Constructive robust stabilization by using square wave dithers: A time-delay approach

Jin Zhang and Emilia Fridman, *Fellow, IEEE*

***Abstract*—This paper studies robust stabilization of the second- and third-order (with relative degree 3) linear uncertain systems by a fast-varying square wave dither with high frequency $\frac{1}{\varepsilon}$ and high gain, where $\varepsilon > 0$ is small. In contrast to the existing methods for control by fast oscillations that are all qualitative, we present constructive quantitative results for finding an upper bound on ε that ensures the exponential stability. Our method consists of two steps: 1) we construct appropriate coordinate transformations that cancel the high-gains and lead to a stable averaged system, 2) we apply the time-delay approach to periodic averaging of the system in new coordinates and derive linear matrix inequalities for finding an upper bound on ε. Three numerical examples illustrate the efficiency of the method.**

***Index Terms*—Stabilization by fast oscillations, averaging, time-delay systems**

I. Introduction

The theory of vibrational control was developed to stabilize linear/nonlinear systems by a fast-varying dither (with zero mean value) that depends on a small parameter $\varepsilon > 0$, see e.g. [1]–[6] and the references therein. These works rely on the coordinate transformation introduced in [2] that allows the application of classical averaging (see Chapter 10 in [7]) leading to various stability results. Besides, Brockett's problem of stabilization of linear systems by static output-feedback with a time-varying gain, where the system is not stabilizable by constant gain, was formulated in [8]. Some solutions to this problem were given in [9]–[12] by using sine and cosine dithers. The only solution to Brocket's problem via square wave dither of the form $\operatorname{sgn}\sin(\frac{2\pi t}{\varepsilon})$ was suggested in [13] for the case, where one coordinate transformation from [2] leads to the stable averaged system. Stabilization by square wave dither in the cases, where two or more transformations are needed (e.g. systems with relative degree 3 as studied in [10]) remains an open problem. Moreover, constructive conditions for finding for all kinds of fast-varying dithers, an upper bound on the small parameter that ensures the stability are missing. Till now such bounds could be found from simulations only, which is not reliable for the uncertain systems.

A constructive time-delay approach to periodic averaging was introduced recently in [14]. By using a backward averaging, the system is transformed to a time-delay system where the delay length is equal to the small parameter. Then direct Lyapunov-Krasovskii (L-K) method (see e.g. [15]) leads to linear matrix inequalities (LMIs) that allow to find an efficient upper bound on the small parameter preserving the exponential stability and input-to-state stability (ISS) of the time-delay system (and thus, of the original one). Recently, an improved time-delay approach to periodic averaging has been provided in [16] with fewer terms to be compensated in the L-K analysis leading to less conservative and simpler LMI conditions. However, the averaging method of [14], [16] cannot be directly applied to vibrational control systems due to the high gains multiplying the square wave dithers.

This paper studies robust stabilization of the second- and third-order (with relative degree 3) linear uncertain systems by square wave dithers. In contrast to the existing methods for stabilization by fast-varying dithers that are all qualitative, we present constructive quantitative results for finding an upper bound on ε that ensures the stability. Our method consists of two steps: 1) we construct appropriate coordinate transformations that cancel the high gains and lead to a stable averaged system, 2) we apply the time-delay approach to periodic averaging of the system in new coordinates and derive LMIs for finding an upper bound on ε. The main challenge is the choice of coordinate transformations that leads to efficient averaging with less conservative results. Note that the transformation in [13], [17] seems not to be applicable to the third-order systems, where it is difficult to find the second transformation leading to a stable averaged system. Our stabilizability conditions coincide with those given in [9]–[11] for the case of sine and cosine dithers, but we give the first full solution to the Brocket's problem that includes an upper bound on ε ensuring a desired performance (exponential decay rate).

In the conference version of this paper [17] the results were confined to the second-order systems without uncertainties, whereas the dither had the form of $\operatorname{sgn}\sin(\frac{2\pi t}{\varepsilon})$ as in [13] and the results were essentially more conservative. We summarize the contribution as follows:

1) We consider, for the first time, stabilization of the third-order systems with relative degree 3 by the dither $\operatorname{sgn}\cos(\frac{2\pi t}{\varepsilon})$ that leads to efficient stability conditions;
2) For the second-order systems, we provide less conservative results in the examples compared to [17]. This is due to the new transformation corresponding to the considered dither $\operatorname{sgn}\cos(\frac{2\pi t}{\varepsilon})$ with a smaller amplitude of time-varying terms than in [17].
3) In both cases, we present a novel comparatively to [14], [16] stability analysis via averaging for systems with slowly-varying norm-bounded uncertainties.

Throughout the paper $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices with the induced matrix norm $\|\cdot\|$. The superscript T stands for the transposition, and the notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix are denoted by $*$.

II. Robust stabilization by square wave dithers

In this section, we will study robust static output-feedback stabilization of linear uncertain system

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + Bu(t), \quad y(t) = Cx(t) \tag{1}$$

with the state $x(t) \in \mathbb{R}^n$, where $n = 2$ or 3, the input $u(t) \in \mathbb{R}$, the output $y(t) \in \mathbb{R}$, constant $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$ and $C \in \mathbb{R}^{1\times n}$ to

This work was supported by Israel Science Foundation under Grant 673/19, C. and H. Manderman Chair on System Control at Tel Aviv University, and the Planning and Budgeting Committee (PBC) Fellowship from the Council for Higher Education, Israel. Jin Zhang's work was done when he was with Tel Aviv University. (Corresponding author: Jin Zhang).

Jin Zhang is with the School of Mechatronic Engineering and Automation, Shanghai University, Shanghai 200072, China, and also with the School of Electrical Engineering, Tel Aviv University, Tel Aviv 69978, Israel (e-mail: zhangjin1116@126.com).

Emilia Fridman is with the School of Electrical Engineering, Tel Aviv University, Tel Aviv 69978, Israel (e-mail: emilia@tauex.tau.ac.il).

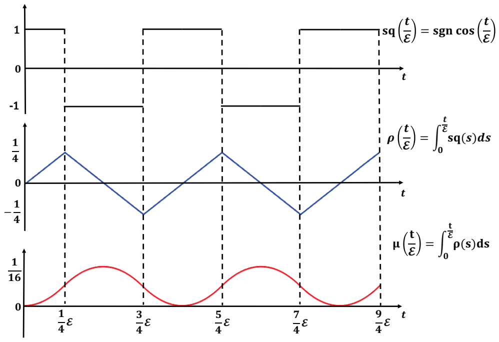

Fig. 1. The square wave dither $\mathbf{sq}(\frac{t}{\varepsilon})$, functions $\boldsymbol{\rho}(\frac{t}{\varepsilon})$ and $\boldsymbol{\mu}(\frac{t}{\varepsilon})$.

be given later, and a time-varying uncertain matrix $\Delta A(t) \in \mathbb{R}^{n\times n}$ satisfying the following inequality

$$\|\Delta A(t)\| \le \sigma_0 \quad \forall t \ge 0. \tag{2}$$

Here $\sigma_0 > 0$ is a small constant. In the case of $\Delta A(t) = 0$, stabilization of system (1) by using sine/cosine wave dithers was studied in [9]- [12], where necessary and sufficient conditions were found that guarantees the stability provided the dither frequency is high enough. However the lower bound on the frequency could be found from simulations only, which is not reliable especially for the uncertain systems that we consider.

Our objective is robust stabilization of the uncertain system (1) by square wave dither with the first constructive and efficient bounds on the dither frequency. We consider for system (1) the novel square wave dither (comparatively to the dither $\mathrm{sgn}\sin(\frac{2\pi t}{\varepsilon})$ in [13], [17]) defined by

$$\begin{aligned}\mathrm{sq}(\tfrac{t}{\varepsilon}) &= \mathrm{sgn}\cos(\tfrac{2\pi t}{\varepsilon})\\ &= \begin{cases}1, & \frac{t}{\varepsilon}\in[j,j+\frac14) \text{ or } [j+\frac34,j+1),\\ -1, & \frac{t}{\varepsilon}\in[j+\frac14,j+\frac34), \qquad j\in\mathbb{N}_0\end{cases}\end{aligned} \tag{3}$$

with a small parameter $\varepsilon > 0$ that is inverse of the dither frequency. See its plot in the top of Fig. 1. Note that if the vibrational control is considered as an open-loop control with a dither to be a control signal (as e.g. in [2]), then this square wave presents a sampled-data implementation of the vibrational control.

A. Second-order linear uncertain system

Consider the second-order linear uncertain system (1) with $x(t) \in \mathbb{R}^2$, $\Delta A(t) \in \mathbb{R}^{2\times 2}$ satisfying (2), and

$$A = \begin{bmatrix}0 & 1\\ a_1 & a_2\end{bmatrix}, \quad B = \begin{bmatrix}0\\ b\end{bmatrix}, \quad C = \begin{bmatrix}c_1 & c_2\end{bmatrix}. \tag{4}$$

Here $a_1 \ge 0$ (implying that A is not Hurwitz), $a_2 < 0$, b, c_1 and $c_2 \ne 0$ are constants. This system with $\Delta A(t) = 0$ may be not stabilizable by a static time-invariant output-feedback, but may be stabilizable by a static time-varying output-feedback [1], [2], [9] (see e.g. Example 2 below borrowed from [12]).

We will study stabilization of system (1), (4) by a fast-varying output-feedback controller

$$u(t) = \tfrac{k}{\varepsilon}\mathrm{sq}(\tfrac{t}{\varepsilon})y(t), \tag{5}$$

where k is a scalar controller gain and $\mathrm{sq}(\frac{t}{\varepsilon})$ is defined by (3). The closed-loop system has the form

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + \tfrac{k}{\varepsilon}\mathrm{sq}(\tfrac{t}{\varepsilon})BCx(t), \quad t \ge 0. \tag{6}$$

System (1), (4) with $\Delta A(t) = 0$ is exponentially stabilizable by $u(t) = \frac{k}{\varepsilon}\cos(\frac{2\pi t}{\varepsilon})y(t)$ with appropriate k and small enough $\varepsilon > 0$ iff the following holds [9]: $a_2 < 0$ and

$$a_1c_2^2 - a_2c_1c_2 - c_1^2 < 0. \tag{7}$$

We will show that the same conditions guarantee stabilization by the square wave dither (3). This will be done by using the coordinate transformation from [2] and application of the time-delay approach to averaging (see e.g. [14], [16]) of the transformed system. Note that the direct averaging is not applicable to (6) (and (21), (25) below) since the averaged system is unstable for non-Hurwitz A. Moreover, the term $\frac{k}{\varepsilon}\mathrm{sq}(\frac{t}{\varepsilon})BCx(t)$ cannot be compensated in the L-K analysis since it is of order $\mathcal{O}(\frac{1}{\varepsilon})$ (i.e. large for small $\varepsilon > 0$) provided x is of order $\mathcal{O}(1)$.

To cancel the large term $\frac{k}{\varepsilon}\mathrm{sq}(\frac{t}{\varepsilon})BCx(t)$, we consider the following generating equation

$$\tfrac{d}{dt}\phi(\tfrac{t}{\varepsilon}) = \tfrac{k}{\varepsilon}\mathrm{sq}(\tfrac{t}{\varepsilon})BC\phi(\tfrac{t}{\varepsilon}), \quad t \ge 0. \tag{8}$$

Denote by $\Phi(\frac{t}{\varepsilon}, s)$ the fundamental matrix of equation (8). For the transformation of system (6), we will employ

$$\Phi(\tfrac{t}{\varepsilon},0) = e^{k\rho(\frac{t}{\varepsilon})BC} = \begin{bmatrix}1 & 0\\ \frac{c_1}{c_2}e^{kbc_2\rho(\frac{t}{\varepsilon})} - \frac{c_1}{c_2} & e^{kbc_2\rho(\frac{t}{\varepsilon})}\end{bmatrix}, \tag{9}$$

where

$$\rho(\tfrac{t}{\varepsilon}) = \begin{cases}\frac{t}{\varepsilon} - j, & \frac{t}{\varepsilon}\in[j,j+\frac14),\\ -\frac{t}{\varepsilon}+j+\frac12, & \frac{t}{\varepsilon}\in[j+\frac14,j+\frac34),\\ \frac{t}{\varepsilon}-j-1, & \frac{t}{\varepsilon}\in[j+\frac34,j+1)\end{cases} \tag{10}$$

with $j \in \mathbb{N}_0$. See the plot of ρ in the middle of Fig. 1. It is clear that $\rho(\frac{t}{\varepsilon})$ (and thus $\Phi(\frac{t}{\varepsilon},0)$ in (9)) is ε-periodic. Note that for the case of $c_2 = 0$ in (4), we can obtain the corresponding $\Phi(\frac{t}{\varepsilon},0)$ by using the limit of (9) as c_2 approaches zero:

$$\Phi(\tfrac{t}{\varepsilon},0) = \begin{bmatrix}1 & 0\\ kbc_1\rho(\frac{t}{\varepsilon}) & 1\end{bmatrix}.$$

Introduce the coordinate transformation

$$x(t) = \Phi(\tfrac{t}{\varepsilon},0)\zeta(t), \quad t \ge 0. \tag{11}$$

Since $\|\Phi(\frac{t}{\varepsilon},0)\|$ and $\|\Phi^{-1}(\frac{t}{\varepsilon},0)\|$ are uniformly bounded for all $t \ge 0$, this coordinate transformation is stability preserving. Taking the derivative with respect to t in (11) and using the relation $\frac{d}{dt}\Phi(\frac{t}{\varepsilon},0) = \frac{k}{\varepsilon}\mathrm{sq}(\frac{t}{\varepsilon})BC\Phi(\frac{t}{\varepsilon},0)$ $\forall t \ge 0$, we obtain

$$\dot{x}(t) = \tfrac{k}{\varepsilon}\mathrm{sq}(\tfrac{t}{\varepsilon})BC\Phi(\tfrac{t}{\varepsilon},0)\zeta(t) + \Phi(\tfrac{t}{\varepsilon},0)\dot{\zeta}(t), \quad t \ge 0. \tag{12}$$

Substituting the right-hand side of (6) for $\dot{x}(t)$ and taking into account that matrix $\Phi(\frac{t}{\varepsilon},0)$ is nonsingular for all $t \ge 0$, we obtain for $t \ge 0$

$$\dot{\zeta}(t) = [\mathcal{A}(\tfrac{t}{\varepsilon}) + \Delta\mathcal{A}(t)]\zeta(t), \quad \mathcal{A}(\tfrac{t}{\varepsilon}) = \Phi^{-1}(\tfrac{t}{\varepsilon},0)A\Phi(\tfrac{t}{\varepsilon},0) \tag{13}$$

with

$$\Delta\mathcal{A}(t) = \Phi^{-1}(\tfrac{t}{\varepsilon},0)\Delta A(t)\Phi(\tfrac{t}{\varepsilon},0). \tag{14}$$

Then the averaged system of (13) with $\Delta A(t) = 0$ has the form

$$\dot{\zeta}_{av}(t) = \mathcal{A}_{av}\zeta_{av}(t), \tag{15}$$

where $\zeta_{av}(t) \in \mathbb{R}^2$ and

$$\begin{aligned}\mathcal{A}_{av} &= \tfrac{1}{\varepsilon}\int_0^{\varepsilon}\mathcal{A}(\tfrac{s}{\varepsilon})ds\\ &= \begin{bmatrix}\mathbf{a}_{11} & \frac{4}{kbc_2}\sinh(\frac{kbc_2}{4})\\ \mathbf{a}_{21} & \mathbf{a}_{22}\end{bmatrix},\\ \mathbf{a}_{11} &= \tfrac{4c_1}{kbc_2^2}\sinh(\tfrac{kbc_2}{4}) - \tfrac{c_1}{c_2},\\ \mathbf{a}_{21} &= \tfrac{4(a_1c_2^2 - a_2c_1c_2 - 2c_1^2)}{kbc_2^3}\sinh(\tfrac{kbc_2}{4}) + \tfrac{a_2c_1}{c_2} + \tfrac{2c_1^2}{c_2^2},\\ \mathbf{a}_{22} &= a_2 - \tfrac{4c_1}{kbc_2^2}\sinh(\tfrac{kbc_2}{4}) + \tfrac{c_1}{c_2}.\end{aligned} \tag{16}$$

Matrix $\mathcal{A}_{av}$ has the following characteristic polynomial:

$$\begin{array}{l} s^2 - a_2 s - \left[\frac{a_2 c_1}{c_2} + \frac{c_1^2}{c_2^2} + \frac{8(\cosh(\frac{kbc_2}{2})-1)}{k^2b^2c_2^4} \right. \\ \left. \times (a_1 c_2^2 - a_2 c_1 c_2 - c_1^2)\right]. \end{array} \quad (17)$$

By using the Routh-Hurtwiz stability criterion, we arrive at:

Lemma 1: Let $a_2 < 0$. Matrix $\mathcal{A}_{av}$ given by (16) is Hurtwiz iff

$$\frac{a_2c_1}{c_2} + \frac{c_1^2}{c_2^2} + \frac{8(\cosh(\frac{kbc_2}{2})-1)}{k^2b^2c_2^4}(a_1c_2^2 - a_2c_1c_2 - c_1^2) < 0. \quad (18)$$

Moreover, if (7) holds, then inequality (18) is always feasible for large enough $|k|$.

Indeed, the following holds for $k \neq 0$:

$$0 < \frac{8(\cosh(\frac{kbc_2}{2})-1)}{k^2b^2c_2^4} = \frac{1}{c_2^2} + \frac{2}{c_2^2}\sum_{i=2}^{\infty}\frac{1}{(2i)!}(\frac{kbc_2}{2})^{2(i-1)}.$$

Supposing that as in [9] (7) holds, we obtain

$$\begin{array}{l} 0 \overset{(7)}{>} \frac{8(\cosh(\frac{kbc_2}{2})-1)}{k^2b^2c_2^4}(a_1c_2^2 - a_2c_1c_2 - c_1^2) \\ = a_1 - \frac{a_2c_1}{c_2} - \frac{c_1^2}{c_2^2} \\ + \frac{2(a_1c_2^2 - a_2c_1c_2 - c_1^2)}{c_2^2}\sum_{i=2}^{\infty}\frac{1}{(2i)!}(\frac{kbc_2}{2})^{2(i-1)}. \end{array}$$

Thus, if (7) holds, the feasibility of

$$a_1 + \frac{2(a_1c_2^2 - a_2c_1c_2 - c_1^2)}{c_2^2}\sum_{i=2}^{\infty}\frac{1}{(2i)!}(\frac{kbc_2}{2})^{2(i-1)} < 0$$

(thus, of inequality (18)) is always guaranteed for large enough $|k|$.

From (17) it follows that as $|k|$ increases the product of two eigenvalues of matrix $\mathcal{A}_{av}$ given by (16) increases whereas their sum is fixed as a_2. Thus, when $|k|$ is large enough, the real parts of eigenvalues are $\frac{a_2}{2}$ implying that the decay rate $\alpha \geq 0$ to be found via LMIs of Theorem 1 or Corollary 1 below should be smaller than $-\frac{a_2}{2}$ (which is independent of k). However, the fast-varying matrix $\mathcal{A}(\frac{t}{\varepsilon})$ defined in (13) corresponding to a larger $|k|$ belongs to a larger polytope in the presentation (34) below leading to more conservative results in the second-order examples (as well as the third-order example) below. We will show how to select an appropriate k in the design, see Remark 3 below.

B. Third-order linear uncertain system

We now consider the third-order linear uncertain system (1) with $x(t) \in \mathbb{R}^3$, $\Delta A(t) \in \mathbb{R}^{3\times3}$ satisfying (2), and

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}, \quad C = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}^T. \quad (19)$$

Here $\sigma_0 > 0$, a_1, a_2, $a_3 < 0$, b and c are constants. For small enough $\sigma_0 > 0$, system (1), (19) is exponentially stabilizable by a static time-invariant output-feedback iff $a_2 < 0$ and $a_3 < 0$ [10]. Therefore, our main interest of this paper is to study stabilization of system (1), (19) subject to $a_2 \geq 0$ (implying that A is not Hurwitz) and $a_3 < 0$. Moreover, it is clear that the relative degree of system (1), (19) is 3 since [18]

$$CB = CAB = 0, \quad CA^2B \neq 0.$$

Remark 1: If the vector $C = \begin{bmatrix} c & c_2 & c_3 \end{bmatrix}$ has non-zero c_2 or c_3, it may happen that controller (5) (the same as for the second-order system) stabilizes the system for small enough $\varepsilon > 0$ (see e.g. [13] and the example of Section 4.1 therein). However, when $c_2 = c_3 = 0$ (in the case of relative degree 3), the averaged system (15) with the characteristic polynomial

$$s^3 - a_3 s^2 - a_2 s - a_1$$

is unstable for $a_2 \geq 0$. Indeed, by the Routh-Hurtwiz criterion, the system is stable iff the following holds:

$$a_3 < 0, \quad a_2 < 0, \quad a_1 < 0, \quad a_2a_3 + a_1 > 0,$$

which contradicts to $a_2 \geq 0$. Thus, controller (5) does not stabilize (1), (19) with $\Delta A(t) = 0$.

We design for system (1), (19) with relative degree 3 a fast-varying output-feedback controller

$$u(t) = \frac{k}{\varepsilon^2}\mathrm{sq}(\tfrac{t}{\varepsilon})y(t) \quad (20)$$

that leads to the following closed-loop system

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + \frac{k}{\varepsilon^2}\mathrm{sq}(\tfrac{t}{\varepsilon})BCx(t), \quad t \geq 0. \quad (21)$$

From [10], it follows that system (1), (19) with relative degree 3 and $\Delta A(t) = 0$ is exponentially stabilizable by $u(t) = \frac{k}{\varepsilon^2}\sin(\frac{2\pi t}{\varepsilon})y(t)$ with appropriate k and small enough $\varepsilon > 0$ iff $a_3 < 0$. We will show that the same condition $a_3 < 0$ guarantees stabilization by the square wave dither (3).

Following the second-order case, we consider

$$\frac{d}{dt}\phi_0(\tfrac{t}{\varepsilon^2}) = \frac{k}{\varepsilon^2}\mathrm{sq}(\tfrac{t}{\varepsilon})BC\phi_0(\tfrac{t}{\varepsilon^2}), \quad t \geq 0. \quad (22)$$

Then the fundamental matrix $\Phi_0(\frac{t}{\varepsilon^2}, 0)$ of equation (22) is obtained as

$$\Phi_0(\tfrac{t}{\varepsilon^2}, 0) = e^{\frac{k}{\varepsilon}\rho(\frac{t}{\varepsilon})BC} = I + \frac{k}{\varepsilon}\rho(\tfrac{t}{\varepsilon})BC \quad (23)$$

with $\rho(\cdot)$ given by (10), where we used $e^X = \sum_{i=0}^{\infty}\frac{1}{i!}X^i$ for all $X \in \mathbb{R}^{n\times n}$ with the fact $CB = 0$. Introduce the following stability preserving coordinate transformation

$$x(t) = \Phi_0(\tfrac{t}{\varepsilon^2}, 0)\zeta_0(t), \quad t \geq 0. \quad (24)$$

By using (23) with the fact $CAB = 0$, we transform system (21) to the following system:

$$\begin{array}{l} \dot{\zeta}_0(t) = [A + \Phi_0^{-1}(\frac{t}{\varepsilon^2}, 0)\Delta A(t)\Phi_0(\frac{t}{\varepsilon^2}, 0) \\ \quad + \frac{k}{\varepsilon}\rho(\frac{t}{\varepsilon})(ABC - BCA)]\zeta_0(t), \quad t \geq 0. \end{array} \quad (25)$$

We further consider

$$\dot{\phi}(\tfrac{t}{\varepsilon}) = \frac{k}{\varepsilon}\rho(\tfrac{t}{\varepsilon})(ABC - BCA)\phi(\tfrac{t}{\varepsilon}) \quad (26)$$

with $\rho(\cdot)$ given by (10). The fundamental matrix $\Phi(\frac{t}{\varepsilon}, 0)$ of equation (26) is obtained as

$$\Phi(\tfrac{t}{\varepsilon}, 0) = e^{k\mu(\frac{t}{\varepsilon})(ABC - BCA)}, \quad (27)$$

where

$$\mu(\tfrac{t}{\varepsilon}) = \begin{cases} \frac{1}{2}(\frac{t}{\varepsilon} - j)^2, & \frac{t}{\varepsilon} \in [j, j + \frac{1}{4}), \\ \frac{1}{16} - \frac{1}{2}(\frac{t}{\varepsilon} - j - \frac{1}{2})^2, & \frac{t}{\varepsilon} \in [j + \frac{1}{4}, j + \frac{3}{4}), \\ \frac{1}{2}(\frac{t}{\varepsilon} - j - 1)^2, & \frac{t}{\varepsilon} \in [j + \frac{3}{4}, j + 1) \end{cases} \quad (28)$$

with $j \in \mathbb{N}_0$. See the plot of μ in the bottom of Fig. 1. It is clear that $\mu(\frac{t}{\varepsilon})$ (and thus $\Phi(\frac{t}{\varepsilon}, 0)$ in (27)) is ε-periodic.

Remark 2: If we consider the dither $\mathrm{sgn}\sin(\frac{2\pi t}{\varepsilon})$ as in [13], [17] and apply the corresponding transformation (24) we will obtain $\Phi(\frac{t}{\varepsilon}, 0)$ given by (27), where

$$\mu(\tfrac{t}{\varepsilon}) = \begin{cases} \frac{1}{4}j + \frac{1}{2}(\frac{t}{\varepsilon} - j)^2, & \mu \in [j, j + \frac{1}{2}), \\ \frac{1}{4}(j+1) - \frac{1}{2}(\frac{t}{\varepsilon} - j - 1)^2, & \mu \in [j + \frac{1}{2}, j + 1) \end{cases}$$

with $j \in \mathbb{N}_0$. Clearly, $\mu(\frac{t}{\varepsilon})$ is monotonically increasing that leads to a non-periodic matrix $\mathcal{A}(\frac{t}{\varepsilon})$ (see e.g. the entry $\mathcal{A}_{11}(\frac{t}{\varepsilon}) = kbc\mu(\frac{t}{\varepsilon})$ of $\mathcal{A}(\frac{t}{\varepsilon})$). The averaging is not applicable to this matrix $\mathcal{A}(\frac{t}{\varepsilon})$. However, our square wave (3) allows to avoid this difficulty (see the ε-periodic $\mu(\frac{t}{\varepsilon})$ given by (28)).

By introducing the coordinate transformation

$$\zeta_0(t) = \Phi(\tfrac{t}{\varepsilon}, 0)\zeta(t), \quad t \geq 0, \tag{29}$$

we transform (25) to system (13) with

$$\Delta\mathcal{A}(t) = \Phi^{-1}(\tfrac{t}{\varepsilon}, 0)\Phi_0^{-1}(\tfrac{t}{\varepsilon^2}, 0)\Delta A(t)\Phi_0(\tfrac{t}{\varepsilon^2}, 0)\Phi(\tfrac{t}{\varepsilon}, 0). \tag{30}$$

Here $\Phi_0(\frac{t}{\varepsilon^2}, 0)$ and $\Phi(\frac{t}{\varepsilon}, 0)$ are, respectively, from (23) and (27). Then the averaged system of (13) with $\Delta A(t) = 0$ is given by (15) with $\zeta_{av}(t) \in \mathbb{R}^3$ and

$$\mathcal{A}_{av} = \begin{bmatrix} \frac{kbc}{32} & 1 & 0 \\ -\frac{23k^2b^2c^2 - 320a_3kbc}{10240} & -\frac{kbc}{16} & 1 \\ \mathbf{a}_{31} & \mathbf{a}_{32} & a_3 + \frac{kbc}{32} \end{bmatrix},$$
$$\mathbf{a}_{31} = a_1 - \frac{39k^3b^3c^3 + 368a_3k^2b^2c^2 - 15360(a_2 + a_3^2)kbc}{491520},$$
$$\mathbf{a}_{32} = a_2 - \frac{23k^2b^2c^2 + 640a_3kbc}{10240}. \tag{31}$$

This system has the following characteristic polynomial:

$$s^3 - a_3s^2 - (a_2 - \tfrac{k^2b^2c^2}{640})s - a_1 - \tfrac{a_3k^2b^2c^2}{1920}.$$

By using the Routh-Hurtwiz stability criterion, we arrive at:

Lemma 2: Let $a_3 < 0$. Matrix $\mathcal{A}_{av}$ given by (31) is Hurtwiz iff

$$a_2 - \tfrac{k^2b^2c^2}{640} < 0, \quad a_1 + \tfrac{a_3k^2b^2c^2}{1920} < 0,$$
$$a_1 + a_3(a_2 - \tfrac{k^2b^2c^2}{960}) > 0. \tag{32}$$

Moreover, inequalities (32) always hold for large enough $|k|$.

III. Robust stability analysis

In this section, we will present a robust stability analysis of uncertain system (13) by developing a time-delay approach to periodic averaging [14], [16]. Note that in the latter works the uncertainty $\Delta A(t)$ was not considered in the systems under study. In this sense we present a novel stability analysis for systems with norm-bounded uncertainties. We formulate the constructive LMI conditions for finding the upper bound ε^* that preserves the exponential stability of the original system for all $\varepsilon \in (0, \varepsilon^*]$. As mentioned above, quantitative bounds were missing in all the previous works (see e.g. [1]–[6], [9]–[13]).

A. A time-delay model

Consider system (13), where $\zeta(t) \in \mathbb{R}^n$ and $a_n < 0$ with $n = 2$ or 3. Let k be subject to (18) or (32) such that $\mathcal{A}_{av}$ given either by (16) or by (31) is Hurwitz. Given small $\sigma_0 > 0$ in (2), let a small enough $\sigma > 0$ (independent on $\varepsilon \in (0, \varepsilon^*]$) be the upper bound on $\Delta\mathcal{A}(t)$ given either by (14) or by (30) for all $t \geq 0$, i.e.

$$\|\Delta\mathcal{A}(t)\| \leq \sigma \quad \forall t \geq 0. \tag{33}$$

Indeed, this upper bound can be found by using (2), (14) and (30):

$$n = 2: \quad \|\Delta\mathcal{A}(t)\| \leq \sigma_0\|\Phi^{-1}(\tfrac{t}{\varepsilon}, 0)\|\|\Phi(\tfrac{t}{\varepsilon}, 0)\| \leq \sigma,$$
$$n = 3: \quad \|\Delta\mathcal{A}(t)\| \leq \sigma_0\|\Phi^{-1}(\tfrac{t}{\varepsilon}, 0)\|\|\Phi_0^{-1}(\tfrac{t}{\varepsilon^2}, 0)\| \times\|\Phi_0(\tfrac{t}{\varepsilon^2}, 0)\|\|\Phi(\tfrac{t}{\varepsilon}, 0)\| \leq \sigma.$$

If particularly $\Delta A(t) = \Delta a(t)I$ with $\Delta a(t) \in \mathbb{R}$ satisfying $|\Delta a(t)| \leq \sigma_0$ for all $t \geq 0$ (see e.g. (61) below), then $\sigma = \sigma_0$. Moreover, from (13) it follows that all entries $\mathcal{A}_{ij}(\frac{t}{\varepsilon})$ of $\mathcal{A}(\frac{t}{\varepsilon})$ are uniformly bounded for $t \geq 0$. Then $\mathcal{A}(\frac{t}{\varepsilon})$ can be presented as a convex combination of the constant matrices $\mathcal{A}_i$ for all $t \geq \varepsilon$:

$$\mathcal{A}(\tfrac{t}{\varepsilon}) = \textstyle\sum_{i=1}^N \rho_i(\tfrac{t}{\varepsilon})\mathcal{A}_i, \quad \rho_i(\tfrac{t}{\varepsilon}) \geq 0, \quad \sum_{i=1}^N \rho_i(\tfrac{t}{\varepsilon}) = 1 \tag{34}$$

with some integer $N \geq 2$.

Following [14], [16], we will apply the time-delay approach to periodic averaging of system (13). Namely, we integrate both sides of system (13) over $[t-\varepsilon, t]$ for $t \geq \varepsilon$, i.e.

$$\tfrac{\zeta(t) - \zeta(t-\varepsilon)}{\varepsilon} = \tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t [\mathcal{A}(\tfrac{s}{\varepsilon}) + \Delta\mathcal{A}(s)]\zeta(s)ds. \tag{35}$$

We present the left-hand side of (35) as

$$\tfrac{\zeta(t) - \zeta(t-\varepsilon)}{\varepsilon} = \tfrac{d}{dt}[\zeta(t) - G(t)] + \tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t \Delta\mathcal{A}(s)\zeta(s)ds - \Delta\mathcal{A}(t)\zeta(t), \tag{36}$$

where

$$G(t) = \tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t (s - t + \varepsilon)\mathcal{A}(\tfrac{s}{\varepsilon})\zeta(s)ds. \tag{37}$$

Note that the term $G(t)$ depends on the nominal part $\mathcal{A}(\frac{t}{\varepsilon})\zeta(t)$ only (that is the fast-varying term to be "averaged") and not on the whole right-hand part of (13) as in [14]. Then we obtain

$$\tfrac{d}{dt}[\zeta(t) - G(t)] = \Delta\mathcal{A}(t)\zeta(t) + \tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t \mathcal{A}(\tfrac{s}{\varepsilon})[\zeta(s) - \zeta(t) + \zeta(t)]ds, \quad t \geq \varepsilon. \tag{38}$$

We present

$$\tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t \mathcal{A}(\tfrac{s}{\varepsilon})[\zeta(s) - \zeta(t)]ds = -\tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t \mathcal{A}(\tfrac{s}{\varepsilon})\int_s^t \dot{\zeta}(\theta)d\theta ds.$$

Thus, we transform system (13) to the following time-delay system:

$$\dot{z}(t) = [\mathcal{A}_{av} + \Delta\mathcal{A}(t)]\zeta(t) - Y(t), \quad t \geq \varepsilon, \tag{39}$$

where

$$z(t) = \zeta(t) - G(t),$$
$$Y(t) = \tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t \mathcal{A}(\tfrac{s}{\varepsilon})\int_s^t \dot{\zeta}(\theta)d\theta ds,$$
$$\dot{\zeta}(\theta) = [\mathcal{A}(\tfrac{\theta}{\varepsilon}) + \Delta\mathcal{A}(\theta)]\zeta(\theta) \tag{40}$$

with $G(t)$ given by (37). Note that system (39) with notations (40) is a neutral type system. Comparatively to the averaged system (15), system (39) with $\Delta\mathcal{A}(t) = 0$ has the additional terms $G(t)$ and $Y(t)$ that are both of order $\mathcal{O}(\varepsilon)$ provided $\zeta(t)$ and $\dot{\zeta}(t)$ are of order $\mathcal{O}(1)$. Thus, for small $\varepsilon > 0$ system (39) with $\Delta\mathcal{A}(t) = 0$ can be considered as a perturbation of system (15). If $\zeta(t)$ is a solution to system (13), then it satisfies the time-delay system (39). Therefore, the stability of the time-delay system (39) guarantees the stability of system (13) (and thus, of systems (6) and (21)).

B. LMI conditions: L-K method

Theorem 1: Let $a_n < 0$ with $n = 2$ or 3, and k satisfy (18) or (32) (resulting in Hurwitz $\mathcal{A}_{av}$ given either by (16) or by (31)). Assume that (33) and (34) hold. Given matrices $\mathcal{A}_i$ $(i = 1, \ldots, N, \ N \geq 2)$ and scalars $\sigma > 0$, $\alpha > 0$, $\varepsilon^* > 0$, let there exist $n \times n$ matrices $P > 0$, $R > 0$ and scalars $h > 0$, $\lambda > 0$ that satisfy the following LMIs:

$$\left[\begin{array}{c|cc} \Xi & \begin{matrix}\sqrt{\varepsilon^*}\mathcal{A}_i^TR \\ 0 \\ 0\end{matrix} & \begin{matrix}\sqrt{\varepsilon^*}h\mathcal{A}_i^T\Omega \\ 0 \\ \sqrt{\varepsilon^*}h\Omega\end{matrix} \\ \hline \begin{matrix} * \\ * \end{matrix} & \begin{matrix} -R \\ * \end{matrix} & \begin{matrix} 0 \\ -h\Omega \end{matrix} \end{array}\right] < 0, \quad i = 1, \ldots, N, \tag{41}$$

where

$$\Xi = \begin{bmatrix} \Xi_{11} & -\mathcal{A}_{av}^TP - 2\alpha P & -P & P \\ * & -\frac{4}{\varepsilon^*}e^{-2\alpha\varepsilon^*}R + 2\alpha P & P & -P \\ * & * & -\frac{2h}{\varepsilon^*}e^{-2\alpha\varepsilon^*}I & 0 \\ * & * & * & -\lambda I \end{bmatrix},$$
$$\Xi_{11} = P\mathcal{A}_{av} + \mathcal{A}_{av}^TP + 2\alpha P + \lambda\sigma^2 I,$$
$$\Omega = \int_0^1 \mathcal{A}^T(\tau)\mathcal{A}(\tau)d\tau. \tag{42}$$

Then systems (6) and (21) are exponentially stable with a decay rate α for all $\varepsilon \in (0, \varepsilon^*]$, meaning that there exists $M > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ the solutions of (6) and (21) initialized by $x(0)$ satisfy the following inequality:

$$|x(t)|^2 \leq Me^{-2\alpha t}|x(0)|^2 \quad \forall t \geq 0 \tag{43}$$

Moreover, LMIs (41) are always feasible for small enough $\varepsilon^* > 0$, $\sigma > 0$ and $\alpha > 0$, i.e. systems (6) and (21) are exponentially stable for small enough $\varepsilon > 0$ and $\sigma_0 > 0$ with a small enough decay rate $\alpha = \alpha_0 > 0$.

Proof: Choose the following Lyapunov functional [16]

$$V(t) = V_P(t) + V_R(t) + V_h(t), \quad t \geq \varepsilon, \tag{44}$$

where

$$\begin{aligned} V_P(t) &= z^T(t)Pz(t), \\ V_R(t) &= \tfrac{1}{\varepsilon}\int_{t-\varepsilon}^{t} e^{-2\alpha(t-s)}(s-t+\varepsilon)^2 \\ &\quad \times \zeta^T(s)\mathcal{A}^T(\tfrac{s}{\varepsilon})R\mathcal{A}(\tfrac{s}{\varepsilon})\zeta(s)ds, \\ V_h(t) &= \tfrac{h}{\varepsilon}\int_{t-\varepsilon}^{t}\int_{s}^{t} e^{-2\alpha(t-\theta)}(s-t+\varepsilon)|\mathcal{A}(\tfrac{s}{\varepsilon})\dot{\zeta}(\theta)|^2 d\theta ds \end{aligned} \tag{45}$$

with $P > 0$, $R > 0$ and $h > 0$. This functional is positive-definite for all $\varepsilon \in [0, \varepsilon^*]$:

$$\begin{aligned} V(t) &\geq V_P(t) + V_R(t) \\ &\geq \begin{bmatrix} \zeta(t) \\ G(t) \end{bmatrix}^T \begin{bmatrix} P & -P \\ * & P + e^{-2\alpha\varepsilon^*}R \end{bmatrix} \begin{bmatrix} \zeta(t) \\ G(t) \end{bmatrix} \geq \bar{c}_1|\zeta(t)|^2, \end{aligned}$$

where we applied Jensen's inequality (i.e. (3.87) in [15]), and $\bar{c}_1 = \lambda_{\min}(\begin{bmatrix} P & -P \\ * & P+e^{-2\alpha\varepsilon^*}R \end{bmatrix})$. As in [16], [19], the latter directly gives the bound on $|\zeta(t)|$ instead of $|z(t)|$ implying that there is no need to verify the stability of $z(t) = 0$ in (40). Following the proof of Theorem 1 in [16], we obtain for all $\varepsilon \in (0, \varepsilon^*]$

$$\begin{aligned} &\dot{V}(t) + 2\alpha V(t) \leq \eta^T(t)\Xi\eta(t) + \varepsilon^*[\zeta^T(t)\mathcal{A}^T(\tfrac{t}{\varepsilon})R\mathcal{A}(\tfrac{t}{\varepsilon})\zeta(t) \\ &\quad + \tfrac{h}{\varepsilon^2}\dot{\zeta}^T(t)\int_{t-\varepsilon}^{t}(s-t+\varepsilon)\mathcal{A}^T(\tfrac{s}{\varepsilon})\mathcal{A}(\tfrac{s}{\varepsilon})ds\dot{\zeta}(t)], \quad t \geq \varepsilon, \end{aligned} \tag{46}$$

where Ξ is given by (42) and

$$\eta^T(t) = [\zeta^T(t), G^T(t), Y^T(t), \zeta^T(t)\Delta\mathcal{A}^T(t)].$$

Taking into account that the following holds with Ω defined in (42):

$$\begin{aligned} &\tfrac{h}{\varepsilon^2}\int_{t-\varepsilon}^{t}(s-t+\varepsilon)\mathcal{A}^T(\tfrac{s}{\varepsilon})\mathcal{A}(\tfrac{s}{\varepsilon})ds \\ &\leq h\int_{t-\varepsilon}^{t}\mathcal{A}^T(\tfrac{s}{\varepsilon})\mathcal{A}(\tfrac{s}{\varepsilon})ds \\ &= h\int_{\frac{t}{\varepsilon}-1}^{\frac{t}{\varepsilon}}\mathcal{A}^T(\tau)\mathcal{A}(\tau)d\tau = h\Omega, \end{aligned}$$

from (46) we obtain

$$\begin{aligned} &\dot{V}(t) + 2\alpha V(t) \leq \eta^T(t)\Xi\eta(t) \\ &\quad + \varepsilon^*[\zeta^T(t)\mathcal{A}^T(\tfrac{t}{\varepsilon})R\mathcal{A}(\tfrac{t}{\varepsilon})\zeta(t) + h\dot{\zeta}^T(t)\Omega\dot{\zeta}(t)], \quad t \geq \varepsilon. \end{aligned} \tag{47}$$

Moreover, from (13) and (34) it follows that

$$\dot{\zeta}(t) = [\textstyle\sum_{i=1}^{N}\rho_i(\tfrac{t}{\varepsilon})\mathcal{A}_i + \Delta\mathcal{A}(t)]\zeta(t), \ \mathcal{A}(\tfrac{t}{\varepsilon}) = \textstyle\sum_{i=1}^{N}\rho_i(\tfrac{t}{\varepsilon})\mathcal{A}_i. \tag{48}$$

Substituting (48) into (47) and applying further Schur complements to (47), we conclude that if

$$\left[\begin{array}{c|cc} \Xi & \begin{matrix} \sqrt{\varepsilon^*}\sum_{i=1}^{N}\rho_i(\tfrac{t}{\varepsilon})\mathcal{A}_i^T R \\ 0 \\ 0 \end{matrix} & \begin{matrix} \sqrt{\varepsilon^*}h\sum_{i=1}^{N}\rho_i(\tfrac{t}{\varepsilon})\mathcal{A}_i^T\Omega \\ 0 \\ \sqrt{\varepsilon^*}h\Omega \end{matrix} \\ \hline * & -R & 0 \\ * & * & -h\Omega \end{array}\right] < 0, \tag{49}$$

we have

$$\dot{V}(t) + 2\alpha V(t) \leq 0 \quad \forall t \geq \varepsilon \tag{50}$$

yielding the following bound for solutions of (13):

$$\bar{c}_1|\zeta(t)|^2 \leq V(t) \leq e^{-2\alpha(t-\varepsilon)}V(\varepsilon) \quad \forall t \geq \varepsilon. \tag{51}$$

Note that $V(\varepsilon)$ in (44) is upper bounded for all $\varepsilon \in (0, \varepsilon^*]$

$$V(\varepsilon) \leq \bar{c}_2[|\zeta(\varepsilon)|^2 + \textstyle\int_0^{\varepsilon}|\dot{\zeta}(s)|^2 ds]$$

with some ε-independent $\bar{c}_2 > 0$. For $t \in [0, \varepsilon]$, $\zeta(t)$ satisfies (13), where using (33) we have

$$\|\mathcal{A}(\tfrac{t}{\varepsilon}) + \Delta\mathcal{A}(t)\| \leq \|\mathcal{A}(\tfrac{t}{\varepsilon})\| + \sigma \leq a$$

for some $a > 0$ since all the entries of $\mathcal{A}(\frac{t}{\varepsilon})$ are uniformly bounded for $t \geq 0$. Therefore, $\frac{d}{dt}|\zeta(t)|^2 \leq 2a|\zeta(t)|^2$ for $t \in [0, \varepsilon]$ yielding

$$|\zeta(t)| \leq e^{at}|\zeta(0)|, \quad |\dot{\zeta}(t)| \leq ae^{at}|\zeta(0)|.$$

Thus, $V(\varepsilon)$ for all $\varepsilon \in (0, \varepsilon^*]$ is further upper bounded as

$$V(\varepsilon) \leq \bar{c}_2(e^{2a\varepsilon} + a^2\textstyle\int_0^{\varepsilon} e^{2as}ds)|\zeta(0)|^2 \leq \bar{c}_3|\zeta(0)|^2 \tag{52}$$

with some ε-independent $\bar{c}_3 > 0$. Moreover, $\Phi(0,0) = \Phi_0(0,0) = I$. For some ε-independent $\bar{c}_4 > 0$, when $n = 2$, via (11) we have for all $t \geq 0$

$$|\zeta(0)| = |x(0)|, \quad |x(t)| \leq \|\Phi(\tfrac{t}{\varepsilon}, 0)\||\zeta(t)| \leq \bar{c}_4|\zeta(t)| \tag{53}$$

and when $n = 3$, via (24) and (29) we have for all $t \geq 0$

$$\begin{aligned} &|\zeta(0)| = |\zeta_0(0)| = |x(0)|, \\ &|x(t)| \leq \|\Phi_0(\tfrac{t}{\varepsilon^2}, 0)\||\zeta_0(t)| \\ &\qquad \leq \|\Phi_0(\tfrac{t}{\varepsilon^2}, 0)\|\|\Phi(\tfrac{t}{\varepsilon}, 0)\||\zeta(t)| \leq \bar{c}_4|\zeta(t)|. \end{aligned} \tag{54}$$

Then (43) follows from (51), (52) and (53) or (54).

The feasibility of the strict LMIs (41) with $\alpha = 0$ implies the feasibility of (41) with the same decision variables and a small enough $\alpha = \alpha_0 > 0$, and thus guarantees exponential stability of system (6) with a small enough decay rate. Moreover, as in [16] the feasibility of LMIs (41) is always guaranteed for small enough $\varepsilon^* > 0$, $\sigma > 0$ and $\alpha > 0$ provided $\mathcal{A}_{av}$ is Hurwitz. □

Remark 3: For the choice of the controller gain k, we suggest the following algorithm: i) given matrices (4) (or (19)) we first find from (18) (or (32)) the minimum value of $|k|$ such that matrix $\mathcal{A}_{av}$ given by (16) (or (31)) is Hurwitz. ii) Using the obtained minimum value of $|k|$, we verify the feasibility of LMIs of Theorem 1 with $\alpha = \sigma = 0$ to find the maximum value of ε^*. iii) Save $\hat{k} = k_0 = |k|$ and $\hat{\varepsilon} = \varepsilon^*$. Then we enlarge $|k|$ until e.g. $5k_0$ with a fixed step e.g. 1 and repeat item ii). If $\hat{\varepsilon}$ is larger than the newly obtained ε^*, update $\hat{k} = |k|$ and $\hat{\varepsilon} = \varepsilon^*$, otherwise not. Finally, we choose $k = \hat{k}$ or $k = -\hat{k}$.

Remark 4: Note that in [16] the uncertainty was fast-varying and it was in $\mathcal{A}(\frac{t}{\varepsilon})$ that resulted in a larger polytope in presentation (34) because of the uncertainty. In the present paper polytope (34) is defined by the uncertainty-independent $\mathcal{A}(\frac{t}{\varepsilon})$. This leads to fewer LMIs in the stability conditions and less conservative results.

In Theorem 1, we have N vertices-dependent LMIs that involve much numerical complexity (see e.g. $N = 8$ vertices in Example 3 below). Alternatively, we can derive a single, but more conservative in examples, LMI. Since all entries $\mathcal{A}_{ij}(\frac{t}{\varepsilon})$ of $\mathcal{A}(\frac{t}{\varepsilon})$ in (13) are uniformly bounded for all $t \geq 0$, there exist some constants $\bar{\mathcal{A}}_{ij} \geq 0$ such that

$$|A_{ij}(\tfrac{t}{\varepsilon})| \leq \bar{\mathcal{A}}_{ij} \quad \forall t \geq 0. \tag{55}$$

Thus,

$$\begin{aligned} \|\mathcal{A}(\tfrac{t}{\varepsilon})\| &\leq \mathrm{trace}\{\mathcal{A}^T(\tfrac{t}{\varepsilon})\mathcal{A}(\tfrac{t}{\varepsilon})\} = \textstyle\sum_{i=1}^{n}\sum_{j=1}^{n}\mathcal{A}_{ij}^2(\tfrac{t}{\varepsilon}) \\ &\leq \textstyle\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{\mathcal{A}}_{ij}^2 \quad \forall t \geq 0. \end{aligned} \tag{56}$$

Choose $R = rI > 0$ in $V_R(t)$ given by (45) to be scalar matrix. Then using (56) we obtain the following upper bounds on the last two quadratic terms in the right-hand side of (46) for $t \geq \varepsilon$:

$$\varepsilon^*\zeta^T(t)\mathcal{A}^T(\tfrac{t}{\varepsilon})R\mathcal{A}(\tfrac{t}{\varepsilon})\zeta(t) \leq \varepsilon^* r\textstyle\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{\mathcal{A}}_{ij}^2|\zeta(t)|^2, \tag{57}$$

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and
content may change prior to final publication. Citation information: DOI 10.1109/TAC.2022.3230772

$$
\begin{array}{l}
\frac{\varepsilon^* h}{\varepsilon^2}\dot{\zeta}^T(t)\int_{t-\varepsilon}^t (s-t+\varepsilon)\mathcal{A}^T(\frac{s}{\varepsilon})\mathcal{A}(\frac{s}{\varepsilon})ds\dot{\zeta}(t)\\
\leq \frac{\varepsilon^* h}{2}\sum_{i=1}^n\sum_{j=1}^n \bar{\mathcal{A}}_{ij}^2\|\mathcal{A}(\frac{t}{\varepsilon})+\Delta\mathcal{A}(t)\|^2|\zeta(t)|^2\\
\leq \varepsilon^* h\sum_{i=1}^n\sum_{j=1}^n \bar{\mathcal{A}}_{ij}^2(\sum_{i=1}^n\sum_{j=1}^n \bar{\mathcal{A}}_{ij}^2+\sigma^2)|\zeta(t)|^2,
\end{array}
\quad (58)
$$

where we substituted (13) for $\dot{\zeta}(t)$ and used Young's inequality

$$
\begin{array}{rl}
\frac{1}{2}\|\mathcal{A}(\frac{t}{\varepsilon})+\Delta\mathcal{A}(t)\|^2 \leq & \|\mathcal{A}(\frac{t}{\varepsilon})\|^2+\sigma^2\\
\leq & \sum_{i=1}^n\sum_{j=1}^n \bar{\mathcal{A}}_{ij}^2+\sigma^2 \quad \forall t\geq\varepsilon.
\end{array}
$$

From (46), (57) and (58) we find

$$
\begin{array}{l}
\dot{V}(t)+2\alpha V(t)\leq \eta^T(t)\Xi\eta(t)+\varepsilon^*\sum_{i=1}^n\sum_{j=1}^n \bar{\mathcal{A}}_{ij}^2\\
\quad\times[r+h(\sum_{i=1}^n\sum_{j=1}^n \bar{\mathcal{A}}_{ij}^2+\sigma^2)]|\zeta(t)|^2, \quad t\geq\varepsilon.
\end{array}
\quad (59)
$$

Thus, following arguments in the proof of Theorem 1 we arrive at the following simpler conditions with a single LMI:

Corollary 1: Let $a_n < 0$ with $n = 2$ or 3, and k satisfy (18) or (32) (resulting in Hurwitz $\mathcal{A}_{av}$ given either by (16) or by (31)). Assume that (33) and (55) hold. Given scalars $\bar{\mathcal{A}}_{ij}$ $(i,j=1,\dots,n)$, $\sigma > 0$, $\alpha > 0$ and $\varepsilon^* > 0$, let there exist $n\times n$ matrix $P > 0$ and scalars $r > 0$, $h > 0$, $\lambda > 0$ that satisfy the following LMI:

$$
\hat{\Xi} < 0, \quad (60)
$$

where $\hat{\Xi}$ is obtained from Ξ in (42) with Ξ_{11} and R changed by $\Xi_{11}+\varepsilon^*\sum_{i=1}^n\sum_{j=1}^n \bar{\mathcal{A}}_{ij}^2[r+h(\sum_{i=1}^n\sum_{j=1}^n \bar{\mathcal{A}}_{ij}^2+\sigma^2)]$ and rI. Then systems (6) and (21) are exponentially stable with a decay rate α for all $\varepsilon\in(0,\varepsilon^*]$, meaning that there exists $M > 0$ such that for all $\varepsilon\in(0,\varepsilon^*]$ the solutions of (6) and (21) initialized by $x(0)$ satisfy (43). Moreover, LMI (60) is always feasible for small enough $\varepsilon^* > 0$, $\sigma > 0$ and $\alpha > 0$, i.e. systems (6) and (21) are exponentially stable for small enough $\varepsilon > 0$ and $\sigma_0 > 0$ with a small enough decay rate $\alpha=\alpha_0 > 0$.

Remark 5: The dither (3) leads to a smaller amplitude of ρ defined in (10) with $|\rho|\leq\frac{1}{4}$ compared to $|\rho|\leq\frac{1}{2}$ in [17]. Consequently, we obtain a smaller polytope in presentation (34) (see e.g the polytope in Example 1 with vertices corresponding to $\rho\in\{-\frac{1}{4},\frac{1}{4}\}$ and $\rho^2\in\{0,\frac{1}{16}\}$ to be compared with $\rho\in\{0,\frac{1}{2}\}$ and $\rho^2\in\{0,\frac{1}{4}\}$ in [17]) that leads to less conservative results for the second-order systems.

Remark 6: The direct Lyapunov method (see e.g in [15]) is applicable not only to the stability but also to the performance analysis. In the presence of locally essentially bounded disturbances ISS can be proved similar to [14]. Thus, under our LMI conditions, ISS is always guaranteed for large enough γ independent on $\varepsilon\in(0,\varepsilon^*]$.

Remark 7: As in [9]–[12], we can consider stabilization by sine/cosine wave dithers. This will lead to a periodic fundamental matrix (see e.g. [12]). After the coordinate transformation [2], we will obtain system (13) with a different but periodic matrix $\mathcal{A}(\frac{t}{\varepsilon})$ that allows to employ further the time-delay approach to averaging [14], [16]. Thus, our constructive method is also applicable to the sine/cosine wave dithers.

IV. NUMERICAL EXAMPLES

To illustrate the efficiency of the method, we will present three examples in the presence of uncertainties, where the uncertainty takes the following form

$$
\Delta A(t)=\sigma_0\sin(t)I. \quad (61)
$$

The latter satisfies (2). From (14) and (30), we arrive at (33) with $\sigma=\sigma_0$.

Example 1: [17] Consider system (1), (4) with

$$
a_1=52.973, \quad a_2=-5, \quad b=c_1=1, \quad c_2=0 \quad (62)
$$

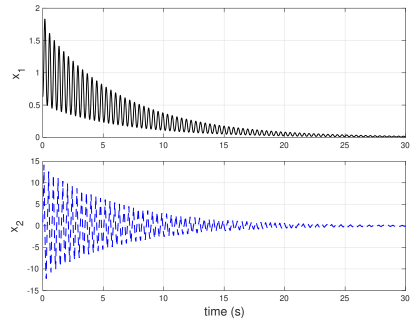

Fig. 2. State trajectory of (6) with (62), $\boldsymbol{k}=\mathbf{57}$ and $\boldsymbol{\Delta A(t)=0}$ $\boldsymbol{\forall t\geq 0}$ when $\boldsymbol{\varepsilon=0.39}$.

and $\Delta A(t)$ given by (61) under a fast-varying output feedback controller (5). By using the coordinate transformation (11), we obtain (13) with

$$
\mathcal{A}(\tfrac{t}{\varepsilon})=\begin{bmatrix} k\rho(\frac{t}{\varepsilon}) & 1\\ 52.973-5k\rho(\frac{t}{\varepsilon})-k^2\rho^2(\frac{t}{\varepsilon}) & -5-k\rho(\frac{t}{\varepsilon})\end{bmatrix} \quad (63)
$$

with $\rho(\cdot)$ given by (10), and $\Delta\mathcal{A}(t)$ in (14) satisfies (33) with $\sigma=\sigma_0$. It is clear that $\mathcal{A}(\frac{t}{\varepsilon})$ in (63) belongs to uncertain polytope with four vertices (that are omitted here) corresponding to $\rho\in\{-\frac{1}{4},\frac{1}{4}\}$ and $\rho^2\in\{0,\frac{1}{16}\}$. We obtain

$$
\mathcal{A}_{av}=\begin{bmatrix} 0 & 1\\ 52.973-\frac{k^2}{48} & -5\end{bmatrix}. \quad (64)
$$

Using Remark 3, we choose the controller gain as $k=57$. The upper bounds on entries of $\mathcal{A}(\frac{t}{\varepsilon})$ are given by

$$
\bar{\mathcal{A}}_{11}=14.25, \quad \bar{\mathcal{A}}_{12}=1, \quad \bar{\mathcal{A}}_{21}=221.3395, \quad \bar{\mathcal{A}}_{22}=19.25.
$$

From (42) and (63) we obtain

$$
\Omega=10^3\times\begin{bmatrix}5.6417 & 0.412\\ 0.412 & 0.09377\end{bmatrix}. \quad (65)
$$

By verifying the feasibility of LMIs of Theorem 1 in the four vertices and of LMI of Corollary 1 with different decay rates α and $\sigma=\sigma_0$, and using (64), (65), we find the upper bounds ε^* (see Table I) that guarantee the exponential stability of system (6), (61) with (62) and $k=57$ for all $\varepsilon\in(0,\varepsilon^*]$. It is clear that LMIs of Theorem 1 lead to an essentially larger upper bound than that via LMI of Corollary 1, but the improvement is achieved on the account of numerical complexity (see Table I). Note that comparatively to [17] an improvement on the upper bound is achieved due to the smaller amplitude of time-varying terms in fundamental matrix by our square wave dither (3). Moreover, the case of $\sigma\neq 0$ was not studied in [17].

Numerical simulations under an arbitrary initial condition $|x(0)|_\infty\leq 1$ show that system (6) with (62), $k=57$ and $\Delta A(t)=0$ $\forall t\geq 0$ is stable for a larger upper bound $\varepsilon^*=0.39$ (to be compared with the theoretical $\varepsilon^*=0.18\times 10^{-2}$), see Fig. 2, which may illustrate the conservatism of the proposed method.

Example 2: [12] Consider system (1), (4) with

$$
a_1=b=c_1=-c_2=1, \quad a_2=-\tfrac{1}{2} \quad (66)
$$

and $\Delta A(t)$ given by (61) under a fast-varying output feedback controller (5). This system with $\Delta A(t)=0$ is not stabilizable by a static time-invariant output-feedback controller. By using the coordinate transformation (11), we obtain (13) where

$$
\mathcal{A}(\tfrac{t}{\varepsilon})=\begin{bmatrix} -e^{-k\rho(\frac{t}{\varepsilon})}+1 & e^{-k\rho(\frac{t}{\varepsilon})}\\ -e^{-k\rho(\frac{t}{\varepsilon})}-\frac{1}{2}e^{k\rho(\frac{t}{\varepsilon})}+\frac{5}{2} & e^{-k\rho(\frac{t}{\varepsilon})}-\frac{3}{2}\end{bmatrix} \quad (67)
$$

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2022.3230772

TABLE I
MAXIMUM VALUE OF ε^* AND NUMERICAL COMPLEXITY OF LMIS

	Example 1 ($n=2$, $N=4$)		Example 2 ($n=2$, $N=4$)		Example 3 ($n=3$, $N=8$)		No. LMI lines	No. dec. vars
(α, σ)	(10^{-6}, 0)	(0.2, 0.2)	(10^{-6}, 0)	(0.01, 0.02)	(10^{-6}, 0)	(0.01, 0.02)		
[17]	0.05×10^{-3}	–	0.04×10^{-3}	–	–	–	$4nN$	n^2+n+1
Th. 1	1.80×10^{-3}	1.31×10^{-3}	9.71×10^{-3}	7.96×10^{-3}	2.40×10^{-2}	1.04×10^{-2}	$6nN$	n^2+n+2
Cor. 1	0.02×10^{-3}	0.01×10^{-3}	0.96×10^{-3}	0.79×10^{-3}	0.36×10^{-2}	0.15×10^{-2}	$4n$	$0.5(n^2+n)+3$

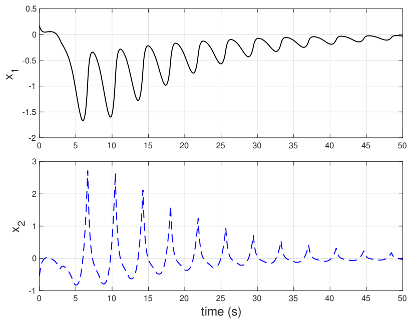

Fig. 3. State trajectory of (6) with (66), $k=9$ and $\Delta A(t)=0\ \forall t\geq 0$ when $\varepsilon=3.8$.

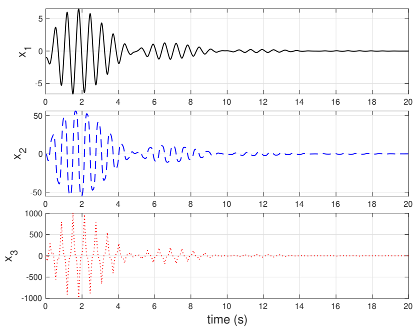

Fig. 4. State trajectory of (21) with (70), $k=14$ and $\Delta A(t)=0$ $\forall t\geq 0$ when $\varepsilon=12$.

with $\rho(\cdot)$ given by (10), and $\Delta\mathcal{A}(t)$ in (14) satisfies (33) with $\sigma=\sigma_0$. It is clear that $\mathcal{A}(\frac{t}{\varepsilon})$ in (67) belongs to uncertain polytope with four vertices (that are omitted here) corresponding to $(-\rho,\rho)\in\{-\frac{1}{4},\frac{1}{4}\}\times\{-\frac{1}{4},\frac{1}{4}\}$. We obtain

$$\mathcal{A}_{av}=\begin{bmatrix}-\frac{4}{k}\sinh(\frac{k}{4})+1, & \frac{4}{k}\sinh(\frac{k}{4})\\ -\frac{6}{k}\sinh(\frac{k}{4})+\frac{5}{2} & \frac{4}{k}\sinh(\frac{k}{4})-\frac{3}{2}\end{bmatrix}. \tag{68}$$

Using Remark 3, we choose the controller gain as $k=9$. The upper bounds on entries of $\mathcal{A}(\frac{t}{\varepsilon})$ are given by

$$\bar{\mathcal{A}}_{11}=8.4877,\ \bar{\mathcal{A}}_{12}=9.4877,\ \bar{\mathcal{A}}_{21}=11.7316,\ \bar{\mathcal{A}}_{22}=7.9877.$$

From (42) and (67) we obtain

$$\Omega=\begin{bmatrix}10.9444 & -12.2628\\ -12.2628 & 15.9964\end{bmatrix}. \tag{69}$$

By verifying the feasibility of LMIs in Theorem 1 in the four vertices and of LMI in Corollary 1 with different α and $\sigma=\sigma_0$, and using (68), (69), we find the upper bounds ε^* (see Table I) that guarantee the exponential stability of (6), (61) with (66) and $k=9$ for all $\varepsilon\in(0,\varepsilon^*]$. Compared with LMI of Corollary 1, LMIs of Theorem 1 lead to a larger upper bound on the account of numerical complexity. Moreover, our conditions lead to an essentially larger upper bound than [17].

Numerical simulations under an arbitrary initial condition $|x(0)|_\infty\leq 1$ show that system (6) with (66), $k=9$ and $\Delta A(t)=0$ $\forall t\geq 0$ is stable for a larger upper bound $\varepsilon^*=3.8$, see Fig. 3.

Example 3: Consider system (1), (19) with

$$a_1=a_2=0,\quad -a_3=b=c=1 \tag{70}$$

and $\Delta A(t)$ given by (61) under a fast-varying output-feedback controller (20). Note that due to $a_2=0$, this system with $\Delta A(t)=0$ is not stabilizable by a static time-invariant output-feedback controller. By using the two successive coordinate transformations (24) and (29), we obtain (13) where

$$\begin{aligned}&\mathcal{A}(\tfrac{t}{\varepsilon})=\begin{bmatrix}k\mu(\frac{t}{\varepsilon}) & 1 & 0\\ -k\mu(\frac{t}{\varepsilon})-\frac{3}{2}k^2\mu^2(\frac{t}{\varepsilon}) & -2k\mu(\frac{t}{\varepsilon}) & 1\\ \mathcal{A}_{31}(\frac{t}{\varepsilon}) & \mathcal{A}_{32}(\frac{t}{\varepsilon}) & -1+k\mu(\frac{t}{\varepsilon})\end{bmatrix},\\ &\mathcal{A}_{31}(\tfrac{t}{\varepsilon})=k\mu(\tfrac{t}{\varepsilon})+\tfrac{1}{2}k^2\mu^2(\tfrac{t}{\varepsilon})-k^3\mu^3(\tfrac{t}{\varepsilon}),\\ &\mathcal{A}_{32}(\tfrac{t}{\varepsilon})=2k\mu(\tfrac{t}{\varepsilon})-\tfrac{3}{2}k^2\mu^2(\tfrac{t}{\varepsilon}),\end{aligned} \tag{71}$$

with $\mu(\cdot)$ given by (28), and $\Delta\mathcal{A}(t)$ in (30) satisfies (33) with $\sigma=\sigma_0$. It is clear that $\mathcal{A}(\frac{t}{\varepsilon})$ in (71) belongs to uncertain polytope with eight vertices (that are omitted here) corresponding to $\mu^i=0$ or $\mu^i=\frac{1}{16^i}$ $(i=1,2,3)$. We obtain

$$\begin{aligned}&\mathcal{A}_{av}=\begin{bmatrix}\frac{k}{32}, & 1 & 0\\ -\frac{23k^2+320k}{10240} & -\frac{k}{16} & 1\\ \mathbf{a}_{31} & -\frac{23k^2-640k}{10240} & -1+\frac{k}{32}\end{bmatrix},\\ &\mathbf{a}_{31}=-\frac{39k^3-368k^2-15360k}{491520}.\end{aligned} \tag{72}$$

Using Remark 3, we choose the controller gain as $k=14$. The upper bounds on entries of $\mathcal{A}(\frac{t}{\varepsilon})$ are given by

$$\begin{aligned}&\bar{\mathcal{A}}_{11}=0.875,\quad \bar{\mathcal{A}}_{12}=\bar{\mathcal{A}}_{23}=\bar{\mathcal{A}}_{33}=1,\quad \bar{\mathcal{A}}_{13}=0,\\ &\bar{\mathcal{A}}_{21}=2.0234,\quad \bar{\mathcal{A}}_{22}=\bar{\mathcal{A}}_{32}=1.75,\quad \bar{\mathcal{A}}_{31}=1.2578.\end{aligned}$$

From (28), (42) and (71) we obtain

$$\Omega=\begin{bmatrix}1.8099 & 1.8928 & -1.0107\\ 1.8928 & 2.4213 & -1.0494\\ -1.0107 & -1.0494 & 1.4185\end{bmatrix}. \tag{73}$$

By verifying the feasibility of LMIs of Theorem 1 in the eight vertices and LMI of Corollary 1 with different α and $\sigma=\sigma_0$, and using (72), (73), we find the upper bounds ε^* (see Table I) that guarantee the exponential stability of (21), (61) with (70) and $k=14$ for all $\varepsilon\in(0,\varepsilon^*]$. Compared with LMI of Corollary 1, LMIs of Theorem 1 lead to a larger upper bound on the account of numerical complexity.

Numerical simulations under an arbitrary initial condition $|x(0)|_\infty\leq 1$ show that system (21) with (70), $k=14$ and $\Delta A(t)=0$ $\forall t\geq 0$ is stable for a larger upper bound $\varepsilon^*=12$, see Fig. 4.

Remark 8: As expected, in all three examples numerical simulations show that the systems are stabilizable by the dither $\operatorname{sgn}\sin(\frac{2\pi t}{\varepsilon})$ from [13], [17] with the same maximum values of ε^* as in the present paper (with $\operatorname{sgn}\cos(\frac{2\pi t}{\varepsilon})$).

V. CONCLUSIONS

We have given the first constructive solution for stabilization of the second- and third-order (with relative degree 3) linear uncertain systems by using a fast-varying square wave dither, where the bounds on the dither frequencies that guarantee the stability are found from LMIs. Extension of the results to higher-order and nonlinear systems as well as to the act-and-wait control [20] may be topics for future research.

References

[1] S. Meerkov, "Principle of vibrational control: theory and applications," *IEEE Transactions on Automatic Control*, vol. 25, no. 4, pp. 755–762, 1980.

[2] R. Bellman, J. Bentsman, and S. Meerkov, "Vibrational control of nonlinear systems: Vibrational stabilizability," *IEEE Transactions on Automatic Control*, vol. 31, no. 8, pp. 710–716, 1986.

[3] J. Bentsman, "Vibrational control of a class of nonlinear systems by nonlinear multiplicative vibrations," *IEEE Transactions on Automatic Control*, vol. 32, no. 8, pp. 711–716, 1987.

[4] J. Bentsman, K. S. Hong, and J. Fakhfakh, "Vibrational control of nonlinear time lag systems: vibrational stabilization and transient behavior," *Automatica*, vol. 27, no. 3, pp. 491–500, 1991.

[5] F. Bullo, "Averaging and vibrational control of mechanical systems," *SIAM Journal on Control and Optimization*, vol. 41, no. 2, pp. 542–562, 2002.

[6] X. Cheng, Y. Tan, and I. Mareels, "On robustness analysis of linear vibrational control systems," *Automatica*, vol. 87, pp. 202–209, 2018.

[7] H. K. Khalil, *Nonlinear Systems*. Prentice Hall, 3rd ed., 2002.

[8] R. Brockett, "A stabilization problem," in *Open problems in mathematical systems and control theory*, pp. 75–78, Springer, 1999.

[9] L. Moreau and D. Aeyels, "Stabilization by means of periodic output feedback," in *Proceedings of the 38th IEEE Conference on Decision and Control*, pp. 108–109, IEEE, 1999.

[10] L. Moreau and D. Aeyels, "A note on stabilization by periodic output feedback for third-order systems," in *Proceedings of 14th International Symposium of Mathematical Theory of Networks and Systems (MTNS), Perpignan*, pp. 19–23, 2000.

[11] L. Moreau and D. Aeyels, "Periodic output feedback stabilization of single-input single-output continuous-time systems with odd relative degree," *Systems & Control Letters*, vol. 51, no. 5, pp. 395–406, 2004.

[12] W. Michiels, S.-I. Niculescu, and L. Moreau, "Using delays and time-varying gains to improve the static output feedback stabilizability of linear systems: a comparison," *IMA Journal of Mathematical Control and Information*, vol. 21, no. 4, pp. 393–418, 2004.

[13] J. Allwright, A. Astolfi, and H. Wong, "A note on asymptotic stabilization of linear systems by periodic, piecewise constant, output feedback," *Automatica*, vol. 41, no. 2, pp. 339–344, 2005.

[14] E. Fridman and J. Zhang, "Averaging of linear systems with almost periodic coefficients: A time-delay approach," *Automatica*, vol. 122, p. 109287, 2020.

[15] E. Fridman, *Introduction to time-delay systems: analysis and control*. Birkhauser, 2014.

[16] J. Zhang and E. Fridman, "L_2-analysis via time-delay approach to periodic averaging with stochastic extension," *Automatica*, vol. 137, p. 110126, 2022.

[17] J. Zhang and E. Fridman, "A time-delay approach to vibrational control with square wave dithers," *IFAC-PapersOnLine*, vol. 54, no. 18, pp. 35–40, 2021.

[18] C. J. Chien and C. Y. Yao, "An output-based adaptive iterative learning controller for high relative degree uncertain linear systems," *Automatica*, vol. 40, no. 1, p. 145–153, 2004.

[19] E. Fridman and L. Shaikhet, "Simple LMIs for stability of stochastic systems with delay term given by Stieltjes integral or with stabilizing delay," *Systems & Control Letters*, vol. 124, pp. 83–91, 2019.

[20] G. Stépán and T. Insperger, "Stability of time-periodic and delayed systems–a route to act-and-wait control," *Annual Reviews in Control*, vol. 30, no. 2, pp. 159–168, 2006.