ is the reaction coefficient. We consider Neumann actuation with a control input $u(t)$ and non-local measurement of the form

$$y(t)=\langle c,z(\cdot,t)\rangle,\quad c\in L^2(0,1). \tag{2.2}$$

Below, we prove the existence and uniqueness of a classical solution to (2.1) (see proof after (2.14)). Therefore, we can present the solution as

$$z(x,t)\overset{L^2}{=}\sum_{n=0}^\infty z_n(t)\phi_n(x),\quad z_n(t)=\langle z(\cdot,t),\phi_n\rangle. \tag{2.3}$$

with $\phi_n(t)$, $n\in\mathbb{Z}_+$ given in (1.2) (see e.g Christofides (2001), Karafyllis and Krstic (2018)). Differentiating $z_n(t)$ and substituting $z_t=z_{xx}+qz$ we have

$$\dot z_n(t)=\int_0^1 z_t(x,t)\phi_n(x)dx=\int_0^1 z_{xx}(x,t)\phi_n(x)dx+qz_n(t).$$

Integrating by parts twice and using the boundary conditions for z and ϕ_n we find

$$\int_0^1 z_{xx}(x,t)\phi_n(x)dx=-\lambda_n z_n(t)+\phi_n(1)u(t)$$

which leads to

$$\begin{aligned}&\dot z_n(t)=(-\lambda_n+q)z_n(t)+b_nu(t),\quad t\geq 0,\\ &b_0=1,\quad b_n=(-1)^n\sqrt{2},\ n\in\mathbb{Z}_+.\end{aligned} \tag{2.4}$$

In particular, note that

$$b_n\neq 0,\quad n\in\mathbb{Z}_+ \tag{2.5}$$

and for $N\geq 0$ the following holds:

$$\sum_{n=N+1}^\infty b_n^2\lambda_n^{-1}=\frac{2}{\pi^2}\sum_{n=N+1}^\infty\frac{1}{n^2}\leq\frac{2}{\pi^2N}. \tag{2.6}$$

Let $\delta>0$ be a desired decay rate. Since $\lim_{n\to\infty}\lambda_n=\infty$, there exists some $N_0\in\mathbb{Z}_+$ such that

$$-\lambda_n+q<-\delta,\quad n>N_0. \tag{2.7}$$

Let $N\geq N_0+1$, where N will define the dimension of the observer, whereas N_0 will be the dimension of the controller. We construct a N-dimensional observer of the form

$$\hat z(x,t):=\sum_{n=0}^N \hat z_n(t)\phi_n(x) \tag{2.8}$$

where $\hat z_n(t)$ satisfy the ODEs for $t\geq 0$

$$\begin{aligned}\dot{\hat z}_n(t)\ &=(-\lambda_n+q)\hat z_n(t)+b_nu(t)\\ &\quad -l_n\left[\left\langle\sum_{n=0}^N\hat z_n(t)\phi_n,c\right\rangle-y(t)\right],\\ \hat z_n(0)\ &=0,\quad 0\leq n\leq N.\end{aligned} \tag{2.9}$$

Here l_n, $0\leq n\leq N$ are scalars, and $l_{N_0+1}=\cdots=l_N=0$. This choice will lead to a reduced-order closed-loop system (see (2.25), (2.26)) with omitted ODEs for $\hat z_{N_0+1},\ldots,\hat z_N$ and will not deteriorate the performance of the closed-loop system. Let

$$\begin{aligned}&A_0=\operatorname{diag}\{-\lambda_i+q\}_{i=0}^{N_0},\ L_0=\operatorname{col}\{l_i\}_{i=0}^{N_0}\\ &B_0=\operatorname{col}\{b_i\}_{i=0}^{N_0},\ C_0=\left[c_0,\ldots,c_{N_0}\right],\ c_n=\langle c,\phi_n\rangle.\end{aligned} \tag{2.10}$$

Assume that

$$c_n \neq 0, \quad 0 \leq n \leq N_0. \tag{2.11}$$

By the Hautus lemma (A_0, C_0) is observable. We choose $L_0 = [l_0, \ldots, l_{N_0}]^T$ which satisfies the Lyapunov inequality:

$$P_o(A_0 - L_0C_0) + (A_0 - L_0C_0)^T P_o < -2\delta P_o \tag{2.12}$$

with $0 < P_o \in \mathbb{R}^{(N_0+1)\times(N_0+1)}$. By the Hautus lemma, (A_0, B_0) is controllable due to (2.5). Let $K_0 \in \mathbb{R}^{1\times(N_0+1)}$ satisfy the Lyapunov inequality

$$P_c(A_0 + B_0K_0) + (A_0 + B_0K_0)^T P_c < -2\delta P_c, \tag{2.13}$$

where $0 < P_c \in \mathbb{R}^{(N_0+1)\times(N_0+1)}$. We propose a controller

$$u(t) = K_0 \hat{z}^{N_0}(t), \quad \hat{z}^{N_0}(t) = \left[\hat{z}_0(t), \ldots, \hat{z}_{N_0}(t)\right]^T \tag{2.14}$$

which is based on the N-dimensional observer (2.9). Note that (2.9) implies $u(0) = 0$.

For well-posedness we introduce the change of variables $w(x,t) = z(x,t) - \frac{1}{2}x^2 u(t)$ leading to the equivalent PDE

$$\begin{aligned} &w_t(x,t) = w_{xx}(x,t) + qw(x,t) + f(x,t), \ x \in [0,1], \ t \geq 0, \\ &f(x,t) = -\tfrac{1}{2}x^2\dot{u}(t) + \left(\tfrac{q}{2}x^2 + 1\right)u(t), \\ &w_x(0,t) = 0, \quad w_x(1,t) = 0. \end{aligned} \tag{2.15}$$

Consider the operator

$$\begin{aligned} &\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \subseteq L^2(0,1) \to L^2(0,1), \quad \mathfrak{A}h = -h'', \\ &\mathcal{D}(\mathfrak{A}) = \left\{h \in H^2(0,1) | h'(0) = h'(1) = 0\right\}. \end{aligned} \tag{2.16}$$

It is well known that $\mathfrak{A}$ generates a strongly continuous semigroup on $L^2(0,1)$ (Pazy, 1983). Let $\mathbb{G} = L^2(0,1) \times \mathbb{R}^{N+1}$ be a Hilbert space with the norm $\|\cdot\|_{\mathbb{G}} = \sqrt{\|\cdot\|^2 + |\cdot|^2}$. Defining the state $\xi(t) = \mathrm{col}\left\{w(\cdot,t), \hat{z}^N(t)\right\}$, where

$$\hat{z}^N(t) = \mathrm{col}\left\{\hat{z}_0(t), \ldots, \hat{z}_N(t)\right\} \tag{2.17}$$

the closed-loop system (2.9), (2.14) and (2.15) can be presented as

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(t) + \mathrm{diag}\left\{\mathfrak{A}, \mathfrak{B}\right\}\xi(t) = \mathrm{col}\left\{f_1(\xi), f_2(\xi)\right\}$$

where

$$\begin{aligned} &\mathfrak{B}\xi_2 = \begin{bmatrix} -(A_0 + B_0K_0 - L_0C_0) & L_0C_1 \\ -B_1K_0 & -A_1 \end{bmatrix}\xi_2, \quad \xi_2 \in \mathbb{R}^{N+1}, \\ &f_1(\xi) = \begin{bmatrix} q & \upsilon & \frac{x^2}{2}K_0L_0C_1 \end{bmatrix}\xi - \frac{x^2}{2}K_0L_0\langle c, \xi_1\rangle, \\ &f_2(\xi) = \mathrm{col}\left\{L_0\langle c, \xi_1\rangle + \tfrac{1}{2}\left\langle c, x^2\right\rangle K_0\xi_2, 0\right\}, \\ &\upsilon = \left(\tfrac{q}{2}x^2 + 1\right)K_0 - \tfrac{x^2}{2}K_0(A_0 + B_0K_0 - L_0C_0) \\ &\quad + \tfrac{1}{2}\left\langle c, x^2\right\rangle L_0K_0. \end{aligned}$$

f_1 and f_2 are linear and, therefore, continuously differentiable. Let $z(\cdot,0) = w(\cdot,0) \in H^1(0,1)$. By Theorem 6.1.5 in Pazy (1983), there exists a unique classical solution

$$\xi \in C\left([0,\infty); \mathbb{G}\right) \cap C^1\left((0,\infty); \mathbb{G}\right) \tag{2.18}$$

satisfying $\xi(t) \in \mathcal{D}(\mathfrak{A}) \times \mathbb{R}^{N+1}$, $t > 0$. Applying $z(x,t) = w(x,t) + \frac{1}{2}x^2u(t)$, (2.1) and (2.9), subject to (2.14), have a unique classical solution such that $z \in C([0,\infty), L^2(0,1)) \cap C^1((0,\infty), L^2(0,1))$ and $z(\cdot,t) \in H^2(0,1)$ with $z_x(0,t) = 0$, $z_x(1,t) = u(t)$ for $t \in [0,\infty)$.

Let

$$e_n(t) = z_n(t) - \hat{z}_n(t), \ 0 \leq n \leq N \tag{2.19}$$

be the estimation error. The last term on the right-hand side of (2.9) can be written as

$$\begin{aligned} &\int_0^1 c(x)\left[\textstyle\sum_{n=1}^N \hat{z}_n(t)\phi_n(x) - \sum_{n=1}^\infty z_n(t)\phi_n(x)\right]dx \\ &= -\textstyle\sum_{n=0}^N c_ne_n(t) - \zeta(t), \ \zeta(t) = \sum_{n=N+1}^\infty c_nz_n(t). \end{aligned} \tag{2.20}$$

Then the error equations for $0 \leq n \leq N$ and $t \geq 0$ are

$$\dot{e}_n(t) = (-\lambda_n + q)e_n(t) - l_n\left(\textstyle\sum_{n=1}^N c_ne_n(t) + \zeta(t)\right). \tag{2.21}$$

Using the Young inequality, we obtain the bound

$$\zeta^2(t) \quad \leq \|c\|_N^2 \textstyle\sum_{n=N+1}^\infty z_n^2(t). \tag{2.22}$$

Denote

$$\begin{aligned} &e^{N_0}(t) = \mathrm{col}\left\{e_n(t)\right\}_{n=1}^{N_0}, \ e^{N-N_0}(t) = \mathrm{col}\left\{e_n(t)\right\}_{n=N_0+1}^{N}, \\ &\hat{z}^{N-N_0}(t) = \mathrm{col}\left\{\hat{z}_n(t)\right\}_{n=N_0+1}^N, \ \mathcal{L}_0 = \mathrm{col}\left\{L_0, -L_0\right\}, \\ &\mathcal{K}_0 = \begin{bmatrix} K_0, & 0_{1\times(N_0+1)} \end{bmatrix}, \ A_1 = \mathrm{diag}\left\{-\lambda_i + q\right\}_{i=N_0+1}^N, \\ &B_1 = \begin{bmatrix} b_{N_0+1}, \ldots, b_N \end{bmatrix}^T, \ C_1 = \begin{bmatrix} c_{N_0+1}, \ldots, c_N \end{bmatrix}, \end{aligned} \tag{2.23}$$

and

$$F_0 = \begin{bmatrix} A_0 + B_0K_0 & L_0C_0 \\ 0 & A_0 - L_0C_0 \end{bmatrix}, \ X_0(t) = \begin{bmatrix} \hat{z}^{N_0}(t) \\ e^{N_0}(t) \end{bmatrix}. \tag{2.24}$$

From (2.3), (2.9), (2.10), (2.14), (2.21), (2.23) and (2.24) we observe that $e^{N-N_0}(t)$ satisfies

$$\dot{e}^{N-N_0}(t) = A_1e^{N-N_0}(t) \tag{2.25}$$

and is exponentially decaying, whereas the *reduced-order* closed-loop system

$$\begin{aligned} &\dot{X}_0(t) = F_0X_0(t) + \mathcal{L}_0C_1e^{N-N_0}(t) + \mathcal{L}_0\zeta(t), \\ &\dot{z}_n(t) = (-\lambda_n + q)z_n(t) + b_n\mathcal{K}_0X_0(t), \ n > N. \end{aligned} \tag{2.26}$$

with $\zeta(t)$ subject to (2.22) does not depend on $\hat{z}^{N-N_0}(t)$. Moreover, $\hat{z}^{N-N_0}(t)$ satisfies

$$\dot{\hat{z}}^{N-N_0}(t) = A_1\hat{z}^{N-N_0}(t) + B_1\mathcal{K}_0X_0(t) \tag{2.27}$$

and is exponentially decaying provided $X_0(t)$ is exponentially decaying. Therefore, for stability of (2.1) under the control law (2.14) it is sufficient to show stability of the reduced-order system (2.26). The latter can be considered as a *singularly perturbed system* with the slow state $X_0(t)$ and the fast infinite-dimensional state $z_n(t)$, $n > N$.

Note that in Katz and Fridman (2020a), the full-order closed-loop system with the states $X_0, \hat{z}^{N-N_0}, e^{N-N_0}, z_n \quad (n > N)$ was considered, leading to full-order LMI conditions for stability. In the present paper we derive stability conditions for the reduced-order system (2.26) in terms of reduced-order LMI (see (2.29)) for finding N and the exponential decay rate δ. Differently from Katz and Fridman (2020a), the dimension of this LMI will not grow with N. Its feasibility for large N will follow directly from the application of Schur complements. Moreover, if this LMI is feasible for N, it will be feasible for $N+1$. To prove the exponential L^2-stability of the closed-loop system we employ the Lyapunov function

$$\begin{aligned} &V(t) = V_0(t) + p_e\left|e^{N-N_0}(t)\right|^2, \\ &V_0(t) = |X_0(t)|_{P_0}^2 + \textstyle\sum_{n=N+1}^\infty z_n^2(t) \end{aligned} \tag{2.28}$$

where $0 < P_0 \in \mathbb{R}^{(2N_0+2)\times(2N_0+2)}$ and $0 < p_e \in \mathbb{R}$. Note that $V(t)$ allows to compensate $\zeta(t)$ using (2.22), whereas V_0 corresponds to (2.26) with $e^{N-N_0} = 0$.

Theorem 2.1. *Consider* (2.1) *with measurement* (2.2) *where* $c \in L^2(0,1)$ *satisfies* (2.11) *and* $z(\cdot,0) \in L^2(0,1)$. *Let the control law* *be given by (2.14). Let $\delta > 0$ be a desired decay rate, $N_0 \in \mathbb{Z}_+$ satisfy (2.7) and $N \geq N_0 + 1$. Assume that L_0 and K_0 are obtained using (2.12) and (2.13), respectively. Let there exist $0 < P_0 \in \mathbb{R}^{(2N_0+2)\times(2N_0+2)}$ and a scalar $\alpha > 0$ such that the following LMI holds:*

$$\begin{bmatrix} \Phi_0 & P_0\mathcal{L}_0 & 0 \\ * & -2\left(\lambda_{N+1}-q-\delta\right)\|c\|_N^{-2} & 1 \\ * & * & -\frac{\alpha\|c\|_N^2}{\lambda_{N+1}} \end{bmatrix} < 0, \qquad (2.29)$$
$$\Phi_0 = P_0F_0 + F_0^TP_0 + 2\delta P_0 + \tfrac{2\alpha}{\pi^2 N}\mathcal{K}_0^T\mathcal{K}_0.$$

Then the solution $z(x,t)$ of (2.1) subject to the control law (2.14) and the corresponding observer $\hat{z}(x,t)$ given by (2.8), (2.9) satisfy the following inequalities:

$$\|z(\cdot,t)\| + \left\|z(\cdot,t)-\hat{z}(\cdot,t)\right\| \leq Me^{-\delta t}\,\|z(\cdot,0)\| \qquad (2.30)$$

for some constant $M \geq 1$. Moreover, LMI (2.29) is always feasible if N is large enough and feasibility of (2.29) for N implies its feasibility for $N+1$.

Proof. We begin by deriving LMI conditions which guarantee $\dot{V} + 2\delta V \leq 0$, thereby implying (2.30). Differentiating $V_0(t)$ along (2.26) we obtain

$$\begin{aligned}&\dot{V}_0 + 2\delta V_0 = X_0^T(t)\left[P_0F_0 + F_0^TP_0 + 2\delta P_0\right]X_0(t)\\ &+2X_0^T(t)P_0\mathcal{L}_0\zeta(t) + 2\textstyle\sum_{n=N+1}^{\infty}(-\lambda_n+q+\delta)z_n^2(t)\\ &+2\textstyle\sum_{n=N+1}^{\infty}z_n(t)b_n\mathcal{K}_0X_0(t) + 2X_0^T(t)P_0\mathcal{L}_0C_1e^{N-N_0}(t).\end{aligned} \qquad (2.31)$$

The Young inequality implies

$$\begin{aligned}&2\sum_{n=N+1}^{\infty} z_n(t)b_n\mathcal{K}_0X_0(t) = 2\sum_{n=N+1}^{\infty}\lambda_n^{\frac{1}{2}}z_n(t)\frac{b_n}{\lambda_n^{\frac{1}{2}}}\mathcal{K}_0X_0(t)\\ &\overset{(2.6)}{\leq} \frac{1}{\alpha}\sum_{n=N+1}^{\infty}\lambda_nz_n^2(t) + \frac{2\alpha}{\pi^2N}\left|\mathcal{K}_0X_0(t)\right|^2,\end{aligned} \qquad (2.32)$$

where $\alpha > 0$. From monotonicity of λ_n, $n \in \mathbb{Z}_+$ we have

$$\begin{aligned}&2\textstyle\sum_{n=N+1}^{\infty}\left(-\lambda_n+q+\delta+\frac{1}{2\alpha}\lambda_n\right)z_n^2(t)\\ &\overset{(2.22)}{\leq} 2\left(-\lambda_{N+1}+q+\delta+\tfrac{1}{2\alpha}\lambda_{N+1}\right)\|c\|_N^{-2}\,\zeta^2(t)\end{aligned} \qquad (2.33)$$

provided $-\lambda_{N+1}+q+\delta+\frac{1}{2\alpha}\lambda_{N+1} \leq 0$. Differentiating $p_e\left|e^{N-N_0}(t)\right|^2$ we have

$$\begin{aligned}&\tfrac{d}{dt}\left[p_e\left|e^{N-N_0}(t)\right|^2\right] + 2\delta p_e\left|e^{N-N_0}(t)\right|^2\\ &\quad= 2p_e\left(e^{N-N_0}(t)\right)^T(A_1+\delta I)\,e^{N-N_0}(t).\end{aligned} \qquad (2.34)$$

Let $\eta(t) = \mathrm{col}\left\{X_0(t), \zeta(t), e^{N-N_0}(t)\right\}$. From (2.31)–(2.34)

$$\dot{V} + 2\delta V \leq \eta^T(t)\Psi\eta(t) \leq 0 \qquad (2.35)$$

if

$$\begin{aligned}&\Psi = \begin{bmatrix}\Omega_1 & \Omega_2\\ * & 2p_e(A_1+\delta I)\end{bmatrix} < 0,\ \ \Omega_2 = \begin{bmatrix}P_0\mathcal{L}_0C_1\\ 0\end{bmatrix},\\ &\Omega_1 = \begin{bmatrix}\Phi_0 & P_0\mathcal{L}_0\\ * & -2\left(\lambda_{N+1}-q-\delta-\frac{1}{2\alpha}\lambda_{N+1}\right)\|c\|_N^{-2}\end{bmatrix}.\end{aligned} \qquad (2.36)$$

We now show feasibility of (2.36) for large N. Note that $A_1 + \delta I < 0$ by (2.7). By Schur complement, $\Psi < 0$ iff

$$\Omega_1 - \tfrac{1}{2p_e}P_0\mathcal{L}_0C_1\left(A_1+\delta I\right)^{-1}C_1^T\mathcal{L}_0^TP_0 < 0 \qquad (2.37)$$

Taking $p_e \to \infty$ in (2.37) (p_e does not appear in Ω_1), we find that $\Psi < 0$ iff $\Omega_1 < 0$ and the latter is equivalent, by Schur complement, to (2.29). Thus, (2.29) guarantees (2.35) implying the exponential stability of the closed-loop system (2.25)–(2.27) and (2.30).

To prove the feasibility of (2.29) for large N, choose $\alpha = 1$ and $N_1 \in \mathbb{N}$ such that for $N \geq N_1$, we have $\Phi_0 < 0$ in (2.29) for some $P_0 > 0$. This is possible since $\|\mathcal{K}_0\|$ is independent of N and F_0 is Hurwitz (see (2.12), (2.13) and (2.24)). By increasing N_1 we can also assume that for $N \geq N_1$ we have $\frac{1}{2}\lambda_{N+1} - q - \delta > 0$. By Schur complement $\Omega_1 < 0$ iff

$$\Phi_0 + \frac{\|c\|_N^2}{\lambda_{N+1}-2q-2\delta}P_0\mathcal{L}_0\mathcal{L}_0^TP_0 < 0. \qquad (2.38)$$

Since $\|\mathcal{L}_0\|$ is independent of N, $\lambda_{N+1} \overset{N\to\infty}{\longrightarrow} \infty$ and $\|c\|_N^2 \overset{N\to\infty}{\longrightarrow} 0$, by increasing N_1 if needed, (2.38) holds. Finally, note that by replacing N with $N+1$ in (2.38), the positive terms on the left-hand side decrease, whereas $P_0F_0 + F_0^TP_0 + 2\delta P_0$ is unchanged. This shows that feasibility for N implies feasibility for $N+1$. □

Remark 2.1. The reduced-order LMIs can be derived similarly for other parabolic PDEs (including heat equations with variable diffusion and reaction coefficients as in Katz and Fridman (2020a) and Kuramoto–Sivashinsky equation (KSE) as in Katz and Fridman (2020b)): for the reduced-order closed-loop system (without $\hat{z}^{N-N_0}$) the Lyapunov function of the form $V(t) = V_0(t) + p_e|e^{N-N_0}(t)|^2$ should be employed, where $p_e > 0$ is large and V_0 corresponds to the reduced-order closed-loop system with the omitted e^{N-N_0}. Then for $p_e \to \infty$ the reduced-order LMI will be obtained. Moreover, it can be shown that for the mentioned above PDEs the similar controller under Neumann actuation and non-local measurement leads to L^2 convergence without dynamic extension. This allows treating fast-varying input/output delays as presented in Section 3.

3. Delayed L^2-stabilization

We consider the delayed reaction–diffusion system

$$\begin{aligned}&z_t(x,t) = z_{xx}(x,t) + qz(x,t),\\ &z_x(0,t) = 0, \quad z_x(1,t) = u(t-\tau_u(t)),\end{aligned} \qquad (3.1)$$

under delayed Neumann actuation and delayed non-local measurement

$$y(t) = \left\langle z(\cdot, t-\tau_y(t)), c\right\rangle, \quad c \in L^2(0,1). \qquad (3.2)$$

Here $z(\cdot, t-\tau_y(t)) = z(\cdot,0)$ for $t - \tau_y(t) \leq 0$ and $\tau_y(t) \geq 0$ is a *known* continuously differentiable output delay with locally Lipschitz derivative from the interval

$$0 < \tau_m \leq \tau_y(t) \leq \tau_M. \qquad (3.3)$$

The lower bound on $\tau_y(t)$ is required for well-posedness only. The continuously differentiable input delay $\tau_u(t)$ belongs to the known interval

$$\tau_u(t) \in [r, r+\theta_M], \quad t \geq 0 \qquad (3.4)$$

where $r > 0$ and has locally Lipschitz derivative. We assume that there exist *unique* $t_y^* \in [\tau_m, \tau_M]$ and $t_u^* \in [r, r+\theta_M]$ such that $t_y^* - \tau_y(t_y^*) = t_u^* - \tau_u(t_u^*) = 0$. Henceforth the dependence of $\tau_y(t)$ and $\tau_u(t)$ on t will be suppressed.

We present the solution of (3.1) as (2.3). Then (2.4) has the form

$$\begin{aligned}&\dot{z}_n(t) = (-\lambda_n+q)z_n(t) + b_nu(t-\tau_u)\\ &b_0 = 1, \quad b_n = (-1)^n\sqrt{2}, \quad n = 0,1,\ldots.\end{aligned} \qquad (3.5)$$

Let $\delta > 0$. There exists some $N_0 \in \mathbb{Z}_+$ such that (2.7) holds. N_0 will define the dimension of the controller, whereas $N \geq N_0 + 1$ will be the dimension of the observer. To derive stability conditions in terms of the reduced-order LMIs, in Sections 3.1 and 3.2 we consider the case of known input delay and construct a

N-dimensional observer of the form (2.8), where $\hat{z}_n(t)$ satisfy the ODEs

$$\begin{aligned}\dot{\hat{z}}_n(t) &= (-\lambda_n + q)\hat{z}_n(t) + b_n u(t-\tau_u)\\ &\quad - l_n\left[\left\langle \hat{z}(\cdot, t-\tau_y), c\right\rangle - y(t)\right], \quad t \geq 0,\\ \hat{z}_n(t) &= 0, \quad t \leq 0, \qquad 0 \leq n \leq N.\end{aligned} \tag{3.6}$$

Here l_n $(0 \leq n \leq N)$ are scalars and $l_{N_0+1} = \cdots = l_N = 0$. In Section 3.3 we consider unknown τ_u, where $u(t-\tau_u)$ in the observer Eq. (3.6) is replaced by $u(t-r)$.

Recall the notations (2.10). Under the assumption (2.11), (A_0, C_0) is observable. Let $L_0 = [l_0, \ldots, l_{N_0}]^T$ satisfy the Lyapunov inequality (2.12) for some $0 < P_o \in \mathbb{R}^{(N_0+1)\times(N_0+1)}$. Similarly, (2.5) implies that (A_0, B_0) is controllable. Let $K_0 \in \mathbb{R}^{1\times(N_0+1)}$ satisfy (2.13) for some $0 < P_c \in \mathbb{R}^{(N_0+1)\times(N_0+1)}$.

Let $z(\cdot, 0) \in \mathcal{D}(\mathfrak{A})$. In Sections 3.1–3.3 well-posedness of the closed-loop systems follows arguments similar to (2.15)–(2.18), together with the step method. These arguments are standard and we refer the reader to the arXiv version of this paper (Katz, Basre, & Fridman, 2021b), where they are explicitly shown. These arguments lead to existence of a unique classical solution $z \in C([0,\infty), L^2(0,1)) \cap C^1((0,\infty)\setminus S, L^2(0,1))$, where $S = \left\{\tau_u^* + j\tau_m\right\}_{j=0}^{\infty}$. Moreover, $z(\cdot, t) \in H^2(0,1)$ with $z_x(0,t) = 0$, $z_x(1,t) = u(t-\tau_u(t))$ for $t \in [0,\infty)$. Here, the details are omitted due to space limitations.

3.1. Stabilization robust with respect to delays

We propose the control law (2.14), which is based on the N-dimensional observer (2.8), (3.6). Recall the estimation error given in (2.23). The last term on the right-hand side of (3.6) can be written as

$$\left\langle \hat{z}(\cdot, t-\tau_y), c\right\rangle - y(t) = -\sum_{n=0}^{N} c_n e_n(t-\tau_y) - \zeta(t-\tau_y) \tag{3.7}$$

with $\zeta(t)$ given in (2.20) and satisfies (2.22). Then the error equations for $t \geq 0$ and $0 \leq n \leq N_0$ are

$$\begin{aligned}\dot{e}_n(t) &= (-\lambda_n + q)e_n(t) - l_n\Big(\textstyle\sum_{n=1}^{N} c_n e_n(t-\tau_y)\\ &\quad + \zeta(t-\tau_y)\Big), \quad e_n(t) = \langle z_0, \phi_n\rangle, \quad t \leq 0.\end{aligned} \tag{3.8}$$

Recall the notations (2.10), (2.23) and (2.24) and let

$$\begin{aligned}&\mathcal{B}_0 = \mathrm{col}\left\{B_0, 0_{(N_0+1)\times 1}\right\}, \ \mathcal{C}_0 = [0_{1\times(N_0+1)}, C_0],\\ &\Upsilon_y(t) = X_0(t-\tau_y) - X_0(t), \ \Upsilon_r(t) = X_0(t-r) - X_0(t),\\ &\Upsilon_u(t) = X_0(t-\tau_u) - X_0(t-r).\end{aligned} \tag{3.9}$$

As in the non-delayed case, here $e^{N-N_0}(t) = e^{A_1 t}e(0)$ satisfies (2.25). Substituting $e^{N-N_0}(t-\tau_y) = e^{-A_1\tau_y}e^{N-N_0}(t)$, the reduced-order (i.e. decoupled from $\hat{z}^{N-N_0}(t)$) closed-loop system is governed by

$$\begin{aligned}\dot{X}_0(t) = \ & F_0 X_0(t) + \mathcal{B}_0\mathcal{K}_0\left[\Upsilon_u(t) + \Upsilon_r(t)\right] + \mathcal{L}_0\mathcal{C}_0\Upsilon_y(t)\\ &+ \mathcal{L}_0\zeta(t-\tau_y) + \mathcal{L}_0 C_1 e^{-A_1\tau_y}e^{N-N_0}(t),\end{aligned} \tag{3.10}$$

$$\begin{aligned}\dot{z}_n(t) = \ & (-\lambda_n + q)z_n(t) + b_n\mathcal{K}_0 X_0(t)\\ &+ b_n\mathcal{K}_0\left[\Upsilon_u(t) + \Upsilon_r(t)\right], \quad n > N,\end{aligned}$$

with $\zeta(t)$ subject to (2.22), where $e^{N-N_0}(t)$ is an exponentially decaying input. Note that $\hat{z}^{N-N_0}(t)$ satisfies

$$\dot{\hat{z}}^{N-N_0}(t) = A_1\hat{z}^{N-N_0}(t) + B_1\mathcal{K}_0 X_0(t-\tau_u) \tag{3.11}$$

and is exponentially decaying provided $X_0(t)$ is exponentially decaying. For L^2-stability analysis of (3.10), (2.25) we fix $\delta_0 > \delta$ and define the Lyapunov functional

$$W(t) := V(t) + \sum_{i=0}^{2} V_{S_i}(t) + \sum_{i=0}^{2} V_{R_i}(t), \tag{3.12}$$

where $V(t)$ is given by (2.28) and

$$\begin{aligned}&V_{S_0}(t) := \textstyle\int_{t-r}^{t} e^{-2\delta_0(t-s)}\left|\mathcal{K}_0 X_0(s)\right|_{S_0}^2 ds,\\ &V_{R_0}(t) := r\textstyle\int_{-r}^{0}\int_{t+\theta}^{t} e^{-2\delta_0(t-s)}\left|\mathcal{K}_0\dot{X}_0(s)\right|_{R_0}^2 ds d\theta\\ &V_{S_1}(t) := \textstyle\int_{t-r-\theta_M}^{t-r} e^{-2\delta_0(t-s)}\left|\mathcal{K}_0 X_0(s)\right|_{S_1}^2 ds,\\ &V_{R_1}(t) := \theta_M\textstyle\int_{-r-\theta_M}^{-r}\int_{t+\theta}^{t} e^{-2\delta_0(t-s)}\left|\mathcal{K}_0\dot{X}_0(s)\right|_{R_1}^2 ds d\theta\\ &V_{S_2}(t) := \textstyle\int_{t-\tau_M}^{t} e^{-2\delta_0(t-s)}\left|X_0(s)\right|_{S_2}^2 ds,\\ &V_{R_2}(t) := \tau_M\textstyle\int_{-\tau_M}^{0}\int_{t+\theta}^{t} e^{-2\delta_0(t-s)}\left|\dot{X}_0(s)\right|_{R_2}^2 ds d\theta.\end{aligned} \tag{3.13}$$

Here $S_2, R_2 > 0$ are square matrices of order $2N_0+2$ and $S_0, R_0, S_1, R_1 > 0$ are scalars. V_{S_0} and V_{R_0} are introduced to compensate $\Upsilon_r(t)$. V_{S_1} and V_{R_1} are used to compensate $\Upsilon_u(t)$. V_{S_2} and V_{R_2} are used to compensate $\Upsilon_y(t)$. Finally, to compensate $\zeta(t-\tau_y)$ we will use Halanay's inequality:

Lemma 3.1 (*Halanay's inequality*)..

Let $0 < \delta_1 < \delta_0$ and let $W : [t_0 - \tau_M, \infty) \longrightarrow [0,\infty)$ be an absolutely continuous function that satisfies

$$\dot{W}(t) + 2\delta_0 W(t) - 2\delta_1 \sup_{-\tau_M\leq\theta\leq 0} W(t+\theta) \leq 0, \quad t \geq t_0.$$

Then $W(t) \leq \exp\big(-2\delta_{\tau_M}(t-t_0)\big)\sup_{-\tau_M\leq\theta\leq 0} W(t_0+\theta)$, $t \geq t_0$, where $\delta_{\tau_M} > 0$ is a unique positive solution of

$$\delta_{\tau_M} = \delta_0 - \delta_1\exp(2\delta_{\tau_M}\tau_M). \tag{3.14}$$

To state the main result of this section, we employ the following notations for $G_1 \in \mathbb{R}$ and $G_2 \in \mathbb{R}^{2(N_0+1)\times 2(N_0+1)}$ and $0 < \alpha, \alpha_1, \alpha_2 \in \mathbb{R}$:

$$\begin{aligned}&\Psi_0 = \left[\begin{array}{c|c}\Theta & \Sigma_1 \quad \Sigma_2\\ \hline * & \mathrm{diag}\left\{\Gamma_1, \Gamma_2\right\}\end{array}\right], \quad \Theta = \begin{bmatrix}\Phi_{\mathrm{delay}} & P_0\mathcal{L}_0\\ * & -2\delta_1\|c\|_N^{-2}\end{bmatrix},\\ &\Sigma_1 = \begin{bmatrix}P_0\mathcal{L}_0\mathcal{C}_0 - 2\delta_1 P_0 - \varepsilon_M S_2 & -\varepsilon_M S_2\\ 0 & 0\end{bmatrix},\\ &\Sigma_2 = \begin{bmatrix}P_0\mathcal{B}_0 - \varepsilon_{r,M}\mathcal{K}_0^T S_1 & \Xi_1 & -\varepsilon_{r,M}\mathcal{K}_0^T S_1\\ 0 & 0 & 0\end{bmatrix},\\ &\Gamma_1 = \begin{bmatrix}-2\delta_1 P_0 - \varepsilon_M(R_2+S_2) & -\varepsilon_M(S_2+G_2)\\ & -\varepsilon_M(R_2+S_2)\end{bmatrix},\end{aligned} \tag{3.15}$$

$$\begin{aligned}&\Gamma_2 = \begin{bmatrix}-\varepsilon_{r,M}(R_1+S_1) + \frac{2\alpha_1}{\pi^2 N} & -\varepsilon_{r,M}S_1 & -\varepsilon_{r,M}(S_1+G_1)\\ * & \Xi_2 & -\varepsilon_{r,M}S_1\\ * & * & -\varepsilon_{r,M}(R_1+S_1)\end{bmatrix},\\ &\Phi_{\mathrm{delay}} = \Phi_0 + (1-\varepsilon_r)\mathcal{K}_0^T S_0\mathcal{K}_0\\ &\qquad + (\varepsilon_r - \varepsilon_{r,M})\mathcal{K}_0^T S_1\mathcal{K}_0 + (1-\varepsilon_M)S_2,\\ &\Xi_1 = P_0\mathcal{B}_0 - \varepsilon_r\mathcal{K}_0^T S_0 + (\varepsilon_r - \varepsilon_{r,M})\mathcal{K}_0^T S_1,\\ &\Xi_2 = \tfrac{2\alpha_2}{\pi^2 N} - \varepsilon_r(R_0+S_0) + (\varepsilon_r - \varepsilon_{r,M})S_1,\\ &\Lambda_0 = [F_0, \mathcal{L}_0, \mathcal{L}_0\mathcal{C}_0, 0, \mathcal{B}_0, \mathcal{B}_0, 0],\\ &\varepsilon_\tau = e^{-2\delta_0\tau}, \quad \tau \in \{r, \tau_M, r+\theta_M\}.\end{aligned}$$

Theorem 3.1. *Consider (3.1), measurement (3.2) with $c \in L^2(0,1)$ satisfying (2.11), control law (2.14). Let $\delta_0 > \delta > 0$ and $\delta_1 = \delta_0 - \delta$. Let $N_0 \in \mathbb{Z}_+$ satisfy (2.7) and $N \geq N_0 + 1$. Assume that L_0 and K_0 are obtained using (2.12) and (2.13), respectively. Given $r, \theta_M, \tau_M > 0$, let there exist positive definite matrices $P_0, S_2, R_2 \in \mathbb{R}^{2(N_0+1)\times 2(N_0+1)}$, scalars $S_0, R_0, S_1, R_1, \alpha, \alpha_1, \alpha_2 > 0$, $G_1 \in \mathbb{R}$ and*

$G_2 \in \mathbb{R}^{2(N_0+1)\times 2(N_0+1)}$ such that

$$\begin{bmatrix} R_1 & G_1 \\ * & R_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} R_2 & G_2 \\ * & R_2 \end{bmatrix} \geq 0, \\ \left[\begin{array}{c|ccc} -\lambda_{N+1}+q+\delta_0 & 1 & 1 & 1 \\ \hline * & & -\frac{2}{\lambda_{N+1}}\operatorname{diag}\{\alpha,\alpha_1,\alpha_2\} & \end{array}\right] < 0 \tag{3.16}$$

and

$$\Psi_0 + \Lambda_0^T\left[\mathcal{K}_0^T\left(r^2 R_0 + \theta_M^2 R_1\right)\mathcal{K}_0 + \tau_M^2 R_2\right]\Lambda_0 < 0 \tag{3.17}$$

hold. Then the solution $z(x,t)$ to (3.1) under the control law (2.14) and the observer $\hat{z}(x,t)$ defined by (2.8), (3.6) satisfy

$$\|z(\cdot,t)\| + \left\|z(\cdot,t)-\hat{z}(\cdot,t)\right\| \leq Me^{-\delta_{\tau_M} t}\|z(\cdot,0)\| \tag{3.18}$$

for some $M \geq 1$, where $\delta_{\tau_M} > 0$ is defined by (3.14). Moreover, LMIs (3.16), (3.17) are always feasible for large enough N and small enough τ_M, θ_M and r and their feasibility for N implies feasibility for $N+1$.

Proof. Differentiating $V(t)$ along (2.25), (3.10) we obtain

$$\begin{aligned}
&\dot{V}+2\delta V = X_0^T(t)\left[P_0F_0+F_0^TP_0+2\delta_0P_0\right]X_0(t) \\
&+2X_0^T(t)P_0\mathcal{L}_0\zeta(t-\tau_y)+2X_0^T(t)P_0\mathcal{B}_0\mathcal{K}_0\left[\Upsilon_u(t)+\Upsilon_r(t)\right] \\
&+2X_0^T(t)P_0\mathcal{L}_0\mathcal{C}_0\Upsilon_y(t)+2X_0^T(t)P_0\mathcal{L}_0C_1e^{-A\tau_y}e^{N-N_0}(t) \\
&+2\textstyle\sum_{n=N+1}^{\infty}(-\lambda_n+q+\delta_0)z_n^2(t)+2p_e\left|e^{N-N_0}(t)\right|^2_{A_1+\delta_0 I} \\
&+2\textstyle\sum_{n=N+1}^{\infty}z_n(t)b_n\mathcal{K}_0\left[X_0(t)+\Upsilon_u(t)+\Upsilon_r(t)\right].
\end{aligned} \tag{3.19}$$

By arguments similar to (2.32) we have

$$\begin{aligned}
&2\textstyle\sum_{n=N+1}^{\infty}z_n(t)b_n\mathcal{K}_0\left[X_0(t)+\Upsilon_u(t)+\Upsilon_r(t)\right] \\
&\leq\left[\frac{1}{\alpha}+\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right]\textstyle\sum_{n=N+1}^{\infty}\lambda_nz_n^2(t)+\frac{2\alpha}{\pi^2N}\left|\mathcal{K}_0X_0(t)\right|^2 \\
&+\frac{2\alpha_1}{\pi^2N}\left|\mathcal{K}_0\Upsilon_u(t)\right|^2+\frac{2\alpha_2}{\pi^2N}\left|\mathcal{K}_0\Upsilon_r(t)\right|^2.
\end{aligned} \tag{3.20}$$

Differentiation of V_{S_0} and V_{R_0} leads to

$$\begin{aligned}
\dot{V}_{S_0}+2\delta_0V_{S_0} &= \left|\mathcal{K}_0X_0(t)\right|^2_{S_0} \\
&\quad -\varepsilon_r\left|\mathcal{K}_0X_0(t)+\mathcal{K}_0\Upsilon_r(t)\right|^2_{S_0}, \\
\dot{V}_{R_0}+2\delta_0V_{R_0} &= r^2\left|\mathcal{K}_0\dot{X}_0(t)\right|^2_{R_0} \\
&\quad -r\int_{t-r}^{t}e^{-2\delta_0(t-s)}\left|\mathcal{K}_0\dot{X}_0(s)\right|^2_{R_0}ds.
\end{aligned} \tag{3.21}$$

By using Jensen's inequality we have

$$-r\int_{t-r}^{t}e^{-2\delta_0(t-s)}\left|\mathcal{K}_0\dot{X}_0(s)\right|^2_{R_0}ds \leq -\varepsilon_r\left|\mathcal{K}_0\Upsilon_r(t)\right|^2_{R_0}.$$

Let

$$\begin{aligned}
Q_u(t) &= X_0(t-r-\theta_M)-X_0(t-\tau_u), \\
Q_y(t) &= X_0(t-\tau_M)-X_0(t-\tau_y).
\end{aligned} \tag{3.22}$$

Differentiation of V_{S_i} and V_{R_i}, $i\in\{1,2\}$, gives:

$$\begin{aligned}
&\dot{V}_{S_1}+2\delta_0V_{S_1} = \varepsilon_r\left|\mathcal{K}_0\Upsilon_r(t)+\mathcal{K}_0X_0(t)\right|^2_{S_1} \\
&-\varepsilon_{r+\theta_M}\left|\mathcal{K}_0\left(Q_u(t)+\Upsilon_u(t)+\Upsilon_r(t)+X_0(t)\right)\right|^2_{S_1}, \\
&\dot{V}_{S_2}+2\delta_0V_{S_2} = \left|X_0(t)\right|^2_{S_2} \\
&-\varepsilon_{\tau_M}\left|Q_y(t)+\Upsilon_y(t)+X_0(t)\right|^2_{S_2}, \\
&\dot{V}_{R_1}+2\delta_0V_{R_1} = \theta_M^2\left|\mathcal{K}_0\dot{X}_0(t)\right|^2_{R_1} \\
&\qquad -\theta_M\int_{t-r-\theta_M}^{t-r}e^{-2\delta_0(t-s)}\left|\mathcal{K}_0\dot{X}_0(s)\right|^2_{R_1}ds, \\
&\dot{V}_{R_2}+2\delta_0V_{R_2} = \tau_M^2\left|\dot{X}_0(t)\right|^2_{R_2} \\
&\qquad -\tau_M\int_{t-\tau_M}^{t}e^{-2\delta_0(t-s)}\left|\dot{X}_0(s)\right|^2_{R_2}ds.
\end{aligned}$$

By Jensen's and Park's inequalities (see Fridman (2014)) to obtain

$$\begin{aligned}
&-\theta_M\int_{t-r-\theta_M}^{t-r}e^{-2\delta_0(t-s)}\left|\mathcal{K}_0\dot{X}_0(s)\right|^2_{R_1}ds \\
&\leq -\varepsilon_{r+\theta_M}\begin{bmatrix}\mathcal{K}_0\Upsilon_u(t)\\ \mathcal{K}_0Q_u(t)\end{bmatrix}^T\begin{bmatrix}R_1 & G_1\\ * & R_1\end{bmatrix}\begin{bmatrix}\mathcal{K}_0\Upsilon_u(t)\\ \mathcal{K}_0Q_u(t)\end{bmatrix}, \\
&-\tau_M\int_{t-\tau_M}^{t}e^{-2\delta_0(t-s)}\left|\dot{X}_0(s)\right|^2_{R_2}ds \\
&\leq -\varepsilon_{\tau_M}\begin{bmatrix}\Upsilon_y(t)\\ Q_y(t)\end{bmatrix}^T\begin{bmatrix}R_2 & G_2\\ * & R_2\end{bmatrix}\begin{bmatrix}\Upsilon_y(t)\\ Q_y(t)\end{bmatrix}.
\end{aligned}$$

To compensate $\zeta(t-\tau_y)$ we will use Halanay's inequality. Note that

$$\begin{aligned}
&-2\delta_1\sup\nolimits_{-\tau_M\leq\theta\leq 0}W(t+\theta) \leq -2\delta_1V(t-\tau_y(t)) \\
&\overset{(2.22)}{\leq} -2\delta_1\left[\Upsilon_y(t)+X_0(t)\right]^TP_0\left[\Upsilon_y(t)+X_0(t)\right] \\
&-2\delta_1\|c\|_N^{-2}\zeta^2(t-\tau_y)-2\delta_1p_e\left|e^{N-N_0}(t)\right|^2_{e^{-2A_1\tau_y}}
\end{aligned} \tag{3.23}$$

where $\delta_0 = \delta_1+\delta$. Let $\eta(t) = \operatorname{col}\left\{X_0(t), \zeta(t-\tau_y), \Upsilon_y(t), Q_y(t), \mathcal{K}_0\Upsilon_u(t), \mathcal{K}_0\Upsilon_r(t), \mathcal{K}_0Q_u(t), e^{N-N_0}(t)\right\}$. Then due to (3.19)–(3.23) Halanay's inequality

$$\begin{aligned}
&\dot{W}(t)+2\delta_0W(t)-2\delta_1\sup\nolimits_{-\tau_M\leq\theta\leq 0}W(t+\theta) \\
&\leq \eta^T(t)\Psi_1\eta(t)+2\textstyle\sum_{n=N+1}^{\infty}\varpi_nz_n^2(t)\leq 0,\ t\geq 0
\end{aligned}$$

holds if

$$\begin{aligned}
&\varpi_n = -\lambda_n+q+\delta_0+\left[\frac{1}{2\alpha}+\frac{1}{2\alpha_1}+\frac{1}{2\alpha_2}\right]\lambda_n<0,\ n>N, \\
&\Psi_1 = \Psi_{\text{full}}+\Lambda^T\left[\mathcal{K}_0^T\left(r^2R_0+\theta_M^2R_1\right)\mathcal{K}_0+\tau_M^2R_2\right]\Lambda<0.
\end{aligned}$$

Here

$$\Lambda = \left[\Lambda_0, \mathcal{L}_0C_1e^{-A_1\tau_y}\right],\ \ \Gamma_3 = 2p_e(A_1+\delta_0I-\delta_1e^{-2A_1\tau_y}),$$

$$\Psi_{\text{full}} = \left[\begin{array}{c|c}\Psi_0 & \Sigma_3\\ \hline * & \Gamma_3\end{array}\right],\quad \Sigma_3 = \begin{bmatrix}P_0\mathcal{L}_0C_1e^{-A_1\tau_y}\\ 0\end{bmatrix}. \tag{3.24}$$

Monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$ and Schur's complement imply that $\varpi_n < 0$, $n > N$ iff the second LMI in (3.16) holds. We have $\Gamma_3 = 2p_e(A_1+\delta_0I-\delta_1e^{-2A_1\tau_y}) < 0$ due to (2.7). Therefore, by Schur complement for $p_e\to\infty$ we obtain that $\Psi_1<0$ iff (3.17) holds. Hence, feasibility of (3.16), (3.17) and Lemma 3.1 lead to $W(t) \leq \exp\left(-2\delta_{\tau_M}t\right)\sup_{-\tau_M\leq\theta\leq 0}W(\theta)$ for $t\geq 0$. The latter implies (3.18). Finally, note that (3.16) and (3.17) are *reduced-order* LMIs whose dimension is independent of N. By arguments similar to Theorem 3.1 in Katz and Fridman (2021a) it can be shown that (3.16) and (3.17) are feasible for large enough N and small enough τ_M, θ_M, r. Moreover, by Schur complements, the LMIs feasibility for N implies their feasibility for $N+1$. □

3.2. Predictor-based L^2-stabilization: known input delay

In this section we compensate the constant and known part r of τ_u subject to (3.4) by using a classical predictor (Artstein, 1982; Selivanov & Fridman, 2016). Recall the observer (2.8) which satisfies (3.6). Using the notations (2.10),(2.17), (2.23) and (2.24) we obtain

$$\begin{aligned}
\dot{\hat{z}}^{N_0}(t) &= A_0\hat{z}^{N_0}(t)+B_0u(t-\tau_u)+L_0C_0e^{N_0}(t-\tau_y) \\
&\quad +L_0C_1e^{N-N_0}(t-\tau_y)+L_0\zeta(t-\tau_y),\quad t\geq 0.
\end{aligned} \tag{3.25}$$

We propose the following predictor-based control law

$$\bar{z}(t) = e^{A_0r}\hat{z}^{N_0}(t)+\int_{t-r}^{t}e^{A_0(t-s)}B_0u(s)ds,\ \ u(t) = K_0\bar{z}(t) \tag{3.26}$$

Differentiating $\bar{z}(t)$ and using (3.25) we obtain

$$\begin{aligned}\dot{\bar{z}}(t) &= A_0\bar{z}(t) + B_0u(t)\\ &+ e^{A_0r}B_0\left[u(t-\tau_u) - u(t-r)\right] + e^{A_0r}L_0\\ &\times\left[C_0e^{N_0}(t-\tau_y) + C_1e^{N-N_0}(t-\tau_y) + \zeta(t-\tau_y)\right].\end{aligned}$$

We present the reduced-order closed-loop system as

$$\begin{aligned}\dot{\bar{X}}(t) = \;& \bar{F}_0\bar{X}(t) + \bar{\mathcal{B}}_0\mathcal{K}_0\bar{\Upsilon}_u(t) + \bar{\mathcal{L}}_0\mathcal{C}_0\bar{\Upsilon}_y(t)\\ & + \bar{\mathcal{L}}_0\zeta(t-\tau_y) + \bar{\mathcal{L}}_0C_1e^{-A_1\tau_y}e^{N-N_0}(t),\\ \dot{z}_n(t) = \;& (-\lambda_n + q)z_n(t) + b_n\mathcal{K}_0\bar{X}(t)\\ & + b_n\mathcal{K}_0[\bar{\Upsilon}_u(t) + \bar{\Upsilon}_r(t)], \quad n > N, \quad t \geq 0,\end{aligned} \tag{3.27}$$

where

$$\bar{X}(t) = \mathrm{col}\{\bar{z}(t), e^{N_0}(t)\}, \;\; \bar{\Upsilon}_y(t) = \bar{X}(t-\tau_y) - \bar{X}(t),$$

$$\begin{aligned}&\bar{\Upsilon}_u(t) = \bar{X}(t-\tau_u) - \bar{X}(t-r), \;\; \mathcal{C}_0 = [0_{1\times(N_0+1)}, C_0],\\ &\bar{\Upsilon}_r(t) = \bar{X}(t-r) - \bar{X}(t), \;\; \bar{\mathcal{L}}_0 = \mathrm{col}\left\{e^{A_0r}L_0, -L_0\right\},\\ &\bar{Q}_u(t) = \bar{X}(t-r-\theta_M) - \bar{X}(t-\tau_u),\\ &\bar{Q}_y(t) = \bar{X}(t-\tau_M) - \bar{X}(t-\tau_y),\\ &\bar{F}_0 = \begin{bmatrix} A_0 + B_0K_0 & e^{A_0r}L_0C_0\\ 0 & A_0 - L_0C_0\end{bmatrix}, \;\; \bar{\mathcal{B}}_0 = \mathrm{col}\left\{e^{A_0r}B_0, 0\right\}.\end{aligned} \tag{3.28}$$

As in the non-delayed case, here $e^{N-N_0}(t)$ satisfies (2.25) and is exponentially decaying, whereas $\zeta(t)$ satisfies (2.22). From (3.26) we have that exponential decay of $\bar{X}(t)$ implies exponential decay of $X_0(t)$ in (2.24).

For L^2-stability analysis of (3.27), (2.25) we fix $\delta_0 > \delta$ and define the Lyapunov functional (3.12). Here $V(t)$ and $V_{S_i}, V_{R_i}, \; i \in \{0,1,2\}$ are given by (2.28) and (3.13), respectively, with X_0 replaced by $\bar{X}$. To state the main result of this section, let $G_1 \in \mathbb{R}$ and $G_2 \in \mathbb{R}^{2(N_0+1)\times 2(N_0+1)}$ and $0 < \alpha, \alpha_1, \alpha_2 \in \mathbb{R}$. We introduce

$$\begin{aligned}&\bar{\Psi}_0 = \left[\begin{array}{c|c}\bar{\Theta} & \bar{\Sigma}_1 \;\; \bar{\Sigma}_2\\ \hline * & \mathrm{diag}\{\Gamma_1, \Gamma_2\}\end{array}\right], \;\; \bar{\Theta} = \begin{bmatrix}\bar{\Phi} & P_0\bar{\mathcal{L}}_0\\ * & -2\delta_1\|c\|_N^{-2}\end{bmatrix},\\ &\bar{\Sigma}_1 = \begin{bmatrix} P_0\bar{\mathcal{L}}_0\mathcal{C}_0 - 2\delta_1P_0 - \varepsilon_MS_2 & -\varepsilon_MS_2\\ 0 & 0\end{bmatrix},\\ &\bar{\Sigma}_2 = \begin{bmatrix} P_0\bar{\mathcal{B}}_0 - \varepsilon_{r,M}\mathcal{K}_0^TS_1 & \bar{\Xi}_1 & -\varepsilon_{r,M}\mathcal{K}_0^TS_1\\ 0 & 0 & 0\end{bmatrix},\\ &\bar{\Phi} = P_0\bar{F}_0 + \bar{F}_0^TP_0 + 2\delta P_0 + (1-\varepsilon_r)\mathcal{K}_0^TS_0\mathcal{K}_0\\ &\quad + \tfrac{2\alpha}{\pi^2N}\mathcal{K}_0^T\mathcal{K}_0 + (\varepsilon_r - \varepsilon_{r,M})\mathcal{K}_0^TS_1\mathcal{K}_0 + (1-\varepsilon_M)S_2,\\ &\bar{\Xi}_1 = -\varepsilon_r\mathcal{K}_0^TS_0 + (\varepsilon_r - \varepsilon_{r,M})\mathcal{K}_0^TS_1,\\ &\bar{\Lambda}_0 = [\bar{F}_0, \bar{\mathcal{L}}_0, \bar{\mathcal{L}}_0\mathcal{C}_0, 0, \bar{\mathcal{B}}_0, 0, 0]\end{aligned}$$

where $\Gamma_i, \; i \in \{1,2,3\}$ and $\varepsilon_\tau, \; \tau \in \{r, \tau_M, r+\theta_M\}$ are given in (3.15), (3.24).

Theorem 3.2. *Consider (3.1), measurement (3.2) with $c \in L^2(0,1)$ satisfying (2.11), control law (3.26). Let $\delta_0 > \delta > 0$ and $\delta_1 = \delta_0 - \delta$. Let $N_0 \in \mathbb{Z}_+$ satisfy (2.7) and $N \geq N_0 + 1$. Assume that L_0 and K_0 are subject to (2.12) and (2.13), respectively. Given $r, \theta_M, \tau_M > 0$, let there exist positive definite matrices $P_0, S_2, R_2 \in \mathbb{R}^{2(N_0+1)\times 2(N_0+1)}$, scalars $S_0, R_0, S_1, R_1, \alpha, \alpha_1, \alpha_2 > 0$, $G_1 \in \mathbb{R}$ and $G_2 \in \mathbb{R}^{2(N_0+1)\times 2(N_0+1)}$ such that (3.16) and*

$$\bar{\Psi}_0 + \bar{\Lambda}_0^T\left[\mathcal{K}_0^T\left(r^2R_0 + \theta_M^2R_1\right)\mathcal{K}_0 + \tau_M^2R_2\right]\bar{\Lambda}_0 < 0. \tag{3.29}$$

hold. Then the solution $z(x,t)$ to (3.1) under the control law (3.26) and the corresponding observer $\hat{z}(x,t)$ defined by (2.8), (3.6) satisfy (3.18) for some $M > 0$ and $\delta_{\tau_M} > 0$ defined by (3.14). LMIs (3.16) and (3.29) are feasible if N is large enough and τ_M, θ_M, r are small enough. Feasibility of (3.16) and (3.29) for N implies their feasibility for $N+1$.

Proof. The proof is essentially identical to proof of Theorem 3.1. Hence, we only state the differences. Let $\eta(t) = \mathrm{col}\left\{\bar{X}(t), \zeta(t-\tau_y), \bar{\Upsilon}_y(t), \bar{Q}_y(t), \mathcal{K}_0\bar{\Upsilon}_u(t), \;\; \mathcal{K}_0\bar{\Upsilon}_r(t), \mathcal{K}_0\bar{Q}_u(t), e^{N-N_0}(t)\right\}$. By arguments similar to (3.19)–(3.23) we obtain

$$\begin{aligned}&\dot{W}(t) + 2\delta_0W(t) - 2\delta_1\sup_{-\tau_M\leq\theta\leq 0}W(t+\theta)\\ &\leq \eta^T(t)\Psi_2\eta(t) + 2\textstyle\sum_{n=N+1}^{\infty}\varpi_nz_n^2(t) \leq 0, \;\; t \geq 0,\end{aligned} \tag{3.30}$$

if

$$\begin{aligned}&\varpi_n = -\lambda_n + q + \delta_0 + \left[\tfrac{1}{2\alpha} + \tfrac{1}{2\alpha_1} + \tfrac{1}{2\alpha_2}\right]\lambda_n < 0, \;\; n > N,\\ &\Psi_2 = \bar{\Psi} + \bar{\Lambda}^T\left[\mathcal{K}_0^T\left(r^2R_0 + \theta_M^2R_1\right)\mathcal{K}_0 + \tau_M^2R_2\right]\bar{\Lambda} < 0.\end{aligned} \tag{3.31}$$

Here $\bar{\Lambda} = [\bar{\Lambda}_0, \bar{\mathcal{L}}_0C_1e^{-A_1\tau_y}]$ and

$$\bar{\Psi} = \left[\begin{array}{c|c}\bar{\Psi}_0 & \Sigma_3\\ \hline * & \Gamma_3\end{array}\right], \;\; \Sigma_3 = \begin{bmatrix}P_0\bar{\mathcal{L}}_0C_1e^{-A_1\tau_y}\\ 0\end{bmatrix}. \tag{3.32}$$

Monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$ and Schur's complement imply that $\varpi_n < 0, \; n > N$ iff the second LMI in (3.16) holds. Finally, note that (2.7) implies $\Gamma_3 < 0$. By Schur complement and $p_e \to \infty$, $\Psi_2 < 0$ iff (3.29) holds. Note that (3.16) and (3.29) are again of reduced-order (i.e., the dimension is independent of N). □

3.3. Predictor-based L^2-stabilization: unknown input delay

In this section we assume an input delay $\tau_u(t) = r + \theta(t)$ with a known constant part $r > 0$ and unknown $\theta(t) \in [0, \theta_M]$. Since $\theta(t)$ is unknown, the observer (2.8) is designed to satisfy (3.6) with $u(t-\tau_u)$ *replaced* by $u(t-r)$. Therefore, (3.25) is modified as follows:

$$\begin{aligned}\dot{\hat{z}}^{N_0}(t) = \;& A_0\hat{z}^{N_0}(t) + B_0u(t-r) + L_0C_0e^{N_0}(t-\tau_y)\\ & + L_0C_1e^{N-N_0}(t-\tau_y) + L_0\zeta(t-\tau_y)\end{aligned} \tag{3.33}$$

whereas $\hat{z}^{N-N_0}(t)$ satisfies

$$\dot{\hat{z}}^{N-N_0}(t) = A_1\hat{z}^{N-N_0}(t) + B_1u(t-r). \tag{3.34}$$

Furthermore, the estimation error satisfies

$$\begin{aligned}&\dot{e}^{N_0}(t) = A_0e^{N_0}(t) + B_0[u(t-\tau_u) - u(t-r)]\\ &\quad - L_0[C_0e^{N_0}(t-\tau_y) + C_1e^{N-N_0}(t-\tau_y) + \zeta(t-\tau_y)],\\ &\dot{e}^{N-N_0}(t) = A_1e^{N-N_0}(t) + B_1[u(t-\tau_u) - u(t-r)].\end{aligned} \tag{3.35}$$

As in Katz and Fridman (2021a), uncertainty in τ_u leads to *coupling* of $e^{N-N_0}(t)$ with $u(t)$. We propose the predictor-based control law (3.26). Differentiating $\bar{z}(t)$ and using (3.33) we obtain

$$\begin{aligned}&\dot{\bar{z}}(t) = (A_0 + B_0K_0)\bar{z}(t) + e^{A_0r}L_0\\ &\times\left[C_0e^{N_0}(t-\tau_y) + C_1e^{N-N_0}(t-\tau_y) + \zeta(t-\tau_y)\right]\end{aligned} \tag{3.36}$$

Differently from the case of a known τ_u, we introduce

$$\bar{X}(t) = \mathrm{col}\{\bar{z}(t), e^{N_0}(t), e^{N-N_0}(t)\} \tag{3.37}$$

as the closed-loop state, which includes $e^{N-N_0}(t)$. Note that differently from Katz and Fridman (2021a), $\hat{z}^{N-N_0}(t)$ is not a part of $\bar{X}(t)$. Therefore, for a given N, the LMIs subsequently obtained will not be of reduced-order, but are of essentially smaller dimension than in Katz and Fridman (2021a). Recall $\bar{Q}_u(t), \bar{Q}_y(t), \bar{\Upsilon}_u(t), \bar{\Upsilon}_y(t)$

and $\bar{\Upsilon}_r(t)$ given in (3.28) and let

$$\bar{F}=\begin{bmatrix} A_0+B_0K_0 & e^{A_0r}L_0C_0 & e^{A_0r}L_0C_1 \\ 0 & A_0-L_0C_0 & -L_0C_1 \\ 0 & 0 & A_1 \end{bmatrix},$$
$$\bar{\mathcal{L}}=\mathrm{col}\left\{e^{A_0r}L_0,-L_0,0\right\},\ \mathcal{C}=[0_{1\times(N_0+1)},\ C_0,\ C_1],$$
$$\bar{\mathcal{B}}=\mathrm{col}\left\{0_{(N_0+1)\times 1},B_0,B_1\right\},\ \mathcal{K}_0=[K_0,\ 0,\ 0].$$

The closed-loop system is governed by

$$\begin{aligned}\dot{\bar{X}}(t)=&\ \bar{F}\bar{X}(t)+\bar{\mathcal{B}}\mathcal{K}_0\bar{\Upsilon}_u(t)+\bar{\mathcal{L}}\mathcal{C}\bar{\Upsilon}_y(t)+\bar{\mathcal{L}}\zeta(t-\tau_y),\\ \dot{z}_n(t)=&\ (-\lambda_n+q)z_n(t)+b_n\mathcal{K}_0\bar{X}(t)\\ &+b_n\mathcal{K}_0[\bar{\Upsilon}_u(t)+\bar{\Upsilon}_r(t)],\quad n>N,\quad t\geq 0\end{aligned}\tag{3.38}$$

where $\zeta(t)$ satisfies (2.22). From (3.34) follows that $\hat{z}^{N-N_0}(t)$ is exponentially decaying if the closed-loop system (3.38) is exponentially decaying.

For L^2-stability of the closed-loop system (3.38) let $\delta_0>\delta$ and define the Lyapunov functional (3.12) with $V(t)$ replaced by $V_0(t)$, given in (2.28), $V_{S_i}, V_{R_i},\ i\in\{0,1,2\}$ given in (3.13) and $X_0(t)$ is replaced by $\bar{X}(t)$ everywhere. To state the main result of this section, let $G_1\in\mathbb{R}$ and $G_2\in\mathbb{R}^{(N+N_0+2)\times(N+N_0+2)}$ and $0<\alpha,\alpha_1,\alpha_2\in\mathbb{R}$. Let

$$\bar{\Psi}_1=\left[\begin{array}{c|c}\bar{\Psi}_2 & \Sigma_4\ \ \Sigma_5\\ \hline * & \mathrm{diag}\left\{\Gamma_1,\Gamma_2\right\}\end{array}\right],\ \bar{\Psi}_2=\begin{bmatrix}\bar{\Phi}_1 & P_0\bar{\mathcal{L}}\\ * & -2\delta_1\left\|c\right\|_N^{-2}\end{bmatrix}$$
$$\Sigma_4=\begin{bmatrix}P_0\bar{\mathcal{L}}\mathcal{C}-2\delta_1P_0-\varepsilon_MS_2 & -\varepsilon_MS_2\\ 0 & 0\end{bmatrix},$$
$$\Sigma_5=\begin{bmatrix}P_0\bar{\mathcal{B}}-\varepsilon_{r,M}\mathcal{K}_0^TS_1 & \bar{\Xi}_1 & -\varepsilon_{r,M}\mathcal{K}_0^TS_1\\ 0 & 0 & 0\end{bmatrix},$$
$$\bar{\Lambda}_1=[\bar{F},\bar{\mathcal{L}},\bar{\mathcal{L}}\mathcal{C},0,\bar{\mathcal{B}},0,0]$$
$$\begin{aligned}\bar{\Phi}_1=&\ P_0\bar{F}+\bar{F}^TP_0+2\delta P_0+(1-\varepsilon_r)\mathcal{K}_0^TS_0\mathcal{K}_0\\ &+\tfrac{2\alpha}{\pi^2N}\mathcal{K}_0^T\mathcal{K}_0+(\varepsilon_r-\varepsilon_{r,M})\mathcal{K}_0^TS_1\mathcal{K}_0+(1-\varepsilon_M)S_2,\end{aligned}$$

with Γ_1,Γ_2 given in (3.15).

Theorem 3.3. *Consider (3.1) with unknown input delay $\tau_u(t)$, measurement (3.2) with $c\in L^2(0,1)$ satisfying (2.11), control law (3.26). Let $\delta_0>\delta>0$ and $\delta_1=\delta_0-\delta$. Let $N_0\in\mathbb{Z}_+$ satisfy (2.7) and $N\geq N_0+1$. Let L_0 and K_0 satisfy (2.12) and (2.13), respectively. Given $r,\theta_M,\tau_M>0$, let there exist positive definite matrices $P_0,S_2,R_2\in\mathbb{R}^{(N+N_0+2)\times(N+N_0+2)}$, scalars $S_0,R_0,S_1,R_1,\alpha,\alpha_1,\alpha_2>0$, $G_1\in\mathbb{R}$ and $G_2\in\mathbb{R}^{(N+N_0+2)\times(N+N_0+2)}$ such that (3.16) and*

$$\bar{\Psi}_1+\bar{\Lambda}_1^T\left[\mathcal{K}_0^T\left(r^2R_0+\theta_M^2R_1\right)\mathcal{K}_0+\tau_M^2R_2\right]\bar{\Lambda}_1<0\tag{3.39}$$

hold. Then the solution $z(x,t)$ to (3.1) under the control law (3.26) and the observer $\hat{z}(x,t)$ defined by (2.8), (3.33) and (3.34) satisfy (3.18) for some $M>0$ and $\delta_{\tau_M}>0$ defined by (3.14). The LMIs (3.16) and (3.39) are always feasible if N is large enough and τ_M,θ_M,r are small enough.

Proof. The proof is essentially identical to proof of Theorem 3.1. Hence, we only state the differences. Let $\eta(t)=\mathrm{col}\left\{\bar{X}(t),\zeta(t-\tau_y),\bar{\Upsilon}_y(t),\bar{\mu}_y(t),\mathcal{K}_0\bar{\Upsilon}_u(t),\mathcal{K}_0\bar{\Upsilon}_r(t),\mathcal{K}_0\bar{Q}_u(t)\right\}$ Similar to (3.19)–(3.23) we obtain

$$\begin{aligned}&\dot{W}(t)+2\delta_0W(t)-2\delta_1\sup_{-\tau_M\leq\theta\leq 0}W(t+\theta)\\ &\leq\eta^T(t)\Psi_3\eta(t)+2\sum_{n=N+1}^{\infty}\varpi_nz_n^2(t)\leq 0,\ t\geq 0,\end{aligned}$$

if

$$\begin{aligned}&\varpi_n=-\lambda_n+q+\delta_0+\left[\tfrac{1}{2\alpha}+\tfrac{1}{2\alpha_1}+\tfrac{1}{2\alpha_2}\right]\lambda_n<0,\ n>N,\\ &\Psi_3=\bar{\Psi}_1+\bar{\Lambda}_1^T\left[\mathcal{K}_0^T\left(r^2R_0+\theta_M^2R_1\right)\mathcal{K}_0+\tau_M^2R_2\right]\bar{\Lambda}_1<0.\end{aligned}\tag{3.40}$$

Schur's complement imply that $\varpi_n<0,\ n>N$ iff the second LMI in (3.16) holds, whereas $\Psi_3<0$ iff (3.39). □

Table 1
Minimal N that guarantees decay rate δ: non-delayed case.

δ	0.1	1	2	5	7.5
N	3	4	4	4	5
K_0	−5	−5	−7	−13	−18
L_0	5.5	8.33	11.67	21.6	29.8

Table 2
Minimal N for the stability: given r and $\tau_M=\theta_M=10^{-7}$.

r	0.06	0.1	0.14	0.18	0.26	0.3
Theorem 3.1	6	6	14	–	–	–
Theorems 3.2, 3.3	6	6	6	8	12	16

4. Example: temperature control in a rod

Consider control of heat flow in the rod with constant thermal conductivity, mass density, specific heat and reaction coefficient (Christofides, 2001; Curtain & Morris, 2009). The control action effects the heat flow at one end, while keeping the heat flow in the other end fixed. The model of spatiotemporal evolution of the dimensionless rod temperature (denoted by $z(x,t)$) is given by (2.1), where q is the reaction coefficient. We consider $q=3$, which results in an unstable open-loop system. The measurement of the distributed rod temperature is given by (2.2), where $c(x)=\chi_{[0.3,0.9]}(x)$ (i.e., an indicator function of $[0.3,0.9]$). We aim to stabilize the rod temperature at the unstable steady state $z(x,t)=0$.

The observer and controller gains are found from (2.12) and (2.13). For non-delayed stabilization we consider $\delta\in\{0.1,1,2,5\}$ which result in $N_0=0$. For each δ we compute the corresponding gains and find the minimum value of N such that the LMI of Theorem 2.1 holds (see Table 1).

For delayed stabilization we choose $\delta=0$, which results in $N_0=0$. The controller and observer gains are given by

$$K_0=-5.5,\quad L_0=5.5.\tag{4.1}$$

We verify the feasibility of LMIs of Theorems 3.1 (no predictor), 3.2 (predictor, known τ_u) and 3.3 (predictor, unknown τ_u) for $\delta_0=\delta_1$. Since the corresponding LMIs are strict, feasibility with $\delta=0$ implies their feasibility for small enough $\delta_*>0$. In the first test we fix $\tau_M=\theta_M=10^{-7}$ and find the minimal value of N which guarantees the feasibility of the LMIs for increasing values of r. The results are given in Table 2. It is seen that predictor allows to increase the maximal value of r from 0.14 till 0.3. The maximum value of r, with corresponding N, for which the LMIs of Theorems 3.1, 3.2 and 3.3 were found feasible are $r=0.16$ ($N=18$), $r=0.44$ ($N=24$) and $r=0.41$ ($N=26$), respectively.

In the second test we fix $\tau_M=\theta_M$ and find the maximum value of r and the corresponding minimal value of N for which LMIs are feasible. The results are given in Table 3. It is seen that for $\theta_M=\tau_M=0.01$ the LMIs of Theorems 3.2 and 3.3 allow for larger r than in Theorem 3.1. For $\theta_M=\tau_M=0.04$ the same comparison holds only for Theorems 3.1 and 3.2, whereas no feasibility was obtained in Theorem 3.3 due to higher-dimensional LMIs for $N=30$.

Our reduced-order LMIs are feasible for larger values of N than in Katz and Fridman (2021a) (where for $N>9$ we could not verify LMIs) due to a significantly lower computational complexity. A larger N allows larger delays in example. For additional LMI simulations with different gains see Katz et al. (2021a).

For simulations of the solutions to the closed-loop systems we choose observer and controller gains given by (4.1). We fix

Table 3
Maximal r and minimal N that guarantee the stability.

$\tau_M = \theta_M$	0.01	0.04
Theorem 3.1	$r = 0.14$, $N = 30$	$r = 0.12$, $N = 30$
Theorem 3.2	$r = 0.3$, $N = 30$	$r = 0.25$, $N = 30$
Theorem 3.3	$r = 0.25$, $N = 22$	–

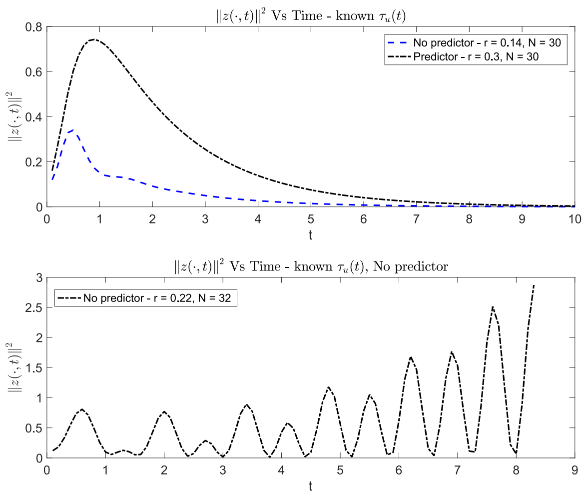

Fig. 1. Simulation results for known τ_u. Top: stability confirming the LMI results. Bottom: instability without predictor.

$\tau_M = \theta_M = 0.01$ and choose the *known* delays $\tau_u(t) = r + 0.01\sin^2(120t)$ and $\tau_y(t) = 0.01\cos^2(120t)$. Note that $\dot{\tau}_y < 1$ and $\dot{\tau}_u < 1$ *does not* hold. We choose $r_{\max}$ and N given in the first column and the first two lines of Table 3. For the initial condition $z(x,0) = 10x^2(1-x)^2$ we do simulations of the closed-loop systems (3.10) (without predictor) and (3.27) (with predictor) and the ODEs satisfied by $\hat{z}^{N-N_0}(t)$. In both cases, we simulate the ODEs of $z_n(t)$ for $N+1 \le n \le 50$. The value of $\zeta(t)$, given by (2.20), is approximated by $\zeta(t) \approx \sum_{n=N+1}^{50} c_n z_n(t)$. Results of the simulations are given at the top of Fig. 1 and confirm our theoretical results. Moreover, a simulation for $r = 0.22$ and $N = 30$ without predictor shows instability (see the bottom of Fig. 1). The use of predictor allows to stabilize for a larger $r = 0.3$ with $N = 30$.

5. Conclusion

We suggested a finite-dimensional observer-based control of the 1D heat equation under Neumann actuation, non-local measurement and fast-varying input/output delays. Reduced-order LMI stability conditions were derived. Classical predictors were used to enlarge the delays.

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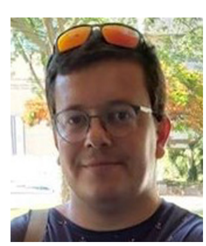

Rami Katz received a B.Sc. degree (summa cum laude) in 2014 and a M.Sc. degree (summa cum laude) in 2016, both in Mathematics, from Tel-Aviv University, Israel. Currently, he is a Ph.D. student at the School of Electrical Engineering, Tel-Aviv University, Israel. His research interests include robust control of time-delay and distributed parameter systems. Rami Katz was a finalist of the Best Student Paper Award at ECC 2021 for the paper "Finite-dimensional control of the heat equation: Dirichlet actuation and point measurement".

Emilia Fridman received the M.Sc. degree from Kuibyshev State University, USSR, in 1981 and the Ph.D. degree from Voronezh State University, USSR, in 1986, all in mathematics. From 1986 to 1992 she was an Assistant and Associate Professor in the Department of Mathematics at Kuibyshev Institute of Railway Engineers, USSR. Since 1993 she has been at Tel Aviv University, where she is currently Professor of Electrical Engineering-Systems. She has held visiting positions at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin (Germany), INRIA in Rocquencourt

(France), Ecole Centrale de Lille (France), Valenciennes University (France), Leicester University (UK), Kent University (UK), CINVESTAV (Mexico), Zhejiang University (China), St. Petersburg IPM (Russia), Melbourne University (Australia), Supelec (France), KTH (Sweden).

Her research interests include time-delay systems, networked control systems, distributed parameter systems, robust control, singular perturbations and nonlinear control. She has published two monographs and about 200 articles in international scientific journals. She serves/served as Associate Editor in Automatica, SIAM Journal on Control and Optimization and IMA Journal of Mathematical Control and Information. In 2014 she was Nominated as a Highly Cited Researcher by Thomson ISI. Since 2018, she has been the incumbent for Chana and Heinrich Manderman Chair on System Control at Tel Aviv University. She is IEEE Fellow since 2019. In 2021 she was recipient of IFAC Delay Systems Life Time Achievement Award and of Kadar Award for outstanding research at Tel Aviv University. She is currently a member of the IFAC Council.