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Brief paper

L_2-gain analysis via time-delay approach to periodic averaging with stochastic extension☆

Jin Zhang *, Emilia Fridman

School of Electrical Engineering, Tel Aviv University, Tel Aviv 69978, Israel

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ABSTRACT

We consider perturbed linear systems with fast-varying coefficients that are piecewise-continuous in time. Recently, a constructive time-delay approach to periodic averaging of such systems was introduced that provided an upper bound on the small parameter preserving their input-to-state stability (ISS). In the present paper, we present an improved time-delay approach and extend it to L_2-gain analysis. By the backward averaging, we transform the system to a modified time-delay system that leads to fewer terms to be compensated in the Lyapunov–Krasovskii (L–K) analysis. As a result we derive less conservative and simpler conditions for ISS and L_2-gain analysis in the form of linear matrix inequalities (LMIs). We further extend our results to stochastic systems where we employ a stochastic extension of Lyapunov functionals that we use for the deterministic case. Two numerical examples (stabilization by vibrational control and by time-dependent switching) illustrate the efficiency of the method.

1. Introduction

Periodic systems are extensively encountered in engineering applications (see e.g. Cheng & Tan, 2018; Christensen & Santos, 2005; Sandberg & Möllerstedt, 2001; Xie & Lam, 2018 and the references therein). Exponential stability and stabilization of periodic piecewise linear systems via a time-dependent homogeneous Lyapunov matrix polynomial were studied in Li et al. (2018). Averaging is one of the efficient methods to study the stability of systems with oscillatory control inputs (Bullo, 2002; Krstić & Wang, 2000; Meerkov, 1980). Results on asymptotic averaging, where the stability of the original system is guaranteed for small enough values of the small parameter if the averaged system is stable, were presented for deterministic systems (Bogoliubov & Mitropolsky, 1961; Khalil, 2002; Teel & Moreau, 2003) and for stochastic systems (Liu & Krstic, 2012). However, these results do not provide an efficient upper bound on the small parameter that guarantees the stability.

Recently, a constructive time-delay approach to periodic averaging was introduced in Fridman and Zhang (2020). It was suggested to use a backward averaging of the system and to present it in the form of a time-delay system, where the delay length is equal to the small parameter. The stability of the time-delay systems guarantees the stability of the original one. Then the direct L–K approach (see e.g. Fridman, 2014) leads to LMI-based conditions that allow to find an efficient upper bound on the small parameter preserving the stability and ISS of the original system. However, the presented results were conservative, whereas L_2-gain analysis and stochastic extension were not studied. It is well known that L–K approach allows to cope with exponential stability analysis of systems with state multiplicative noise. Multiplicative noise may appear due to the system parameters that undergo random perturbations of white noise process and due to nonlinearities (Mao, 2007; Shaikhet, 2013).

In this paper, we consider perturbed linear systems with fast-varying coefficients that are piecewise-continuous in time. We present an improved time-delay approach to periodic averaging and extend it to L_2-gain analysis and to the stochastic systems. We provide LMI-based conditions for finding an upper bound on the small parameter that preserves ISS and guarantees a certain L_2-gain. Two numerical examples (stabilization by vibrational control and by time-dependent switching) illustrate the efficiency of the method. Particularly, essentially larger bounds on the small parameter are obtained via simpler LMIs comparatively to Fridman and Zhang (2020). We summarize the contribution as follows:

(1) We provide less conservative and simpler LMI conditions for the stability and ISS analysis comparatively to Fridman

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* Corresponding author.
E-mail addresses: zhangjin1116@126.com (J. Zhang), emilia@tauex.tau.ac.il (E. Fridman).

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and Zhang (2020). This is achieved by using a novel time-delay model that leads to fewer terms to be compensated in the L–K analysis.
(2) We study, for the first time, L_2-gain analysis. We derive LMIs for the L_2-gain analysis of the time-delay system. These LMIs together with an additional condition guarantee the same L_2-gain for the original system.
(3) For the first time, we extend averaging via the time-delay approach to the stochastic systems. This extension is not straightforward since the deterministic Lyapunov functionals depend on $\dot{x}$ and are not applicable in the stochastic case (Fridman & Shaikhet, 2019; Zhang & Fridman, 2020). We suggest an appropriate stochastic extension of the Lyapunov functionals that leads to constructive LMIs for the ISS and L_2-gain analysis.

Throughout the paper $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices with the induced matrix norm $\|\cdot\|$. The superscript T stands for the vector/matrix transposition, and the notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix are denoted by $*$.

We will employ extended Jensen's inequalities (Solomon & Fridman, 2013):

Lemma 1.1. *Let $\mathcal{G} = \int_a^b f(s)\phi(s)ds$ and $\mathcal{Y} = \int_a^b \int_s^b \phi(\theta)d\theta ds$ with $a \le b$, where $f : [a, b] \to \mathbb{R}$, $\phi : [a, b] \to \mathbb{R}^n$ and the integrations concerned are well defined. Then for any $0 < R \in \mathbb{R}^{n\times n}$ the following inequalities hold:*

$$\mathcal{G}^T R\mathcal{G} \le \int_a^b |f(\theta)|d\theta \int_a^b |f(s)|\phi^T(s)R\phi(s)ds, \tag{1.1}$$

$$\mathcal{Y}^T R\mathcal{Y} \le \frac{(b-a)^2}{2} \int_a^b \int_s^b \phi^T(\theta)R\phi(\theta)d\theta ds. \tag{1.2}$$

2. Improved time-delay approach to periodic averaging: L_2-gain and ISS analysis

Consider a linear system with fast-varying coefficients:

$$\dot{x}(t) = A(\tfrac{t}{\varepsilon})x(t) + Bv(t), \quad t \ge 0, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $A : [0,\infty) \to \mathbb{R}^{n\times n}$ is piecewise-continuous, $B \in \mathbb{R}^{n\times n_v}$ is a constant matrix, $v(t) \in \mathbb{R}^{n_v}$ is the disturbance and $\varepsilon > 0$ is a small parameter. For L_2-gain analysis we consider $v \in L_2[0,\infty)$, whereas for ISS analysis v is locally essentially bounded. Then for any $x(0) \in \mathbb{R}^n$, (2.1) has a unique solution in the sense of Carathéodory (see Theorem 5.1 of Hale, 1980). The latter means that this solution x satisfies the initial condition, it is absolutely continuous and it satisfies (2.1) almost for all $t \ge 0$.

Similar to Fridman and Zhang (2020), we assume:

A1 Assume that the following holds:

$$\begin{aligned}&\tfrac{1}{\varepsilon\mathcal{T}} \int_{t-\varepsilon\mathcal{T}}^t A(\tfrac{s}{\varepsilon})ds = A_{av} + \Delta A(\tfrac{t}{\varepsilon}),\\ &\|\Delta A(\tfrac{t}{\varepsilon})\| \le \sigma \quad \forall \tfrac{t}{\varepsilon} \ge \mathcal{T}\end{aligned} \tag{2.2}$$

with Hurwitz constant matrix A_{av}, period $\mathcal{T} > 0$ and small enough constant $\sigma > 0$. This means that the unperturbed averaged system

$$\dot{x}_{av}(t) = [A_{av} + \Delta A(\tfrac{t}{\varepsilon})]x_{av}(t), \quad x_{av}(t) \in \mathbb{R}^n \tag{2.3}$$

is exponentially stable for small enough $\sigma > 0$ and all $\varepsilon > 0$.

A2 All entries $a_{kj}(\frac{t}{\varepsilon})$ of $A(\frac{t}{\varepsilon})$ are uniformly bounded for $t \ge 0$ with the values from some finite intervals $a_{kj}(\frac{t}{\varepsilon}) \in [a^m_{kj}, a^M_{kj}]$ for $\frac{t}{\varepsilon} \ge \mathcal{T}$.

Differently from **A1** in Fridman and Zhang (2020) with $\mathcal{T} = 1$ in (2.2), in the present paper we consider the periodic averaging over a general period $\mathcal{T}$ in **A1** (that allows to avoid scaling in time in order to have $\mathcal{T} = 1$). We can rewrite (2.2) in **A1** in terms of the stretched time $\tau = \frac{t}{\varepsilon}$ (by changing variable s to $\zeta = \frac{s}{\varepsilon}$ in the integral) as

$$\tfrac{1}{\mathcal{T}} \int_{\tau-\mathcal{T}}^{\tau} A(\zeta)d\zeta = A_{av} + \Delta A(\tau), \quad \|\Delta A(\tau)\| \le \sigma \quad \forall \tau \ge \mathcal{T}.$$

Matrix $\Delta A(\tau)$ may stand for system uncertainty whose norm is upper bounded by a known constant $\sigma > 0$. Under **A2**, $A(\tau)$ can be presented as a convex combination of the constant matrices A_i with the entries a^m_{kj} or a^M_{kj}:

$$\begin{aligned}&A(\tau) = \textstyle\sum_{i=1}^N \rho_i(\tau)A_i \quad \forall \tau \ge \mathcal{T},\\ &\rho_i \ge 0, \quad \textstyle\sum_{i=1}^N \rho_i = 1, \quad 1 \le N \le 2^{n^2}.\end{aligned} \tag{2.4}$$

For a constant a_{kj}, we have $a^m_{kj} = a^M_{kj}$. Note that we study the case where the time-varying parameters stabilize the system. Here treating time-varying terms as norm-bounded or polytopic type uncertainties is not appropriate since without these terms the system is unstable.

In this paper, by using periodic averaging we will present an improved (compared to Fridman & Zhang, 2020) ISS analysis of system (2.1) with locally essentially bounded v. For the first time, we will study L_2-gain analysis of system (2.1) with $v \in L_2[0,\infty)$. Following the time-delay approach to periodic averaging of Fridman and Zhang (2020), we integrate (2.1) on $[t-\varepsilon\mathcal{T}, t]$ for $t \ge \varepsilon\mathcal{T}$. Denote

$$f(\tfrac{t}{\varepsilon}) = A(\tfrac{t}{\varepsilon})x(t) \tag{2.5}$$

and

$$G(t,\varepsilon) \triangleq \tfrac{1}{\varepsilon\mathcal{T}} \int_{t-\varepsilon\mathcal{T}}^t (s-t+\varepsilon\mathcal{T})f(\tfrac{s}{\varepsilon})ds. \tag{2.6}$$

For shortness we omit the dependence on ε throughout this paper (e.g. $G(t) = G(t,\varepsilon)$, $Y(t) = Y(t,\varepsilon)$ in (2.12), etc.). Similar to Fridman and Shaikhet (2016), we can present

$$\begin{aligned}&\tfrac{1}{\varepsilon\mathcal{T}} \int_{t-\varepsilon\mathcal{T}}^t \dot{x}(s)ds = \tfrac{1}{\varepsilon\mathcal{T}}[x(t) - x(t-\varepsilon\mathcal{T})]\\ &= \tfrac{d}{dt}[x(t) - G(t) - \tfrac{1}{\varepsilon\mathcal{T}} \int_{t-\varepsilon\mathcal{T}}^t (s-t+\varepsilon\mathcal{T})Bv(s)ds]\\ &= \tfrac{d}{dt}[x(t) - G(t)] + \tfrac{1}{\varepsilon\mathcal{T}} \int_{t-\varepsilon\mathcal{T}}^t Bv(s)ds - Bv(t).\end{aligned} \tag{2.7}$$

Integrating (2.1), and then adding and subtracting $x(t)$, via (2.7) we arrive at

$$\begin{aligned}&\tfrac{d}{dt}[x(t) - G(t)] = \tfrac{1}{\varepsilon\mathcal{T}} \int_{t-\varepsilon\mathcal{T}}^t A(\tfrac{s}{\varepsilon})[x(s) + x(t)\\ &\quad -x(t)]ds + Bv(t), \quad t \ge \varepsilon\mathcal{T}.\end{aligned} \tag{2.8}$$

We present

$$\begin{aligned}&\tfrac{1}{\varepsilon\mathcal{T}} \int_{t-\varepsilon\mathcal{T}}^t A(\tfrac{s}{\varepsilon})[x(s) - x(t)]ds\\ &= -\tfrac{1}{\varepsilon\mathcal{T}} \int_{t-\varepsilon\mathcal{T}}^t A(\tfrac{s}{\varepsilon}) \int_s^t \dot{x}(\theta)d\theta ds.\end{aligned} \tag{2.9}$$

Denote

$$z(t) \triangleq x(t) - G(t). \tag{2.10}$$

Then under **A1** we transform (2.1) to the following time-delay system for $t \ge \varepsilon\mathcal{T}$:

$$\dot{z}(t) = [A_{av} + \Delta A(\tfrac{t}{\varepsilon})]x(t) - Y(t) + Bv(t), \tag{2.11}$$

where

$$\begin{aligned}&Y(t) \triangleq \tfrac{1}{\varepsilon\mathcal{T}} \int_{t-\varepsilon\mathcal{T}}^t A(\tfrac{s}{\varepsilon}) \int_s^t \dot{x}(\theta)d\theta ds,\\ &\dot{x}(\theta) = A(\tfrac{\theta}{\varepsilon})x(\theta) + Bv(\theta).\end{aligned} \tag{2.12}$$

A simpler derivation of the time-delay model will be presented in Section 3. Note that system (2.11) is a kind of neutral type system. Moreover, compared with the averaged system (2.3), system (2.11) with $v \equiv 0$ and $\dot{x}(\theta) = A(\frac{\theta}{\varepsilon})x(\theta) + Bv(\theta)$ has the additional terms $-G(t)$ and $-Y(t)$ that are both of the order $O(\varepsilon)$

provided x and $\dot{x}$ are $O(1)$. Thus, for small $\varepsilon > 0$, system (2.11) with $v \equiv 0$ can be considered as a perturbation of system (2.3). If given $v(t)$, the function $x(t)$ is a solution to system (2.1) then it satisfies the time-delay system (2.11). Therefore, the stability and ISS of the time-delay system guarantee the stability and ISS of the original system.

Remark 2.1. Note that in Fridman and Zhang (2020), for the ISS analysis $G(t)$-term has a form of (2.6) with $f(\frac{s}{\varepsilon})$ replaced by $\dot{x}(s)$ that leads to the additional disturbance term $\frac{1}{\varepsilon\mathcal{T}}\int_{t-\varepsilon\mathcal{T}}^{t} Bv(s)ds$, whereas $Y(t)$-term in (2.12) is replaced by $\sum_{i=1}^{N} A_iY_i(t)$ with $Y_i(t) \triangleq \frac{1}{\varepsilon\mathcal{T}}\int_{t-\varepsilon\mathcal{T}}^{t}\rho_i(\frac{s}{\varepsilon})\int_s^t \dot{x}(\theta)d\theta ds$. The $Y_i(t)$-terms are further compensated by N integral terms $V_{H_i}(t)$, $i = 1, \dots, N$ (see (2.22) of Fridman & Zhang, 2020). Comparatively to Fridman and Zhang (2020), system (2.11) has novel $G(t)$-term depending on $f(\frac{s}{\varepsilon})$ only and a single $Y(t)$-term that lead to fewer terms to be compensated in the L–K analysis. The latter significantly simplifies the LMIs and improves the results in the examples.

We will now present a L–K method for system (2.11) leading to LMIs that allow to find an upper bound ε^* on ε preserving the exponential stability and the corresponding performance of perturbed system (2.1) for all $\varepsilon \in (0, \varepsilon^*]$. Consider the following Lyapunov functional

$$V_1(t) = V_P(t) + V_R(t) + V_H(t), \quad t \geq \varepsilon\mathcal{T}, \tag{2.13}$$

where

$$\begin{aligned} V_P(t) &= z^T(t)Pz(t),\\ V_R(t) &= \tfrac{1}{\varepsilon\mathcal{T}}\int_{t-\varepsilon\mathcal{T}}^{t} e^{-2\alpha(t-s)}(s-t+\varepsilon\mathcal{T})^2 f^T(\tfrac{s}{\varepsilon})Rf(\tfrac{s}{\varepsilon})ds,\\ V_H(t) &= \tfrac{1}{\varepsilon\mathcal{T}}\int_{t-\varepsilon\mathcal{T}}^{t}\int_s^t e^{-2\alpha(t-\theta)}(s-t+\varepsilon\mathcal{T})\\ &\quad \times \dot{x}^T(\theta)A^T(\tfrac{s}{\varepsilon})HA(\tfrac{s}{\varepsilon})\dot{x}(\theta)d\theta ds \end{aligned} \tag{2.14}$$

with $n \times n$ matrices $P > 0$, $R > 0$, $H > 0$ and a scalar $\alpha \geq 0$. Here $V_R(t)$ compensates $G(t)$-term whereas $V_H(t)$ is employed to compensate $Y(t)$-term. By Jensen's inequality (3.87) in Fridman (2014), we have for all $\varepsilon \in (0, \varepsilon^*]$

$$\begin{aligned} V_1(t) &\geq V_P(t) + V_R(t)\\ &\geq \begin{bmatrix} x(t)\\ G(t)\end{bmatrix}^T \begin{bmatrix} P & -P\\ * & P+e^{-2\alpha\varepsilon^*\mathcal{T}}R\end{bmatrix}\begin{bmatrix} x(t)\\ G(t)\end{bmatrix} \geq c_1|x(t)|^2, \end{aligned} \tag{2.15}$$

where

$$c_1 = \lambda_{\min}(\begin{bmatrix} P & -P\\ * & P+e^{-2\alpha\varepsilon^*\mathcal{T}}R\end{bmatrix}). \tag{2.16}$$

Thus, $V_1(t)$ is positive-definite.

Consider next the controlled output

$$y(t) = Cx(t), \quad t \geq 0, \quad y(t) \in \mathbb{R}^l, \tag{2.17}$$

where $C \in \mathbb{R}^{l\times n}$ is a constant matrix. Given $\gamma > 0$, for neutral system (2.11) we employ the following performance index:

$$J_{\varepsilon\mathcal{T}} = \int_{\varepsilon\mathcal{T}}^{\infty}\left[|Cx(t)|^2 - \gamma^2|v(t)|^2\right]dt. \tag{2.18}$$

We will derive LMI conditions that guarantee $J_{\varepsilon\mathcal{T}} < 0$ for all $0 \neq v \in L_2[\varepsilon\mathcal{T}, \infty)$ along the solutions of (2.11) starting from the zero initial condition $x(t) \equiv 0$ $\forall t \in [0, \varepsilon\mathcal{T}]$, meaning that the neutral system (2.11), (2.17) has L_2-gain less than γ. Given $\gamma > 0$, we define the following performance index for the original system (2.1):

$$J = \int_0^{\infty}\left[|Cx(t)|^2 - \gamma^2|v(t)|^2\right]dt. \tag{2.19}$$

System (2.1), (2.17) has L_2-gain less than γ if $J < 0$ for all $0 \neq v \in L_2[0, \infty)$ along the solutions of (2.1) starting from $x(0) = 0$.

From **A2**, it follows that there exists a constant $a > 0$ such that $\|A(\frac{t}{\varepsilon})\| \leq a$ holds for all $t \geq 0$. The following lemma gives sufficient conditions for L_2-gain and ISS analysis:

Lemma 2.1. *Given $\varepsilon^* > 0, \alpha \geq 0$ and $\gamma > 0$, let for $V_1(t)$ given by (2.13) the following inequality holds along the solutions of (2.11) for all $\varepsilon \in (0, \varepsilon^*]$:*

$$\begin{aligned} &\dot{V}_1(t) + 2\alpha V_1(t) + |Cx(t)|^2 - \gamma^2|v(t)|^2 < 0\\ &\qquad \forall 0 \neq v(t) \in \mathbb{R}^{n_v} \text{ and } \forall t \geq \varepsilon\mathcal{T}. \end{aligned} \tag{2.20}$$

If (2.20) holds with $\alpha = 0$, then the neutral system (2.11), (2.17) has L_2-gain less than γ for all $\varepsilon \in (0, \varepsilon^*]$. If additionally

$$\varepsilon^* M - \gamma^2 < 0 \tag{2.21}$$

is satisfied with

$$\begin{aligned} M &= \tfrac{\mathcal{T}\|B\|^2}{2a}(e^{2a\varepsilon^*\mathcal{T}} - 1)[\varepsilon^* c_2(2a^2+1) + \|C\|^2]\\ &\quad + c_2\|B\|^2(2 + \mathcal{T}e^{2a\varepsilon^*\mathcal{T}}),\\ c_2 &= \max\{2\lambda_{\max}(P), a^2\mathcal{T}\lambda_{\max}(2P+R), a^2\mathcal{T}\lambda_{\max}(H)\}, \end{aligned} \tag{2.22}$$

where a is the upper bound of $\|A(\frac{t}{\varepsilon})\|$, then the original system (2.1), (2.17) has L_2-gain less than γ for all $\varepsilon \in (0, \varepsilon^*]$. Moreover, if (2.20) holds with $\alpha > 0$ and $C = 0$, then system (2.1) is ISS, i.e. there exists $M_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ and locally essentially bounded v, the solutions of system (2.1) initialized by $x(0) \in \mathbb{R}^n$ satisfy the following inequality:

$$|x(t)|^2 \leq M_0 e^{-2\alpha t}|x(0)|^2 + \left[M_0 e^{-2\alpha t} + \tfrac{\gamma^2}{2\alpha c_1}\right]\|v[0,t]\|_{\infty}^2 \tag{2.23}$$

for all $t \geq 0$ with c_1 given by (2.16).

Proof. Let (2.20) hold with $\alpha = 0$. Integration of (2.20) in t from $\varepsilon\mathcal{T}$ to ∞ yields

$$V_1(\infty) - V_1(\varepsilon\mathcal{T}) + \int_{\varepsilon\mathcal{T}}^{\infty}|Cx(t)|^2dt - \gamma^2\int_{\varepsilon\mathcal{T}}^{\infty}|v(t)|^2dt < 0. \tag{2.24}$$

Taking into account $V_1(\varepsilon\mathcal{T}) = 0$ and $V_1(\infty) \geq 0$, (2.24) yields $J_{\varepsilon\mathcal{T}} < 0$ implying that system (2.11), (2.17) has L_2-gain less than γ.

We next study L_2-gain analysis of system (2.1), (2.17). From (2.24), we have

$$J < V_1(\varepsilon\mathcal{T}) + \int_0^{\varepsilon\mathcal{T}}\|C\|^2|x(t)|^2dt - \gamma^2\int_0^{\varepsilon\mathcal{T}}|v(t)|^2dt. \tag{2.25}$$

By using Young's and Jensen's inequalities, we obtain the upper bound on $V_1(\varepsilon\mathcal{T})$ given by (2.13) for all $\varepsilon \in (0, \varepsilon^*]$

$$\begin{aligned} V_1(\varepsilon\mathcal{T}) &\leq 2x^T(\varepsilon\mathcal{T})Px(\varepsilon\mathcal{T}) + \varepsilon\mathcal{T}\int_0^{\varepsilon\mathcal{T}} f^T(\tfrac{s}{\varepsilon})(2P+R)f(\tfrac{s}{\varepsilon})ds\\ &\quad + \int_0^{\varepsilon\mathcal{T}}\int_s^{\varepsilon\mathcal{T}}\dot{x}^T(\theta)A^T(\tfrac{s}{\varepsilon})HA(\tfrac{s}{\varepsilon})\dot{x}(\theta)d\theta ds. \end{aligned}$$

Using (2.5), we have

$$V_1(\varepsilon\mathcal{T}) \leq c_2\left[|x(\varepsilon\mathcal{T})|^2 + \varepsilon\int_0^{\varepsilon\mathcal{T}}(|x(t)|^2 + |\dot{x}(t)|^2)dt\right] \tag{2.26}$$

with c_2 given by (2.22). Note that for $t \in [0, \varepsilon\mathcal{T}]$, $x(t)$ satisfies (2.1). Thus, from (2.1), we find

$$\begin{aligned} |\dot{x}(t)|^2 &\leq [a|x(t)| + \|B\||v(t)|]^2\\ &\leq 2a^2|x(t)|^2 + 2\|B\|^2|v(t)|^2 \quad \forall t \in [0, \varepsilon\mathcal{T}], \end{aligned}$$

where we applied Young's inequality. Therefore,

$$\int_0^{\varepsilon\mathcal{T}}|\dot{x}(t)|^2dt \leq 2a^2\int_0^{\varepsilon\mathcal{T}}|x(t)|^2dt + 2\|B\|^2\int_0^{\varepsilon\mathcal{T}}|v(t)|^2dt. \tag{2.27}$$

Substituting (2.27) into (2.26), we have

$$\begin{aligned} V_1(\varepsilon\mathcal{T}) \leq c_2\Big[& |x(\varepsilon\mathcal{T})|^2 + \varepsilon(2a^2+1)\int_0^{\varepsilon\mathcal{T}}|x(t)|^2dt\\ & + 2\varepsilon\|B\|^2\int_0^{\varepsilon\mathcal{T}}|v(t)|^2dt\Big]. \end{aligned} \tag{2.28}$$

Integrating (2.1) with $x(0) = 0$, we find for all $t \in [0, \varepsilon\mathcal{T}]$

$$|x(t)| \leq a\int_0^t |x(s)|ds + \|B\|\int_0^{\varepsilon\mathcal{T}}|v(s)|ds.$$

The latter, by Gronwall's inequality, implies

$$|x(t)| \le \|B\|e^{at}\int_0^{\varepsilon\mathcal{T}}|v(s)|ds, \quad t\in[0,\varepsilon\mathcal{T}]$$

and, by Jensen's inequality,

$$|x(t)|^2 \le \varepsilon\mathcal{T}\|B\|^2e^{2at}\int_0^{\varepsilon\mathcal{T}}|v(s)|^2ds, \quad t\in[0,\varepsilon\mathcal{T}]. \tag{2.29}$$

Then we have

$$\int_0^{\varepsilon\mathcal{T}}|x(t)|^2dt \le \frac{\varepsilon\mathcal{T}}{2a}\|B\|^2(e^{2a\varepsilon\mathcal{T}}-1)\int_0^{\varepsilon\mathcal{T}}|v(s)|^2ds.$$

Substituting the latter and (2.29) into (2.28) and further into (2.25), we arrive at $J<(\varepsilon M-\gamma^2)\int_0^{\varepsilon\mathcal{T}}|v(t)|^2dt$. Thus, under (2.21) we have $J<0$ for all $\varepsilon\in(0,\varepsilon^*]$ and for all non-zero $v\in L_2[0,\infty)$, i.e. system (2.1), (2.17) has L_2-gain less than γ.

Moreover, if (2.20) holds for $C=0$ and $\alpha>0$, then by comparison principle and by employing (2.15) we obtain the following bound on solutions of (2.1) for all $t\ge\varepsilon\mathcal{T}$:

$$c_1|x(t)|^2 \le V_1(t)\le e^{-2\alpha(t-\varepsilon\mathcal{T})}V_1(\varepsilon\mathcal{T})+\frac{\gamma^2}{2\alpha}\|v[0,t]\|_\infty^2. \tag{2.30}$$

We have along (2.1) for $t\in[0,\varepsilon\mathcal{T}]$ (cf. (2.29))

$$|x(t)|^2\le 2e^{2at}[|x(0)|^2+\varepsilon\mathcal{T}\|B\|^2\int_0^{\varepsilon\mathcal{T}}|v(s)|^2ds].$$

From (2.28) and the latter, it follows that (2.30) implies (2.23) with some ε-independent $M_0>0$. □

We will further derive LMI-based conditions for finding ε^* such that (2.20) along (2.11) and (2.21) hold. As in Fridman and Shaikhet (2019), there is no need to verify the stability of $z(t)=0$ since the bound (2.30) on $|x(t)|$ directly follows from (2.15) and (2.20).

Theorem 2.1. *Consider system (2.1) subject to* **A1** *and* **A2**. *Given matrices A_{av}, A_i $(i=1,\ldots,N)$, B, C, and constants $\sigma>0$, $\alpha\ge0$, $\varepsilon^*>0$, $\mathcal{T}>0$, let there exist $n\times n$ matrices $P>0$, $R>0$, $H>0$ and $\bar H>0$ and scalars $\lambda>0$ and $\gamma>0$ (that becomes a tuning parameter for L_2-gain analysis) satisfying the following LMIs:*

$$\frac{1}{\mathcal{T}^2}\int_{\tau-\mathcal{T}}^{\tau}(\zeta-\tau+\mathcal{T})A^T(\zeta)HA(\zeta)d\zeta\le\bar H \quad \forall\tau\ge\mathcal{T} \tag{2.31}$$

and for $i=1,\ldots,N$

$$\left[\begin{array}{c|cc}\Phi & \begin{matrix}\sqrt{\varepsilon^*\mathcal{T}}A_i^TR\\ 0_{3n\times n}\\ 0_{n_v\times n}\end{matrix} & \begin{matrix}\sqrt{\varepsilon^*\mathcal{T}}A_i^T\bar H\\ 0_{3n\times n}\\ \sqrt{\varepsilon^*\mathcal{T}}B^T\bar H\end{matrix}\\ \hline * & -R & 0_{n\times n}\\ * & * & -\bar H\end{array}\right]<0, \tag{2.32}$$

where Φ is the symmetric matrix composed of

$$\begin{aligned}&\Phi_{11}=PA_{av}+A_{av}^TP+2\alpha P+\lambda\sigma^2I_n+C^TC,\\&\Phi_{12}=-A_{av}^TP-2\alpha P,\quad \Phi_{13}=\Phi_{24}=-P,\\&\Phi_{14}=\Phi_{23}=P,\quad \Phi_{15}=-\Phi_{25}=PB,\\&\Phi_{22}=-\tfrac{4}{\varepsilon^*\mathcal{T}}e^{-2\alpha\varepsilon^*\mathcal{T}}R+2\alpha P,\quad \Phi_{44}=-\lambda I_n,\\&\Phi_{33}=-\tfrac{2}{\varepsilon^*\mathcal{T}}e^{-2\alpha\varepsilon^*\mathcal{T}}H,\quad \Phi_{55}=-\gamma^2I_{n_v}\end{aligned} \tag{2.33}$$

and other blocks are zero matrices. If LMIs (2.32) hold with $\alpha=0$, then system (2.11), (2.17) has L_2-gain less than γ for all $\varepsilon\in(0,\varepsilon^*]$. If additionally (2.21) is satisfied with M defined by (2.22), then system (2.1), (2.17) has L_2-gain less than γ for all $\varepsilon\in(0,\varepsilon^*]$. Moreover, if LMIs (2.32) hold with $\alpha>0$ and $C=0$, then system (2.1) is ISS (i.e. there exists $M_0>0$ such that the solutions of system (2.1) initialized by $x(0)\in\mathbb{R}^n$ satisfy (2.23) for all $\varepsilon\in(0,\varepsilon^*]$ and locally essentially bounded v). LMIs (2.32) are always feasible for small enough $\varepsilon^*>0$, $\frac{1}{\gamma}>0$, $\alpha>0$ and $\|C\|$.

Proof. Consider the functional $V_1(t)$ given by (2.13). Differentiating $V_P(t)$ along (2.11) we have

$$\begin{aligned}\dot V_P(t)=2[x(t)-G(t)]^TP[(A_{av}+\Delta A(\tfrac{t}{\varepsilon}))x(t)\\ -Y(t)+Bv(t)].\end{aligned} \tag{2.34}$$

For the term $V_R(t)$, we find

$$\begin{aligned}&\dot V_R(t)+2\alpha V_R(t)=\varepsilon\mathcal{T}f^T(\tfrac{t}{\varepsilon})Rf(\tfrac{t}{\varepsilon})\\&\quad-\tfrac{2}{\varepsilon\mathcal{T}}\int_{t-\varepsilon\mathcal{T}}^{t}e^{-2\alpha(t-s)}(s-t+\varepsilon\mathcal{T})f^T(\tfrac{s}{\varepsilon})Rf(\tfrac{s}{\varepsilon})ds.\end{aligned}$$

Jensen's inequality (1.1) leads to

$$2G^T(t)RG(t)\le\int_{t-\varepsilon\mathcal{T}}^{t}(s-t+\varepsilon\mathcal{T})f^T(\tfrac{s}{\varepsilon})Rf(\tfrac{s}{\varepsilon})ds.$$

Then

$$\dot V_R(t)+2\alpha V_R(t)\le\varepsilon\mathcal{T}f^T(\tfrac{t}{\varepsilon})Rf(\tfrac{t}{\varepsilon})-\tfrac{4}{\varepsilon\mathcal{T}}e^{-2\alpha\varepsilon\mathcal{T}}G^T(t)RG(t). \tag{2.35}$$

Moreover, we have

$$\begin{aligned}&\dot V_H(t)+2\alpha V_H(t)\\&\le\dot x^T(t)\cdot\tfrac{1}{\varepsilon\mathcal{T}}\int_{t-\varepsilon\mathcal{T}}^{t}(s-t+\varepsilon\mathcal{T})A^T(\tfrac{s}{\varepsilon})HA(\tfrac{s}{\varepsilon})ds\cdot\dot x(t)\\&\quad-\tfrac{1}{\varepsilon\mathcal{T}}e^{-2\alpha\varepsilon\mathcal{T}}\int_{t-\varepsilon\mathcal{T}}^{t}\int_s^t\dot x^T(\theta)A^T(\tfrac{s}{\varepsilon})HA(\tfrac{s}{\varepsilon})\dot x(\theta)d\theta ds.\end{aligned}$$

By changing variable $s=\varepsilon\zeta$ and employing (2.31) we obtain

$$\begin{aligned}&\tfrac{1}{\varepsilon^2\mathcal{T}^2}\int_{t-\varepsilon\mathcal{T}}^{t}(s-t+\varepsilon\mathcal{T})A^T(\tfrac{s}{\varepsilon})HA(\tfrac{s}{\varepsilon})ds\\&=\tfrac{1}{\mathcal{T}^2}\int_{\frac{t}{\varepsilon}-\mathcal{T}}^{\frac{t}{\varepsilon}}(\zeta-\tfrac{t}{\varepsilon}+\mathcal{T})A^T(\zeta)HA(\zeta)d\zeta\le\bar H.\end{aligned} \tag{2.36}$$

Applying further the extended Jensen's inequality (1.2)

$$2Y^T(t)HY(t)\le\int_{t-\varepsilon\mathcal{T}}^{t}\int_s^t\dot x^T(\theta)A^T(\tfrac{s}{\varepsilon})HA(\tfrac{s}{\varepsilon})\dot x(\theta)d\theta ds,$$

we arrive at

$$\dot V_H(t)+2\alpha V_H(t)\le\varepsilon\mathcal{T}\dot x^T(t)\bar H\dot x(t)-\tfrac{2}{\varepsilon\mathcal{T}}e^{-2\alpha\varepsilon\mathcal{T}}Y^T(t)HY(t). \tag{2.37}$$

To compensate $\Delta A(\frac{t}{\varepsilon})x(t)$ in (2.34), from (2.2) we have

$$\lambda[\sigma^2|x(t)|^2-|\Delta A(\tfrac{t}{\varepsilon})x(t)|^2]\ge0 \tag{2.38}$$

with some $\lambda>0$. Then from (2.34)–(2.37), by applying S-procedure where we add (2.38) to $\dot V_1(t)$, we obtain for all $\varepsilon\in(0,\varepsilon^*]$

$$\begin{aligned}&\dot V_1(t)+2\alpha V_1(t)+|Cx(t)|^2-\gamma^2|v(t)|^2\\&\le\dot V_1(t)+2\alpha V_1(t)+|Cx(t)|^2-\gamma^2|v(t)|^2\\&\quad+\lambda[\sigma^2|x(t)|^2-|\Delta A(\tfrac{t}{\varepsilon})x(t)|^2]\\&\le\xi_1^T(t)\Phi\xi_1(t)+\varepsilon^*\mathcal{T}[f^T(\tfrac{t}{\varepsilon})Rf(\tfrac{t}{\varepsilon})+\dot x^T(t)\bar H\dot x(t)].\end{aligned} \tag{2.39}$$

Here $\xi_1^T(t)=[x^T(t),G^T(t),Y^T(t),x^T(t)\Delta A^T(\frac{t}{\varepsilon}),v^T(t)]$ and Φ is composed of (2.33). We substitute into (2.39)

$$\begin{aligned}&f(\tfrac{t}{\varepsilon})=\textstyle\sum_{i=1}^N\rho_i(\tfrac{t}{\varepsilon})A_ix(t),\\&\dot x(t)=\textstyle\sum_{i=1}^N\rho_i(\tfrac{t}{\varepsilon})A_ix(t)+Bv(t).\end{aligned} \tag{2.40}$$

Applying further Schur complements, we conclude that if

$$\left[\begin{array}{c|cc}\Phi & \begin{matrix}\sqrt{\varepsilon^*\mathcal{T}}\sum_{i=1}^N\rho_i(\frac{t}{\varepsilon})A_i^TR\\ 0_{3n\times n}\\ 0_{n_v\times n}\end{matrix} & \begin{matrix}\sqrt{\varepsilon^*\mathcal{T}}\sum_{i=1}^N\rho_i(\frac{t}{\varepsilon})A_i^T\bar H\\ 0_{3n\times n}\\ \sqrt{\varepsilon^*\mathcal{T}}B^T\bar H\end{matrix}\\ \hline * & -R & 0_{n\times n}\\ * & * & -\bar H\end{array}\right]<0, \tag{2.41}$$

then (2.20) holds. LMIs (2.32) imply (2.41) (thus, (2.20)) since (2.41) is affine in $\sum_{i=1}^N\rho_i(\frac{t}{\varepsilon})A_i^T$. Then the result follows from Lemma 2.1.

We show next the feasibility of $\Phi<0$ for small enough $\varepsilon^*>0$, $\frac{1}{\gamma}>0$ and $\|C\|$. Since A_{av} is Hurwitz, there exists $n\times n$ matrix $P>0$ such that for small enough $\alpha>0$ the following holds: $\Phi_0=PA_{av}+A_{av}^TP+2\alpha P<0$. Choose $R=H=\mathcal{T}e^{2\alpha\varepsilon^*\mathcal{T}}I_n$,

$\lambda = \frac{1}{\varepsilon^*}$ and $\sigma = \varepsilon^*$. By applying Schur complements, $\Phi < 0$ is equivalent to

$$\begin{bmatrix} \Phi_0 + \varepsilon^* I_n + C^T C & -A_{av}^T P - 2\alpha P \\ * & -\frac{4}{\varepsilon^*} I_n + 2\alpha P \end{bmatrix} + \begin{bmatrix} P \\ -P \end{bmatrix} (\tfrac{3}{2}\varepsilon^* I_n + \tfrac{1}{\gamma^2} BB^T) \begin{bmatrix} P \\ -P \end{bmatrix}^T < 0.$$

Since $\Phi_0 < 0$, the latter inequality (thus, $\Phi < 0$) is always feasible for small enough $\varepsilon^* > 0$, $\frac{1}{\gamma} > 0$ and $\|C\|$. Finally, applying Schur complements to the last two block-columns and block-rows of LMIs (2.32), we find that LMIs (2.32) hold for small enough $\varepsilon^* > 0$ if $\Phi < 0$ is feasible. Therefore, LMIs (2.32) are always feasible for small enough $\varepsilon^* > 0$, $\frac{1}{\gamma} > 0$, $\alpha > 0$ and $\|C\|$. □

Remark 2.2. In some cases the upper-bounding in (2.31) (and in (3.26)) can be done directly (see the switched system Example 4.2). In others (as in Example 4.1), one can choose $H = hI_n$ with scalar $h > 0$ to be determined, and use the bounding

$$\frac{1}{\mathcal{T}^2} \int_{\tau-\mathcal{T}}^{\tau} (\zeta - \tau + \mathcal{T}) A^T(\zeta) H A(\zeta) d\zeta \le \frac{h}{\mathcal{T}} \int_{\tau-\mathcal{T}}^{\tau} A^T(\zeta) A(\zeta) d\zeta.$$

3. Stability and performance analysis of stochastic systems by periodic averaging

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a probability space, $\{\mathfrak{F}_t, t \ge 0\}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, i.e., $\mathfrak{F}_s \subset \mathfrak{F}_t$ for $s < t$, $\mathbf{P}\{\cdot\}$ be the probability of an event enclosed in the brackets. The mathematical expectation $\mathbf{E}$ of a random variable $\xi = \xi(w)$ on the probability space $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ is defined as $\mathbf{E}\xi = \int_\Omega \xi(w) d\mathbf{P}(w)$. The scalar standard Wiener process (also called Brownian motion) is a stochastic process $w(t)$ with normal distribution satisfying $w(0) = 0$, $\mathbf{E}w(t) = 0$ $(t > 0)$ and $\mathbf{E}w^2(t) = t$ $(t > 0)$ (Shaikhet, 2013).

Consider the following stochastic fast-varying system

$$dx(t) = [A(\tfrac{t}{\varepsilon})x(t) + Bv(t)]dt + Dx(t)dw(t), \quad t \ge 0, \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $A : [0, \infty) \to \mathbb{R}^{n\times n}$ is piecewise-continuous, $B \in \mathbb{R}^{n\times n_v}$ and $D \in \mathbb{R}^{n\times n}$ are constant matrices, $v(t) \in \mathbb{R}^{n_v}$ is the deterministic disturbance, $w(t)$ is the scalar standard Wiener process and $\varepsilon > 0$ is a small parameter. For the well-posedness, we assume $v(t)$ to be continuous.

A solution of (3.1) with the initial condition $x(0)$ is a stochastic process $x(t)$ that satisfies the initial condition and for $t \ge 0$ with probability 1 satisfies the equation

$$x(t) = x(0) + \int_0^t [A(\tfrac{s}{\varepsilon})x(s) + Bv(s)]ds + \int_0^t Dx(s)dw(s).$$

By Theorem 2.3.1 of Mao (2007), there exists a unique solution to system (3.1) and this solution satisfies $\int_0^t \mathbf{E}|x(s)|^2 ds = \mathbf{E}\int_0^t |x(s)|^2 ds < \infty$ for all $t \ge 0$.

Note that in **A1**, term $\Delta A(\tau)$ stems from system uncertainty that now may be included in the multiplicative noise. Thus, for stochastic system (3.1) we assume that $A(\tau)$ is $\mathcal{T}$-periodic and the following holds:

A3 Assume that the following holds:

$$\frac{1}{\varepsilon\mathcal{T}} \int_{t-\varepsilon\mathcal{T}}^t A(\tfrac{s}{\varepsilon}) ds = A_{av} \quad \forall \tfrac{t}{\varepsilon} \ge \mathcal{T} \tag{3.2}$$

with Hurwitz constant matrix A_{av} and period $\mathcal{T} > 0$.

Assume also that **A2** and relation (2.4) hold. We will follow the time-delay approach to periodic averaging of Fridman and Zhang (2020). Let $f(\frac{t}{\varepsilon})$, $G(t)$ and $z(t)$ be defined by (2.5), (2.6) and (2.10) respectively. We have for all $t \ge \varepsilon\mathcal{T}$

$$\begin{aligned} dz(t) &= d[x(t) - \tfrac{1}{\varepsilon\mathcal{T}} \textstyle\int_{t-\varepsilon\mathcal{T}}^t (s - t + \varepsilon\mathcal{T}) A(\tfrac{s}{\varepsilon}) x(s) ds] \\ &= dx(t) + \big[\tfrac{1}{\varepsilon\mathcal{T}} \textstyle\int_{t-\varepsilon\mathcal{T}}^t A(\tfrac{s}{\varepsilon}) x(s) ds - A(\tfrac{t}{\varepsilon}) x(t)\big] dt. \end{aligned} \tag{3.3}$$

By substituting the right-hand side of (3.1) for $dx(t)$ and adding and subtracting $x(t)$, we obtain from (3.3)

$$\begin{aligned} dz(t) = &\big[\tfrac{1}{\varepsilon\mathcal{T}} \textstyle\int_{t-\varepsilon\mathcal{T}}^t A(\tfrac{s}{\varepsilon})(x(s) + x(t) - x(t)) ds \\ &+ Bv(t)\big] dt + Dx(t)dw(t), \quad t \ge \varepsilon\mathcal{T} \end{aligned}$$

that under **A3** leads to the neutral system

$$\begin{aligned} dz(t) = &\big[A_{av}x(t) + \tfrac{1}{\varepsilon\mathcal{T}} \textstyle\int_{t-\varepsilon\mathcal{T}}^t A(\tfrac{s}{\varepsilon})(x(s) - x(t)) ds \\ &+ Bv(t)\big] dt + Dx(t)dw(t), \quad t \ge \varepsilon\mathcal{T}. \end{aligned} \tag{3.4}$$

For a functional $V(t, x(t))$, associated with (3.4), which is continuously differentiable in t and twice continuously differentiable in x, we employ the generator $\mathcal{L}$ (Mao, 2007; Shaikhet, 2013):

$$\begin{aligned} \mathcal{L}V(t, x(t)) = &V_t(t, x(t)) + V_x(t, x(t))[A_{av}x(t) \\ &+ \tfrac{1}{\varepsilon\mathcal{T}} \textstyle\int_{t-\varepsilon\mathcal{T}}^t A(\tfrac{s}{\varepsilon})(x(s) - x(t)) ds + Bv(t)] \\ &+ \tfrac{1}{2}\mathrm{trace}\{x^T(t)D^T V_{xx}(t, x(t))Dx(t)\}, \end{aligned} \tag{3.5}$$

where $V_t = \frac{\partial}{\partial t}V$, $V_x = (\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n})$ and $V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{n\times n}$. Differently from the deterministic case, for the stochastic case we emphasize dependence on $x(t)$ of Lyapunov functional $V(t, x(t))$, which is important for definition of $\mathcal{L}V(t, x(t))$ in (3.5). Denote

$$\tilde{f}(\tfrac{t}{\varepsilon}) = A(\tfrac{t}{\varepsilon})x(t) + Bv(t). \tag{3.6}$$

We present (cf. (2.9))

$$\frac{1}{\varepsilon\mathcal{T}} \int_{t-\varepsilon\mathcal{T}}^t A(\tfrac{s}{\varepsilon})[x(s) - x(t)] ds = -\sum_{i=1}^2 Y_i(t), \tag{3.7}$$

where we substituted the right-hand side of (3.1) for $dx(\theta)$ and

$$\begin{aligned} Y_1(t) &\triangleq \tfrac{1}{\varepsilon\mathcal{T}} \textstyle\int_{t-\varepsilon\mathcal{T}}^t A(\tfrac{s}{\varepsilon}) \int_s^t \tilde{f}(\tfrac{\theta}{\varepsilon}) d\theta ds, \\ Y_2(t) &\triangleq \tfrac{1}{\varepsilon\mathcal{T}} \textstyle\int_{t-\varepsilon\mathcal{T}}^t A(\tfrac{s}{\varepsilon}) \int_s^t Dx(\theta) dw(\theta) ds. \end{aligned} \tag{3.8}$$

By substituting (3.7) into (3.5) we obtain

$$\begin{aligned} \mathcal{L}V(t, x(t)) = &V_t(t, x(t)) + V_x(t, x(t)) \big[A_{av}x(t) + Bv(t) \\ &- \textstyle\sum_{i=1}^2 Y_i(t)\big] + \tfrac{1}{2}\mathrm{trace}\{x^T(t)D^T V_{xx}(t, x(t))Dx(t)\}. \end{aligned} \tag{3.9}$$

Consider the following Lyapunov functional for (3.4):

$$V_2(t) = V(t, x(t)) = V_P(t, x(t)) + V_R(t) + V_H(t) + V_F(t), \tag{3.10}$$

where $V_R(t)$ is given by (2.14), and

$$\begin{aligned} V_P(t, x(t)) &= [x(t) - G(t)]^T P [x(t) - G(t)], \\ V_H(t) &= \tfrac{1}{\varepsilon\mathcal{T}} \textstyle\int_{t-\varepsilon\mathcal{T}}^t \int_s^t e^{-2\alpha(t-\theta)}(s - t + \varepsilon\mathcal{T}) \\ &\quad \times \tilde{f}^T(\tfrac{\theta}{\varepsilon}) A^T(\tfrac{s}{\varepsilon}) H A(\tfrac{s}{\varepsilon}) \tilde{f}(\tfrac{\theta}{\varepsilon}) d\theta ds, \\ V_F(t) &= \tfrac{1}{\varepsilon^2\mathcal{T}^2} \textstyle\int_{t-\varepsilon\mathcal{T}}^t \int_s^t e^{-2\alpha(t-\theta)}(s - t + \varepsilon\mathcal{T}) \\ &\quad \times x^T(\theta) D^T A^T(\tfrac{s}{\varepsilon}) F A(\tfrac{s}{\varepsilon}) Dx(\theta) d\theta ds \end{aligned} \tag{3.11}$$

with $n \times n$ matrices $P > 0$, $H > 0$ and $F > 0$ and a scalar $\alpha \ge 0$. Note that $V_H(t)$ and $V_F(t)$ are employed to compensate $Y_1(t)$ and $Y_2(t)$ respectively. From (2.15), it follows that $V(t, x(t))$ is positive-definite for all $\varepsilon \in (0, \varepsilon^*]$ where due to (2.15) the following holds for some ε-independent $c_1 > 0$:

$$V(t, x(t)) \ge V_P(t, x(t)) + V_R(t) \ge c_1|x(t)|^2. \tag{3.12}$$

For simplicity, we will further use notation $V_2(t) = V(t, x(t))$. The stochastic extension of L_2-gain and ISS analysis will be based on the following Lemma:

Lemma 3.1. *Consider $J_{\varepsilon\mathcal{T}}$ and J defined by* (2.18) *and* (2.19) *respectively. Given $\varepsilon^* > 0, \alpha \ge 0$ and $\gamma > 0$, let for $V_2(t)$ defined by* (3.10) *and $\mathcal{L}V_2(t) = \mathcal{L}V(t, x(t))$ given by* (3.9) *the following inequality holds for all $\varepsilon \in (0, \varepsilon^*]$:*

$$\begin{aligned} &\mathbf{E}\left(\mathcal{L}V_2(t) + 2\alpha V_2(t) + |Cx(t)|^2\right) < \gamma^2 |v(t)|^2 \\ &\qquad \forall 0 \ne v(t) \in \mathbb{R}^{n_v} \text{ and } \forall t \ge \varepsilon\mathcal{T}. \end{aligned} \tag{3.13}$$

If (3.13) holds with $\alpha = 0$, then for all $\varepsilon \in (0, \varepsilon^*]$ the neutral system (3.4), (2.17) has L_2-gain less than or equal to γ meaning that $\mathbf{E}J_{\varepsilon\mathcal{T}} \leq 0$ for all $0 \neq v \in L_2[\varepsilon\mathcal{T}, \infty)$ and all solutions of (3.4) starting from the zero initial condition $x(t) \equiv 0 \ \forall t \in [0, \varepsilon\mathcal{T}]$. If additionally

$$\varepsilon^* \tilde{M} - \gamma^2 < 0 \tag{3.14}$$

holds, where

$$\begin{aligned} \tilde{M} &= \tfrac{\mathcal{T}\|B\|^2}{a^2\varepsilon^*\mathcal{T}+\|D\|^2}(e^{\varepsilon^*\mathcal{T}(a^2\varepsilon^*\mathcal{T}+\|D\|^2)} - 1)[\tilde{c}_2(1 + 2a^2\varepsilon^* \\ &\quad +\varepsilon^*) + \|C\|^2] + \tilde{c}_2\|B\|^2(2 + 3\mathcal{T}e^{3\varepsilon^*\mathcal{T}(a^2\varepsilon^*\mathcal{T}+\|D\|^2)}), \\ \tilde{c}_2 &= \max\{2\lambda_{\max}\{P\}, a^2\mathcal{T}\lambda_{\max}\{2P + R\}, a^2\mathcal{T}\lambda_{\max}\{H\}, \\ &\quad a^2\|D\|^2\lambda_{\max}\{F\}\} \end{aligned} \tag{3.15}$$

and a is the upper bound of $\|A(\frac{t}{\varepsilon})\|$, then for all $\varepsilon \in (0, \varepsilon^*]$ the original system (3.1), (2.17) has L_2-gain less than γ meaning that $\mathbf{E}J < 0$ for all $0 \neq v \in L_2[0, \infty)$ and solutions of (3.1) starting from $x(0) = 0$. Moreover, if (3.13) holds with $\alpha > 0$ and $C = 0$, then system (3.1) is ISS, i.e. there exists $M_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ and locally essentially bounded v, the solutions of system (3.1) initialized by $x(0) \in \mathbb{R}^n$ satisfy the following inequality:

$$\mathbf{E}|x(t)|^2 \leq M_0 e^{-2\alpha t}\mathbf{E}|x(0)|^2 + \left[M_0 e^{-2\alpha t} + \tfrac{\gamma^2}{2\alpha c_1}\right]\|v[0, t]\|_\infty^2 \tag{3.16}$$

for all $t \geq 0$ with c_1 given by (2.16).

Proof. Let (3.13) hold with $\alpha = 0$. The proof of $\mathbf{E}J_{\varepsilon\mathcal{T}} \leq 0$ follows the standard arguments (see e.g. Xu & Chen, 2002). Integration of (3.13) in s from $\varepsilon\mathcal{T}$ to $t > \varepsilon\mathcal{T}$ yields for $0 \neq v \in L_2[\varepsilon\mathcal{T}, t)$

$$\int_{\varepsilon\mathcal{T}}^t \mathbf{E}|Cx(s)|^2 ds - \gamma^2 \int_{\varepsilon\mathcal{T}}^t |v(s)|^2 ds < -\int_{\varepsilon\mathcal{T}}^t \mathbf{E}\mathcal{L}V_2(s)ds. \tag{3.17}$$

From equation (2.8) of Shaikhet (2013), we have

$$-\int_{\varepsilon\mathcal{T}}^t \mathbf{E}\mathcal{L}V_2(s)ds = \mathbf{E}V_2(\varepsilon\mathcal{T}) - \mathbf{E}V_2(t) \leq \mathbf{E}V_2(\varepsilon\mathcal{T}), \tag{3.18}$$

where the inequality follows from $\mathbf{E}V_2(t) \geq 0$ for $t > \varepsilon\mathcal{T}$. Moreover, since

$$\int_{\varepsilon\mathcal{T}}^t \mathbf{E}|Cx(s)|^2 ds \leq \|C\|^2 \int_{\varepsilon\mathcal{T}}^t \mathbf{E}|x(s)|^2 ds < \infty, \quad t > \varepsilon\mathcal{T},$$

by Fubini's theorem (see Theorem 2.39 of Klebaner, 2005) we have

$$\int_{\varepsilon\mathcal{T}}^t \mathbf{E}|Cx(s)|^2 ds = \mathbf{E}\int_{\varepsilon\mathcal{T}}^t |Cx(s)|^2 ds, \quad t > \varepsilon\mathcal{T}. \tag{3.19}$$

Thus, (3.17)–(3.19) imply

$$\mathbf{E}\int_{\varepsilon\mathcal{T}}^t |Cx(s)|^2 ds < \gamma^2 \int_{\varepsilon\mathcal{T}}^t |v(s)|^2 ds + \mathbf{E}V_2(\varepsilon\mathcal{T}) \quad \forall t > \varepsilon\mathcal{T}$$

which leads to

$$\lim_{t\to\infty} \mathbf{E}\int_{\varepsilon\mathcal{T}}^t |Cx(s)|^2 ds \leq \gamma^2 \int_{\varepsilon\mathcal{T}}^\infty |v(s)|^2 ds + \mathbf{E}V_2(\varepsilon\mathcal{T}).$$

By using monotonic convergence theorem (see Theorem 1.2.2 of Mao, 2007), we have

$$\lim_{t\to\infty} \mathbf{E}\int_{\varepsilon\mathcal{T}}^t |Cx(s)|^2 ds = \mathbf{E}\int_{\varepsilon\mathcal{T}}^\infty |Cx(s)|^2 ds.$$

Thus, we arrive at

$$\mathbf{E}J_{\varepsilon\mathcal{T}} = \mathbf{E}\int_{\varepsilon\mathcal{T}}^\infty |Cx(s)|^2 ds - \gamma^2 \int_{\varepsilon\mathcal{T}}^\infty |v(s)|^2 ds \leq \mathbf{E}V_2(\varepsilon\mathcal{T}) \tag{3.20}$$

meaning that $\mathbf{E}J_{\varepsilon\mathcal{T}} \leq 0$ since $\mathbf{E}V_2(\varepsilon\mathcal{T}) = 0$.

Next, we study L_2-gain analysis of the original system (3.1), (2.17). For J defined by (2.19), from (3.20) we have

$$\mathbf{E}J \leq \mathbf{E}V_2(\varepsilon\mathcal{T}) + \mathbf{E}\int_0^{\varepsilon\mathcal{T}} \|C\|^2|x(t)|^2 dt - \gamma^2 \int_0^{\varepsilon\mathcal{T}} |v(t)|^2 dt. \tag{3.21}$$

Note that $V_2(\varepsilon\mathcal{T})$ given by (3.10) is upper bounded for all $\varepsilon \in (0, \varepsilon^*]$

$$V_2(\varepsilon\mathcal{T}) \leq \tilde{c}_2 \left[\, |x(\varepsilon\mathcal{T})|^2 + \int_0^{\varepsilon\mathcal{T}}((1+\varepsilon)|x(t)|^2 + \varepsilon|\tilde{f}(\tfrac{t}{\varepsilon})|^2)dt \,\right]$$

with $\tilde{c}_2$ defined by (3.15). Under **A2**, $\tilde{f}(\frac{t}{\varepsilon})$ defined by (3.6) satisfies the following inequality (cf. (2.27)):

$$\int_0^{\varepsilon\mathcal{T}} |\tilde{f}(\tfrac{t}{\varepsilon})|^2 dt \leq 2a^2 \int_0^{\varepsilon\mathcal{T}} |x(t)|^2 dt + 2\|B\|^2 \int_0^{\varepsilon\mathcal{T}} |v(t)|^2 dt. \tag{3.22}$$

Thus, we have

$$\begin{aligned} V_2(\varepsilon\mathcal{T}) \leq \tilde{c}_2 \Big[\, &|x(\varepsilon\mathcal{T})|^2 + 2\varepsilon\|B\|^2 \int_0^{\varepsilon\mathcal{T}} |v(t)|^2 dt \\ &+(1 + 2a^2\varepsilon + \varepsilon)\int_0^{\varepsilon\mathcal{T}} |x(t)|^2 \,\Big]. \end{aligned} \tag{3.23}$$

Integrating (3.1) with the initial condition $x(0) = 0$, we obtain for all $t \in [0, \varepsilon\mathcal{T}]$

$$|x(t)| \leq a|\int_0^t x(s)ds| + \|B\||\int_0^t v(s)ds| + \|D\||\int_0^t x(s)dw(s)|.$$

Differently from the deterministic case, we cannot apply Gronwall's inequality directly to the latter. Instead, we use Cauchy–Schwarz and Jensen's inequalities:

$$\begin{aligned} |x(t)|^2 &\leq 3a^2|\int_0^t x(s)ds|^2 + 3\|B\|^2|\int_0^t v(s)ds|^2 \\ &\quad +3\|D\|^2|\int_0^t x(s)dw(s)|^2 \\ &\leq 3a^2\varepsilon\mathcal{T}\int_0^t |x(s)|^2 ds + 3\varepsilon\mathcal{T}\|B\|^2 \int_0^{\varepsilon\mathcal{T}} |v(s)|^2 ds \\ &\quad +3\|D\|^2|\int_0^t x(s)dw(s)|^2 \quad \forall t \in [0, \varepsilon\mathcal{T}]. \end{aligned}$$

Since $\int_0^t \mathbf{E}|x(s)|^2 ds < \infty$ for $t \in [0, \varepsilon\mathcal{T}]$, by using Itô isometry property (see Theorem 4.3 in Klebaner, 2005) we have

$$\mathbf{E}|\int_0^t x(s)dw(s)|^2 = \mathbf{E}\int_0^t |x(s)|^2 ds, \quad t \in [0, \varepsilon\mathcal{T}].$$

We obtain

$$\begin{aligned} \mathbf{E}|x(t)|^2 &\leq 3\varepsilon\mathcal{T}\|B\|^2 \int_0^{\varepsilon\mathcal{T}} |v(s)|^2 ds \\ &\quad +3\left(a^2\varepsilon\mathcal{T} + \|D\|^2\right)\int_0^t \mathbf{E}|x(s)|^2 ds, \quad t \in [0, \varepsilon\mathcal{T}], \end{aligned}$$

where we applied Fubini's theorem. By Gronwall's inequality, the following holds for all $\varepsilon \in (0, \varepsilon^*]$ and $t \in [0, \varepsilon\mathcal{T}]$

$$\mathbf{E}|x(t)|^2 \leq 3\varepsilon\mathcal{T}\|B\|^2 e^{3(a^2\varepsilon^*\mathcal{T}+\|D\|^2)t} \int_0^{\varepsilon\mathcal{T}} |v(s)|^2 ds. \tag{3.24}$$

Then we have

$$\int_0^{\varepsilon\mathcal{T}} \mathbf{E}|x(t)|^2 dt \leq \tfrac{\varepsilon\mathcal{T}\|B\|^2}{a^2\varepsilon^*\mathcal{T}+\|D\|^2}(e^{3(a^2\varepsilon^*\mathcal{T}+\|D\|^2)\varepsilon\mathcal{T}} - 1)\int_0^{\varepsilon\mathcal{T}} |v(s)|^2 ds.$$

Substituting the latter and (3.24) into (3.23) and further into (3.21), under (3.14) we arrive at $\mathbf{E}J \leq (\varepsilon^*\tilde{M} - \gamma^2)\int_0^{\varepsilon\mathcal{T}} |v(t)|^2 dt < 0$ for all $0 \neq v \in L_2[0, \infty)$.

Let (3.13) hold with $\alpha > 0$ and $C = 0$. Then the comparison principle implies

$$\mathbf{E}V_2(t) \leq e^{-2\alpha(t-\varepsilon\mathcal{T})}\mathbf{E}V_2(\varepsilon\mathcal{T}) + \tfrac{\gamma^2}{2\alpha}\|v[0, t]\|_\infty^2 \quad \forall t \geq \varepsilon\mathcal{T}. \tag{3.25}$$

We have along (3.1) for $t \in [0, \varepsilon\mathcal{T}]$

$$\mathbf{E}|x(t)|^2 \leq 4e^{4(a^2\varepsilon^*\mathcal{T}+\|D\|^2)t}[\mathbf{E}|x(0)|^2 + \varepsilon\mathcal{T}\|B\|^2 \int_0^{\varepsilon\mathcal{T}} |v(s)|^2 ds].$$

From (3.12), (3.23) and the latter, it follows that (3.25) implies (3.16) with some ε-independent $M_0 > 0$. □

Theorem 3.1. *Consider the system (3.1) subject to **A2** and **A3**. Given matrices A_{av}, A_i $(i = 1, \ldots, N)$, B, C, D, and constants $\alpha \geq 0$, $\varepsilon^* > 0$, $\mathcal{T} > 0$, let there exist $n \times n$ matrices $P > 0$, $R > 0$, $H > 0$, $\bar{H} > 0$, $F > 0$ and $\bar{F} > 0$ and a scalar $\gamma > 0$ (that becomes a tuning parameter for L_2-gain analysis) satisfying that satisfy the following LMIs: (2.31),*

$$\tfrac{1}{\mathcal{T}^2}\int_{\tau-\mathcal{T}}^{\tau}(\zeta - \tau + \mathcal{T})A^T(\zeta)FA(\zeta)d\zeta \leq \bar{F} \quad \forall \tau \geq \mathcal{T} \tag{3.26}$$

and for $i = 1, \ldots, N$

$$\begin{bmatrix} \Xi & \begin{matrix} \sqrt{\varepsilon^* \mathcal{T}} A_i^T R & \sqrt{\varepsilon^* \mathcal{T}} A_i^T \bar{H} \\ 0_{3n\times n} & 0_{3n\times n} \\ 0_{n_v \times n} & \sqrt{\varepsilon^* \mathcal{T}} B^T \bar{H} \end{matrix} \\ \hline \begin{matrix} * \\ * \end{matrix} & \begin{matrix} -R & 0_{n\times n} \\ * & -\bar{H} \end{matrix} \end{bmatrix} < 0, \tag{3.27}$$

where Ξ is the symmetric matrix composed of

$$\begin{aligned} &\Xi_{11} = PA_{av} + A_{av}^T P + 2\alpha P + C^T C + D^T (P + \bar{F}) D, \\ &\Xi_{12} = -A_{av}^T P - 2\alpha P, \quad \Xi_{1i} = -\Xi_{2i} = -P, \quad i = 3, 4, \\ &\Xi_{15} = -\Xi_{25} = PB, \quad \Xi_{22} = -\tfrac{4}{\varepsilon^* \mathcal{T}} e^{-2\alpha \varepsilon^* \mathcal{T}} R + 2\alpha P, \\ &\Xi_{33} = -\tfrac{2}{\varepsilon^* \mathcal{T}} e^{-2\alpha \varepsilon^* \mathcal{T}} H, \\ &\Xi_{44} = -\tfrac{1}{\varepsilon^* \mathcal{T}} e^{-2\alpha \varepsilon^* \mathcal{T}} F, \quad \Xi_{55} = -\gamma^2 I_{n_v} \end{aligned} \tag{3.28}$$

and other blocks are zero matrices. If LMIs (3.27) hold with $\alpha = 0$, then system (3.4), (2.17) has L_2-gain less than or equal to γ. If additionally (3.14) is satisfied with $\tilde{M}$ defined by (3.15), then system (2.1), (2.17) has L_2-gain less than γ for all $\varepsilon \in (0, \varepsilon^*]$. Moreover, if LMIs (3.27) hold with $\alpha > 0$ and $C = 0$, then system (3.1) is ISS (i.e. there exists $M_0 > 0$ such that the solutions of system (3.1) initialized by $x(0) \in \mathbb{R}^n$ satisfy (3.16) for all $\varepsilon \in (0, \varepsilon^*]$ and locally essentially bounded v). LMIs (3.27) are always feasible for small enough $\varepsilon^* > 0$, $\frac{1}{\gamma} > 0$, $\alpha > 0$, $\|C\|$ and $\|D\|$.

Proof. Consider the functional $V_2(t)$ given by (3.10). Using (3.9) we have

$$\begin{aligned} \mathcal{L} V_P(t, x(t)) &= 2[x(t) - G(t)]^T P [A_{av} x(t) - \textstyle\sum_{i=1}^2 Y_i(t) \\ &\quad + Bv(t)] + x^T(t) D^T P D x(t). \end{aligned} \tag{3.29}$$

For the $V_F(t)$-term given by (3.11), we find

$$\begin{aligned} &\mathcal{L} V_F(t) + 2\alpha V_F(t) \\ &= \tfrac{1}{\varepsilon^2 \mathcal{T}^2} x^T(t) D^T \textstyle\int_{t-\varepsilon\mathcal{T}}^t (s - t + \varepsilon \mathcal{T}) A^T(\frac{s}{\varepsilon}) F A(\frac{s}{\varepsilon}) ds D x(t) \\ &\quad - \tfrac{1}{\varepsilon^2 \mathcal{T}^2} \textstyle\int_{t-\varepsilon\mathcal{T}}^t \int_s^t e^{-2\alpha(t-\theta)} x^T(\theta) D^T A^T(\frac{s}{\varepsilon}) F A(\frac{s}{\varepsilon}) D x(\theta) d\theta ds. \end{aligned}$$

By changing variable $s = \varepsilon \zeta$ and employing (3.26) we have

$$\begin{aligned} &\tfrac{1}{\varepsilon^2 \mathcal{T}^2} \textstyle\int_{t-\varepsilon\mathcal{T}}^t (s - t + \varepsilon\mathcal{T}) A^T(\frac{s}{\varepsilon}) F A(\frac{s}{\varepsilon}) ds \\ &= \tfrac{1}{\mathcal{T}^2} \textstyle\int_{\frac{t}{\varepsilon}-\mathcal{T}}^{\frac{t}{\varepsilon}} (\zeta - \frac{t}{\varepsilon} + \mathcal{T}) A^T(\zeta) F A(\zeta) d\zeta \le \bar{F}. \end{aligned} \tag{3.30}$$

By using Jensen's inequality (3.87) in Fridman (2014)

$$\begin{aligned} &\varepsilon \mathcal{T} \mathbf{E} Y_2^T(t) F Y_2(t) \le \mathbf{E} \textstyle\int_{t-\varepsilon\mathcal{T}}^t \int_s^t x^T(\theta) D^T A^T(\frac{s}{\varepsilon}) dw(\theta) \\ &\quad \times F \textstyle\int_s^t A(\frac{s}{\varepsilon}) D x(\theta) dw(\theta) ds \end{aligned}$$

and Itô isometry property

$$\begin{aligned} &\mathbf{E} \textstyle\int_s^t x^T(\theta) D^T A^T(\frac{s}{\varepsilon}) dw(\theta) F \int_s^t A(\frac{s}{\varepsilon}) D x(\theta) dw(\theta) \\ &= \mathbf{E} \textstyle\int_s^t x^T(\theta) D^T A^T(\frac{s}{\varepsilon}) F A(\frac{s}{\varepsilon}) D x(\theta) d\theta, \quad s \in [t - \varepsilon\mathcal{T}, t], \end{aligned}$$

via Fubini's theorem we obtain

$$\begin{aligned} &\mathbf{E} \mathcal{L} V_F(t) + 2\alpha \mathbf{E} V_F(t) \le \mathbf{E} x^T(t) D^T \bar{F} D x(t) \\ &\quad - \tfrac{1}{\varepsilon \mathcal{T}} e^{-2\alpha\varepsilon\mathcal{T}} \mathbf{E} Y_2^T(t) F Y_2(t). \end{aligned} \tag{3.31}$$

In view of (3.29) and (3.31), taking into account (2.35) and (2.37) with $\dot{x}(t)$, $Y(t)$ respectively changed by $\tilde{f}(\frac{t}{\varepsilon})$, $Y_1(t)$, we have for all $\varepsilon \in (0, \varepsilon^*]$

$$\begin{aligned} &\mathbf{E} \mathcal{L} V_2(t) + 2\alpha \mathbf{E} V_2(t) + \mathbf{E} |Cx(t)|^2 - \gamma^2 |v(t)|^2 \\ &\le \mathbf{E} \xi_2^T(t) \Xi \xi_2(t) + \varepsilon^* \mathcal{T} \mathbf{E} [f^T(\tfrac{t}{\varepsilon}) R f(\tfrac{t}{\varepsilon}) + \tilde{f}^T(\tfrac{t}{\varepsilon}) \bar{H} \tilde{f}(\tfrac{t}{\varepsilon})], \end{aligned} \tag{3.32}$$

where $\xi_2^T(t) = [x^T(t), G^T(t), Y_1^T(t), Y_2^T(t), v^T(t)]$, Ξ is composed of (3.28) and $\bar{H}$ is defined by (2.31). Via (2.4) we can present $\tilde{f}(\frac{t}{\varepsilon})$ in (3.6) as follows

$$\tilde{f}(\tfrac{t}{\varepsilon}) = \textstyle\sum_{i=1}^N \rho_i(\frac{t}{\varepsilon}) A_i x(t) + Bv(t). \tag{3.33}$$

By substituting the first equation in (2.40) and (3.33) into (3.32) and applying further Schur complements, we conclude that if

$$\begin{bmatrix} \Xi & \begin{matrix} \sqrt{\varepsilon^* \mathcal{T}} \sum_{i=1}^N \rho_i(\frac{t}{\varepsilon}) A_i^T R & \sqrt{\varepsilon^* \mathcal{T}} \sum_{i=1}^N \rho_i(\frac{t}{\varepsilon}) A_i^T \bar{H} \\ 0_{3n\times n} & 0_{3n\times n} \\ 0_{n_v \times n} & \sqrt{\varepsilon^* \mathcal{T}} B^T \bar{H} \end{matrix} \\ \hline \begin{matrix} * \\ * \end{matrix} & \begin{matrix} -R & 0_{n\times n} \\ * & -\bar{H} \end{matrix} \end{bmatrix} < 0, \tag{3.34}$$

then (3.13) holds. LMIs (3.27) imply (3.34) (thus, (3.13)) since (3.34) is affine in $\sum_{i=1}^N \rho_i(\frac{t}{\varepsilon}) A_i^T$. Then the result follows from Lemma 3.1. The proof of the feasibility of LMIs (3.27) is similar to that of LMIs (2.32). □

4. Examples

Example 4.1 (*Khalil, 2002, Example 10.10: Vibrational Control*)**.** Consider the suspended pendulum with the suspension point that is subject to vertical vibrations of small amplitude and high frequency. We consider a linearized model at the upper equilibrium position (i.e. $x_1 = \pi$, $x_2 = 0$). Following the classical arguments for stochastic systems (see e.g. Shaikhet, 2013), in the linearized system we add multiplicative noise that models the error due to linearization (this error increases for larger state). We therefore consider

$$\begin{aligned} dx(t) &= \begin{bmatrix} \cos\frac{t}{\varepsilon} & 1 \\ \gamma_0^2 - \cos^2\frac{t}{\varepsilon} & -\gamma_0(\beta + \Delta\beta) - \cos\frac{t}{\varepsilon} \end{bmatrix} x(t) dt \\ &\quad + Bv(t) dt + Dx(t) dw(t). \end{aligned} \tag{4.1}$$

with $\gamma_0 > 0$ and $\beta > 0$. For the deterministic case, as in Fridman and Zhang (2020) we consider the uncertainty $\Delta\beta$ that stems from the uncertainties of friction coefficient and satisfies $|\Delta\beta| \le \beta_1$ with $\beta_1 \ge 0$, whereas for the stochastic case we consider $\Delta\beta = 0$. Since $A(\tau)$ in (4.1) with $\Delta\beta = 0$ is 2π-periodic, we verify **A1** with $\mathcal{T} = 2\pi$ and obtain as follows:

$$\begin{aligned} &A_{av} = \begin{bmatrix} 0 & 1 \\ \gamma_0^2 - 0.5 & -\gamma_0\beta \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 0 & 0 \\ 0 & -\gamma_0 \Delta\beta \end{bmatrix}, \\ &\sigma = \gamma_0 \beta_1. \end{aligned} \tag{4.2}$$

It follows from Theorem 10.4 of Khalil (2002) that for $\gamma_0^2 < 0.5$ and small enough ε, system (4.1) with $\Delta\beta = 0$ and $D = 0$ is exponentially stable. We choose $\gamma_0 = 0.2$ and $\beta = 1$ such that A_{av} in (4.2) is Hurwitz. Note that $\cos\tau \in [-1, 1]$ and $\cos^2\tau \in [0, 1]$. Therefore, $A(\tau)$ can be presented as a convex combination (2.4) of $N = 8$ constant matrices:

$$A_i = \begin{cases} \begin{bmatrix} 1 & 1 \\ -0.46 \pm 0.5 & -1.2 \pm 0.2\beta_1 \end{bmatrix}, & i = 1, \ldots, 4, \\ \begin{bmatrix} -1 & 1 \\ -0.46 \pm 0.5 & 0.8 \pm 0.2\beta_1 \end{bmatrix}, & i = 5, \ldots, 8. \end{cases} \tag{4.3}$$

As explained in Remark 2.2, to reduce inequality (2.31) to simple LMI, we assume $H = hI_2 > 0$ to be a scalar matrix. Let matrices Ω_1 and $\Omega_2(\zeta) = [\Omega_{ij}]$ be as follows:

$$\begin{aligned} &\Omega_1 = \mathrm{diag}\{0.0016, 1 + 0.04(1 + \beta_1)^2\}, \\ &\Omega_{11} = 0.92\cos^2\zeta + \cos^4\zeta, \\ &\Omega_{12} = \Omega_{21} = -0.008(1 + \Delta\beta) + 0.96\cos\zeta \\ &\quad + 0.2(1 + \Delta\beta)\cos^2\zeta + \cos^3\zeta, \\ &\Omega_{22} = 0.4(1 + \Delta\beta)\cos\zeta + \cos^2\zeta. \end{aligned}$$

After simple calculations, for all $\tau \geq \mathcal{T} = 2\pi$ we have

$$\begin{aligned} &\tfrac{1}{\mathcal{T}^2}\int_{\tau-\mathcal{T}}^{\tau}(\zeta-\tau+\mathcal{T})A^T(\zeta)HA(\zeta)d\zeta \\ &\leq \tfrac{h}{2}\Omega_1 + \tfrac{h}{\mathcal{T}}\int_{\tau-\mathcal{T}}^{\tau}\Omega_2(\zeta)d\zeta = h\Omega, \\ &\Omega = \begin{bmatrix} 0.8358 & 0.092(1+\Delta\beta) \\ * & 0.02(1+\beta_1)^2+1 \end{bmatrix}. \end{aligned} \tag{4.4}$$

Since $\Delta\beta \in [-\beta_1, \beta_1]$, Ω in (4.4) can be presented as a convex combination $\Omega = \sum_{i=1}^2 \tilde{\rho}_i \Omega^{(i)}$, where $\sum_{i=1}^2 \tilde{\rho}_i = 1$ and $\tilde{\rho}_i \geq 0$, with two constant matrices

$$\Omega^{(i)} = \begin{bmatrix} 0.8358 & 0.092(1\pm\beta_1) \\ * & 0.02(1+\beta_1)^2+1 \end{bmatrix}, \quad i=1,2. \tag{4.5}$$

The latter leads to the choice $\bar{H} = h\sum_{i=1}^2 \tilde{\rho}_i\Omega^{(i)}$ in (2.31).

In the deterministic case, i.e. $D = 0$, we first choose $\beta_1 = \sigma = 0$, where the number of vertices in (4.3) becomes $N = 4$ and $\bar{H} = h\Omega^{(1)}$ with $\beta_1 = 0$. By verifying the feasibility of LMIs (2.32) in the 4 vertices and using $\bar{H} = h\Omega^{(1)}$ with $\beta_1 = 0$, we find the maximum value of ε^* (see Table 4.1) that guarantees the exponential stability (and thus ISS) of system (4.1) with $D = 0$ for all $\varepsilon \in (0, \varepsilon^*]$ either with a small enough decay rate (for $\alpha = 0$) or with a decay rate $\alpha = \frac{1}{10\pi}$. It is clear that our method allows to essentially improve the results of Fridman and Zhang (2020) in terms of essentially larger ε^*, whereas LMIs are not more complicated.

We next choose $\beta_1 = 0.1$ leading to $\sigma = 0.02$ (cf. (4.2)). By verifying the feasibility of LMIs (2.32) in the 16 vertices that correspond to the 8 vertices in (4.3) and $\bar{H} = h\Omega^{(1)}$ or $\bar{H} = h\Omega^{(2)}$, we find the smaller maximum values of ε^* comparatively to the case of $\sigma = 0$ in Table 4.1 (in brackets we show ε^* achieved in Fridman & Zhang, 2020):

$$\alpha = 0: \varepsilon^* = 0.0058\ (0.0013); \quad \alpha = \tfrac{1}{10\pi}: \varepsilon^* = 0.0034\ (0.0007)$$

that guarantee the exponential stability (and thus ISS) of system (4.1) with $D = 0$ for all $\varepsilon \in (0, \varepsilon^*]$ either with a small enough decay rate (for $\alpha = 0$) or with a decay rate $\alpha = \frac{1}{10\pi}$. Note that we first choose $\sigma = 0$ and find the corresponding value of ε^*. Then we choose a small positive $\sigma = 0.02$ that still guarantees a reasonable (but smaller) $\varepsilon^* > 0$.

In the stochastic case, we consider $\Delta\beta = 0$ (thus, $\beta_1 = \sigma = 0$) and $D = 0.05I_2$. From (4.4), it follows that the choice of $\bar{H}$ in (2.31) is $\bar{H} = h\Omega^{(1)}$ with $\beta_1 = 0$. Similarly, for (3.26) we choose $\bar{F} = \eta\Omega^{(1)}$ with $\beta_1 = 0$ and with a scalar $\eta > 0$ to be determined. By verifying the feasibility of LMIs (3.27) with $D = 0.05I_2$ in the 4 vertices (4.3) with $\beta_1 = 0$, and using $\bar{H} = h\Omega^{(1)}$ and $\bar{F} = \eta\Omega^{(1)}$ with $\beta_1 = 0$, we find the maximum value of ε^* (see Table 4.1) that guarantees the exponential stability (and thus ISS) of system (4.1) with $D = 0.05I_2$ for all $\varepsilon \in (0, \varepsilon^*]$ either with a small enough decay rate (for $\alpha = 0$) or with a decay rate $\alpha = \frac{1}{10\pi}$. Note that Fridman and Zhang (2020) is not applicable to the stochastic case.

We further consider L_2-gain analysis of system (4.1) with $B = [1, 0]^T$, $\Delta\beta = 0$, $D = 0.05I_2$ and the controlled output:

$$y(t) = [0.2, 0.1]x(t). \tag{4.6}$$

For $\alpha = 0$ and $\varepsilon^* = 0.002$, by verifying LMIs (3.27) in the vertices (4.3) with $\beta_1 = 0$, and using $\bar{H} = h\Omega^{(1)}$ and $\bar{F} = \eta\Omega^{(1)}$ with $\beta_1 = 0$, we find the minimum value of $\gamma = 1.54$. From (3.15) where we choose $a = \sqrt{\sup \operatorname{trace}\{A^T(\frac{t}{\varepsilon})A(\frac{t}{\varepsilon})\}}$, we find $\tilde{M} = 964$. Clearly, $\varepsilon^*\tilde{M} - \gamma^2 < 0$ holds implying that system (4.1), (4.6) has L_2-gain less than $\gamma = 1.54$. We further perform numerical simulations for (4.1) with $x(0) = 0$, $\varepsilon = 0.002$ and choose disturbances $v_1(t) = \sin(t)$ and $v_2(t) = 1$ if $t \leq 10$ and $v_1(t) = v_2(t) = 0$ otherwise. Simulations confirm our theoretical result that $\mathbf{E}J < 0$ holds with $\gamma = 1.54$. Fig. 1 (left) plots $|x|$ with $v_i(t)$ $(i = 1, 2)$.

Example 4.2 (*Hetel & Fridman, 2013: Stabilization by Fast Switching*)**.** Consider a stochastic switched system

$$dx(t) = \begin{cases} [A_1x(t) + Bv(t)]dt + Dx(t)dw(t), \\ \qquad t \in [k\varepsilon, k\varepsilon + \beta\varepsilon), \\ [A_2x(t) + Bv(t)]dt + Dx(t)dw(t), \\ \qquad t \in [k\varepsilon + \beta\varepsilon, (k+1)\varepsilon), \end{cases} \tag{4.7}$$

where $\varepsilon > 0$, $k = 0, 1, \ldots$ and $\beta \in (0, 1)$, with unstable modes

$$A_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.6 & -0.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.13 & -0.16 \\ -0.33 & 0.03 \end{bmatrix}. \tag{4.8}$$

Then (4.7) can be presented as (3.1) with

$$A(\tau) = \textstyle\sum_{i=1}^2 \rho_i(\tau)A_i, \quad \tau \in [k, k+1), \quad k = 0, 1, \ldots,$$

where $\rho_1(\tau) = \chi_{[k,k+\beta)}(\tau)$ is the indicator function of $[k, k+\beta)$, $\rho_2(\tau) = 1 - \chi_1(\tau)$. Note that since $A(\tau)$ is not continuous, Theorem 10.4 of Khalil (2002) is not applicable here for $D = 0$. It is clear that in this example $A(\tau)$ is $\mathcal{T} = 1$-periodic and $\Delta A = 0$ and $\sigma = 0$. We choose $\beta = 0.4$ that leads to Hurwitz $A_{av} = \beta A_1 + (1-\beta)A_2$. For all $\tau \geq \mathcal{T} = 1$, inequality (2.31) in this example

$$\begin{aligned} &\textstyle\int_{\tau-1}^{\tau}(\zeta-\tau+1)A^T(\zeta)HA(\zeta)d\zeta \\ &\leq \textstyle\int_{\tau-\beta}^{\tau}(\zeta-\tau+1)dsA_1^THA_1 \\ &\quad + \textstyle\int_{\tau-(1-\beta)}^{\tau}(\zeta-\tau+1)dsA_2^THA_2 \\ &= \tfrac{1-(1-\beta)^2}{2}A_1^THA_1 + \tfrac{1-\beta^2}{2}A_2^THA_2 \end{aligned}$$

holds with

$$\bar{H} = \tfrac{1-(1-\beta)^2}{2}A_1^THA_1 + \tfrac{1-\beta^2}{2}A_2^THA_2. \tag{4.9}$$

Similarly, we find that (3.26) holds with

$$\bar{F} = \tfrac{1-(1-\beta)^2}{2}A_1^TFA_1 + \tfrac{1-\beta^2}{2}A_2^TFA_2. \tag{4.10}$$

By verifying the feasibility of LMIs (2.32) with $D = 0$, and (3.27) with $D = 0.05I_2$ in the 2 vertices (4.8) and using (4.9) and (4.10), we find the maximum value of ε^* (see Table 4.1) that guarantees the exponential stability (and thus ISS) of (4.7) for all $\varepsilon \in (0, \varepsilon^*]$ either with a small enough decay rate (for $\alpha = 0$) or with a decay rate $\alpha = 0.005$. In the deterministic case, our method allows to improve (Fridman & Zhang, 2020) by more than 40%, where for ISS LMIs in Fridman and Zhang (2020) have 28 lines and 15 decision variables and our LMIs have the same number of lines and 14 decision variables. In the stochastic case our method leads to efficient results whereas Fridman and Zhang (2020) is not applicable.

Let $B = [0, 1]^T$, $D = 0.05I_2$, $\alpha = 0$ and $\varepsilon^* = 0.002$. By verifying the feasibility of LMIs (3.27) in the two vertices (4.8) and using (4.9) and (4.10), we find the minimum value of $\gamma = 7.34$. From (3.15), we find $\tilde{M} = 1.4554 \times 10^4$. Clearly, $\varepsilon^*\tilde{M} - \gamma^2 < 0$ holds implying that system (4.7), (4.6) has L_2-gain less than $\gamma = 7.34$. By performing numerical simulations of solutions to the stochastic system with $x(0) = 0$, $\varepsilon = 0.002$ and the same disturbances as in Example 4.1, it confirms $\mathbf{E}J < 0$ with $\gamma = 7.34$. Fig. 1 (right) plots $|x|$ with $v_i(t)$ $(i = 1, 2)$.

5. Conclusions

This paper has presented an improved time-delay method to periodic averaging that allows to derive essentially less conservative LMIs for the upper bound on the small parameter preserving the exponential stability, L_2-gain and ISS of linear systems with piecewise-continuous fast-varying coefficients. The method has been extended to systems with multiplicative noise. The suggested method may be applied in the future to various control problems that employ averaging, e.g. to power systems (Krein et al., 1990) and stochastic extremum seeking (Liu & Krstic, 2012).

Table 4.1
Maximum value of ε^* and numerical complexity of LMIs for ISS.

		Example 4.1 ($N=4$)		Example 4.2 ($N=2$)		No. LMI lines	No. dec. vars
		$\alpha=0$	$\alpha=\frac{1}{10\pi}$	$\alpha=0$	$\alpha=0.005$		
$D=0$, $\sigma=0$:	Fridman and Zhang (2020)	0.0031	0.0021	0.1363	0.0930	$nN(N+4)+2n_vN$	$\frac{(N+2)(n^2+n)+6}{2}$
	Theorem 2.1	0.0074	0.0050	0.1920	0.1306	$n(6N+1)+n_vN$	$2n^2+2n+2$
$D=0.05I_2$:	Theorem 3.1	0.0066	0.0043	0.1164	0.0698	$n(6N+1)+n_vN$	$3n^2+3n+1$

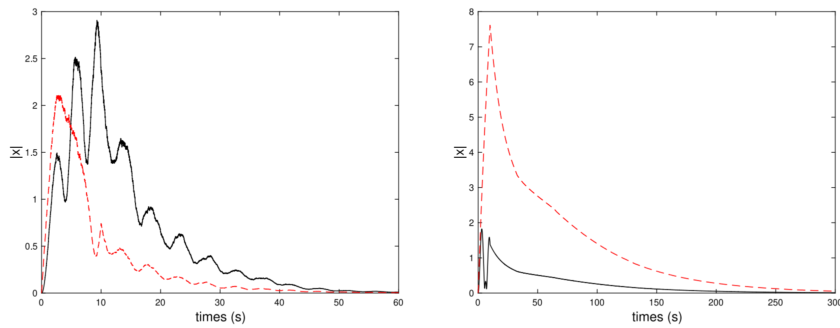

Fig. 1. Dynamics of (4.1) (left) and (4.7) (right) with $v_1(t)$ (black solid line) and with $v_2(t)$ (red dashed line).

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Jin Zhang received the Ph.D. degree in control theory and control engineering from Shanghai University, Shanghai, China, in 2018. Since October 2018, he has been a Post-Doctoral Fellow with the School of Electrical Engineering, Tel Aviv University, Tel Aviv, Israel. His research interests include time-delay systems, networked control systems, and event-triggered control.

Emilia Fridman received the M.Sc. degree from Kuibyshev State University, USSR, in 1981 and the Ph.D. degree from Voronezh State University, USSR, in 1986, all in mathematics. From 1986 to 1992 she was an Assistant and Associate Professor in the Department of Mathematics at Kuibyshev Institute of Railway Engineers, USSR. Since 1993 she has been at Tel Aviv University, where she is currently Professor of Electrical Engineering-Systems. She has held visiting positions at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin (Germany), INRIA in Rocquencourt (France), Ecole Centrale de Lille (France), Valenciennes University (France), Leicester University (UK), Kent University (UK), CINVESTAV (Mexico), Zhejiang University (China), St. Petersburg IPM (Russia), Melbourne University (Australia), Supelec (France), KTH (Sweden).

Her research interests include time-delay systems, networked control systems, distributed parameter systems, robust control, singular perturbations and nonlinear control. She has published 200 articles in international scientific journals and the author of two monographs. She serves/served as Associate Editor in Automatica, SIAM Journal on Control and Optimization and IMA Journal of Mathematical Control and Information. In 2014 she was Nominated as a Highly Cited Researcher by Thomson ISI. Since 2018, she has been the incumbent for Chana and Heinrich Manderman Chair on System Control at Tel Aviv University. She is IEEE Fellow since 2019. She is currently a member of the IFAC Council. In 2021 she was recipient of IFAC Delay Systems Life Time Achievement Award and of Kadar Award for outstanding research at Tel Aviv University.