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Global finite-dimensional observer-based stabilization of a semilinear heat equation with large input delay☆

Rami Katz *, Emilia Fridman

School of Electrical Engineering, Tel-Aviv University, Tel-Aviv, Israel

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ABSTRACT

We study global finite-dimensional observer-based stabilization of a semilinear 1D heat equation with globally Lipschitz semilinearity in the state variable. We consider Neumann actuation and point measurement. Using dynamic extension and modal decomposition we derive nonlinear ODEs for the modes of the state. We propose a controller that is based on a nonlinear finite-dimensional Luenberger observer. Our Lyapunov H^1-stability analysis leads to LMIs, which are shown to be feasible for a large enough observer dimension and small enough Lipschitz constant. Next, we consider the case of a constant input delay $r > 0$. To compensate the delay, we introduce a chain of M sub-predictors that leads to a nonlinear closed-loop ODE system, coupled with nonlinear infinite-dimensional tail ODEs. We provide LMIs for H^1-stability and prove that for any $r > 0$, the LMIs are feasible provided M and the observer dimension N are large enough and the Lipschitz constant is small enough. Numerical examples demonstrate the efficiency of the proposed approach.

1. Introduction

Observer-based control of parabolic PDEs is a challenging problem with numerous applications, including chemical reactors, flame propagation and viscous flow [1]. Output-feedback controllers for PDEs have been constructed by the modal decomposition approach [2–4], the backstepping method [5] and the spatial decomposition approach [6,7]. Constructive finite-dimensional observer-based design for linear 1D parabolic PDEs was introduced in [8,9], via modal decomposition. The challenging problem of efficient finite-dimensional observer-based design for semilinear parabolic PDEs remained open.

State-feedback control of several semilinear PDEs was studied in [10] using backstepping, in [11] using small-gain theorem and in [12] via control Lyapunov functions. Recently, modal-decomposition-based state-feedback was proposed in [13] for global stabilization of heat equation and in [9] for regional stabilization of the Kuramoto–Sivashinsky equation. Finite-dimensional control based on linear observers was proposed in [14] for semilinear parabolic PDEs via modal decomposition. Linear observers should have high gains required to dominate the nonlinearity, which leads to small delays that preserve the stability [15,16].

For ODEs, compensation of input delay can be achieved using three main predictor methods: the classical predictor [17], the PDE-based predictor [18] and sequential sub-predictors (observers of future state) [19]. For delay compensation of input/output delays in the case of nonlinear ODEs see e.g. [20–26] and the references therein. For the semilinear heat equation, by using spatial decomposition, a chain of PDE observers (to compensate output delay) was suggested in [27]. For the linear heat equation, a classical state-feedback predictor via modal decomposition was proposed in [28], whereas a sub-predictor based on PDE observer was suggested in [29]. For linear parabolic PDEs, finite-dimensional observe-based classical predictors and sub-predictors were introduced in [30].

For semilinear parabolic PDEs, efficient finite-dimensional observer-based controller design as well as input delay compensation remained open challenging problems. The goal of this work is to address many of these challenges. We consider global stabilization of a semilinear heat equation under Neumann actuation and point measurement. The semilinearity is assumed to be globally Lipschitz in the state. Using dynamic extension and modal decomposition we derive nonlinear ODEs for the modes of the state. We design a linear controller, which is based on a finite-dimensional *nonlinear* observer. The challenge in the Lyapunov-based analysis is due to the coupling between the finite-dimensional and infinite-dimensional parts of the closed-loop system, introduced by both the semilinearity and the estimation error. Our H^1-stability analysis leads to LMIs, which are shown to be feasible for a large enough observer dimension and small enough Lipschitz constant.

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* Corresponding author.
E-mail addresses: ramikatz@mail.tau.ac.il (R. Katz), emilia@tauex.tau.ac.il (E. Fridman).

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We further consider the case of constant input delay $r>0$ and suggest compensating the delay using chain of M sub-predictors — observers of the future state. We introduce an approximate nonlinearity into the sub-predictor ODEs and provide H^1-stability analysis, where the difference between the approximate nonlinearity and the actual nonlinearity is estimated using the sub-predictor estimation error. We prove that for any $r>0$, the LMIs for the stability analysis are feasible provided M and the observer dimension N are large enough and the Lipschitz constant is small enough. Numerical examples demonstrate the efficiency of the proposed approach.

Notations and preliminaries: $L^2(0,1)$ is the Hilbert space of Lebesgue measurable and square integrable functions $f:[0,1]\to\mathbb{R}$ with the inner product $\langle f,g\rangle := \int_0^1 f(x)g(x)dx$ and induced norm $\|f\|^2 := \langle f,f\rangle$. $H^k(0,1)$ is the Sobolev space of functions $f:[0,1]\to\mathbb{R}$ having k square integrable weak derivatives, with the norm $\|f\|^2_{H^k} := \sum_{j=0}^k \left\|f^{(j)}\right\|^2$. Given $f,g\in L^2(0,1)$, $f\overset{L^2}{=}g$ means that $\|f-g\|=0$. The Euclidean norm on $\mathbb{R}^n$ is denoted by $|\cdot|$. We write $f\in H_0^1(0,1)$ if $f\in H^1(0,1)$ and $f(0)=f(1)=0$. For $P\in\mathbb{R}^{n\times n}$, $P>0$ means that P is symmetric and positive definite. The sub-diagonal elements of a symmetric matrix will be denoted by $*$. For $0<U\in\mathbb{R}^{n\times n}$ and $x\in\mathbb{R}^n$ we denote $|x|_U^2=x^TUx$. $\mathbb{Z}_+$ denotes the nonnegative integers.

Consider the Sturm–Liouville eigenvalue problem

$$\phi''+\lambda\phi=0,\quad x\in(0,1) \tag{1.1}$$

with boundary conditions

$$\phi'(0)=\phi'(1)=0. \tag{1.2}$$

This problem induces a sequence of eigenvalues with corresponding eigenfunctions. The normalized eigenfunctions form a complete orthonormal system in $L^2(0,1)$. The eigenvalues and corresponding eigenfunctions are given by

$$\phi_0(x)\equiv 1,\ \phi_n(x)=\sqrt{2}\cos\left(\sqrt{\lambda_n}x\right),\ \lambda_n=n^2\pi^2,\ n\in\mathbb{Z}_+. \tag{1.3}$$

The following lemmas will be used:

Lemma 1.1 (*[31]*). *Let $h\overset{L^2}{=}\sum_{n=0}^\infty h_n\phi_n$. Then $h\in H^2(0,1)$ with $h'(0)=h'(1)=0$ if and only if $\sum_{n=1}^\infty \lambda_n^2h_n^2<\infty$. Moreover,*

$$\left\|h'\right\|^2=\sum_{n=1}^\infty \lambda_n h_n^2. \tag{1.4}$$

Lemma 1.2 (*Sobolev's Inequality [32]*). *Let $h\in H^1(0,1)$. Then, for all $\Gamma>0$:*

$$\max_{x\in[0,1]}|h(x)|^2\le(1+\Gamma)\|h\|^2+\Gamma^{-1}\left\|h'\right\|^2. \tag{1.5}$$

2. Finite-dimensional observer-based control of a non-delayed semilinear heat equation

2.1. Problem formulation and controller deign

In this section we consider stabilization of the non-delayed semilinear 1D heat equation

$$z_t(x,t)=z_{xx}(x,t)+g\left(t,x,z(x,t)\right),\ t\ge 0 \tag{2.1}$$

where $x\in(0,1)$, $z(x,t)\in\mathbb{R}$. We consider Neumann actuation

$$z_x(0,t)=0,\quad z_x(1,t)=u(t) \tag{2.2}$$

where $u(t)$ is a control input to be designed. We further assume point measurement given by

$$y(t)=z(x_*,t),\quad x_*\in[0,1]. \tag{2.3}$$

Note that $x_*=0$ or $x_*=1$ correspond to boundary measurements. Here $g:\mathbb{R}^3\to\mathbb{R}$ is a locally Lipschitz function which satisfies $g(t,x,0)\equiv 0$ and

$$\sup_{z_1\neq z_2}\frac{|g(t,x,z_1)-g(t,x,z_2)|}{|z_1-z_2|}\le\sigma,\ \forall(t,x)\in\mathbb{R}^2 \tag{2.4}$$

for some $\sigma>0$, independent of $(t,x)\in\mathbb{R}^2$.

Remark 2.1. For simplicity, in the present paper we consider a reaction–diffusion PDE with constant diffusion and reaction coefficients. As in [8], our results can be easily extended to the more general reaction–diffusion PDE

$$\begin{aligned}z_t(x,t)&=\partial_x\left(p(x)z_x(x,t)\right)+q(x)z(x,t)\\&\quad+g(t,x,z(x,t)),\quad x\in[0,1],\ t\ge 0,\end{aligned}$$

where $p(x)$ and $q(x)$ are sufficiently smooth on $(0,1)$.

Let $\psi(x)=-\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)$ and note that it satisfies

$$\begin{aligned}&\psi''(x)=-\mu\psi(x),\quad \mu=\frac{\pi^2}{4},\\&\psi'(0)=0,\ \ \psi'(1)=1,\ \ \|\psi\|^2=\frac{2}{\pi^2}.\end{aligned} \tag{2.5}$$

Furthermore, note that

$$\begin{aligned}&\langle\psi,\phi_0\rangle=\int_0^1\psi(x)dx=\frac{4}{\pi^2},\\&\langle\psi,\phi_n\rangle=-\frac{1}{\lambda_n}\int_0^1\psi(x)\phi_n''(x)dx=\frac{1}{\lambda_n}\phi_n'(1)\\&-\frac{1}{\lambda_n}\int_0^1\psi''(x)\phi_n(x)dx=\frac{\sqrt{2}(-1)^n}{\lambda_n}+\frac{\mu}{\lambda_n}\langle\psi,\phi_n\rangle,\ \ n\ge 1.\end{aligned} \tag{2.6}$$

Similar to [12], we introduce the change of variables

$$w(x,t)=z(x,t)-\psi(x)u(t), \tag{2.7}$$

to obtain the equivalent PDE

$$\begin{aligned}w_t(x,t)&=w_{xx}(x,t)+g\left(t,x,w(x,t)+\psi(x)u(t)\right)\\&\quad-\psi(x)[\dot{u}(t)+\mu u(t)]\end{aligned} \tag{2.8}$$

with

$$w_x(0,t)=w_x(1,t)=0 \tag{2.9}$$

and measurement

$$y(t)=w(x_*,t)+\psi(x_*)u(t). \tag{2.10}$$

We define further the new control input $v(t)$ that satisfies the following relations:

$$\dot{u}(t)=-\mu u(t)+v(t),\quad u(0)=0,\quad t\ge 0.$$

Then (2.8) can be presented as the ODE–PDE system

$$\begin{aligned}&\dot{u}(t)=-\mu u(t)+v(t),\quad t\ge 0,\\&w_t(x,t)=w_{xx}(x,t)+g\left(t,x,w(x,t)+\psi(x)u(t)\right)\\&\qquad-\psi(x)v(t).\end{aligned} \tag{2.11}$$

We will treat further $u(t)$ as an additional state variable.

We present the solution to (2.11) as

$$w(x,t)=\sum_{n=0}^\infty w_n(t)\phi_n(x),\ \ w_n(t)=\langle w(\cdot,t),\phi_n\rangle, \tag{2.12}$$

with $\{\phi_n\}_{n=0}^\infty$ defined in (1.3). By differentiating under the integral sign, integrating by parts and using (1.1) and (1.2) we obtain for $t\ge 0$

$$\begin{aligned}&\dot{w}_n(t)=-\lambda_n w_n(t)+g_n(t)+b_n v(t),\\&w_n(0)=\langle w(\cdot,0),\phi_n\rangle,\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}&g_n(t)=\langle g\left(t,\cdot,w(\cdot,t)+\psi(\cdot)u(t)\right),\phi_n\rangle,\\&b_0\overset{(2.6)}{=}\frac{4}{\pi^2},\quad b_n\overset{(2.6)}{=}\frac{(-1)^{n+1}4\sqrt{2}}{\pi^2(4n^2-1)},\ \ n\ge 1.\end{aligned} \tag{2.14}$$

Note that given $N \in \mathbb{Z}_+$, (2.14) and the integral test for series convergence imply

$$\begin{aligned}&\sum_{n=N+1}^{\infty}\lambda_n b_n^2 = \frac{32}{\pi^2}\sum_{n=N+1}^{\infty}\frac{n^2}{(4n^2-1)^2}\\ &= \frac{2}{\pi^2}\sum_{n=N+1}^{\infty}\frac{1}{n^2}\left(1+\frac{1}{4n^2-1}\right)^2 \le \frac{2\xi_{N+1}}{\pi^2},\\ &\xi_{N+1} = \left(1+\frac{1}{4(N+1)^2-1}\right)^2\frac{1}{N}.\end{aligned} \tag{2.15}$$

Let $\delta > 0$ be a decay rate and let $N_0 \in \mathbb{Z}_+$ satisfy

$$-\lambda_n + \sigma < -\delta, \quad n > N_0. \tag{2.16}$$

N_0 is the number of modes in our controller, whereas $N \in \mathbb{Z}_+$, $N \ge N_0$ is the observer dimension. We construct a finite-dimensional observer of the form

$$\hat{w}(x,t) = \sum_{n=0}^{N}\hat{w}_n(t)\phi_n(x) \tag{2.17}$$

where $\left\{\hat{w}_n(t)\right\}_{n=0}^N$ satisfy the *nonlinear* ODEs

$$\begin{aligned}&\dot{\hat{w}}_n(t) = -\lambda_n\hat{w}_n(t) + \hat{g}_n(t) + b_n v(t)\\ &-l_n\left[\hat{w}(x_*,t) + \psi(x_*)u(t) - y(t)\right],\ 0 \le n \le N\end{aligned} \tag{2.18}$$

with scalar observer gains $\{l_n\}_{n=0}^N$ and

$$\hat{g}_n(t) = \left\langle g\left(t,\cdot,\hat{w}(\cdot,t)+\psi(\cdot)u(t)\right),\phi_n\right\rangle,\ 0 \le n \le N. \tag{2.19}$$

In particular, we approximate the projections of the semilinearity $g(t,x,w(x,t)+\psi(x)u(t))$ onto $\{\phi_n\}_{n=0}^N$ by the projections of the approximate semilinearity $g(t,x,\hat{w}(x,t)+\psi(x)u(t))$ onto $\{\phi_n\}_{n=0}^N$.

Assumption 1. The point $x_* \in [0,1]$ satisfies

$$c_n = \phi_n(x_*) \neq 0, \quad 0 \le n \le N_0. \tag{2.20}$$

It can be easily verified that Assumption 1 holds provided $x_* \notin \left\{\frac{2k-1}{2n} \mid k \in \{1,\dots,n\},\ n \in \{1,\dots,N_0\}\right\}$.

Denote

$$\begin{aligned}&\tilde{A}_0 = \mathrm{diag}\{-\mu, A_0\}, \quad \tilde{B}_0 = \mathrm{col}\{1, B_0\}\\ &A_0 = \mathrm{diag}\{-\lambda_n\}_{n=0}^{N_0}, \quad B_0 = \mathrm{col}\{b_n\}_{n=0}^{N_0}\\ &C_0 = \left[c_0,\dots,c_{N_0}\right], \quad C_1 = \left[c_{N_0+1},\dots,c_N\right],\end{aligned} \tag{2.21}$$

Under Assumption 1, the pair (A_0, C_0) is observable by the Hautus lemma. Let $L_0 = \{l_n\}_{n=0}^{N_0} \in \mathbb{R}^{N_0+1}$ satisfy the Lyapunov inequality

$$P_o(A_0 - L_0C_0) + (A_0 - L_0C_0)^T P_o < -2\delta P_o \tag{2.22}$$

with $0 < P_o \in \mathbb{R}^{(N_0+1)\times(N_0+1)}$. We further choose the remaining gains as $l_n = 0$, $N_0+1 \le n \le N$.

Similarly, by the Hautus lemma, the pair $(\tilde{A}_0, \tilde{B}_0)$ is controllable. Let $K_0 \in \mathbb{R}^{1\times(N_0+2)}$ satisfy

$$P_c(\tilde{A}_0 - \tilde{B}_0K_0) + (\tilde{A}_0 - \tilde{B}_0K_0)^T P_c < -2\delta P_c, \tag{2.23}$$

with $0 < P_c \in \mathbb{R}^{(N_0+2)\times(N_0+2)}$. We propose the controller

$$v(t) = -K_0\hat{w}^{N_0}(t), \quad \hat{w}^{N_0}(t) = \mathrm{col}\left\{u(t), \hat{w}_n(t)\right\}_{n=0}^{N_0} \tag{2.24}$$

which is based on the finite-dimensional observer (2.17).

2.2. Well-posedness of the closed-loop system

For well-posedness of the closed-loop system (2.7), (2.18) subject to (2.24), consider the operator

$$\begin{aligned}&\mathcal{A}: \mathcal{D}(\mathcal{A}) \to L^2(0,1),\ \mathcal{A} = -\partial_{xx},\\ &\mathcal{D}(\mathcal{A}) = \left\{h \in H^2(0,1) \mid h'(0) = h'(1) = 0\right\}.\end{aligned}$$

Let $\theta > 0$ and $\mathcal{A}_\theta = \mathcal{A} + \theta I$. Given $h \in \mathcal{D}(\mathcal{A}_\theta) = \mathcal{D}(\mathcal{A})$, integration by parts gives $\langle \mathcal{A}_\theta h, h\rangle = \left\|h'\right\|^2 + \theta\left\|h\right\|^2$. Hence, $\langle \mathcal{A}_\theta h, h\rangle > 0$. Since $-\mathcal{A}_\theta$ is diagonalizable, by Section 2.6 in [33], the spectrum of $-\mathcal{A}_\theta$ is given by $\sigma(-\mathcal{A}_\theta) = \{-\lambda_n - \theta\}_{n=0}^\infty \subset (-\infty, 0)$. Thus, $\{\mu \in \mathbb{C}\ |\mathrm{Re}(\mu) > 0\} \subseteq \rho(-\mathcal{A}_\theta)$, where $\rho(-\mathcal{A}_\theta)$ is the resolvent set of $-\mathcal{A}_\theta$. By [33], $-\mathcal{A}_\theta$ generates an analytic semigroup on $L^2(0,1)$. Moreover, by Section 3.4 in [33] and positivity of $\mathcal{A}_\theta$, there exists a unique positive root $\mathcal{A}_\theta^{\frac{1}{2}}$ where $\mathcal{D}\left(\mathcal{A}_\theta^{\frac{1}{2}}\right) \subseteq L^2(0,1)$ is the completion of $\mathcal{D}(\mathcal{A}_\theta) \subseteq L^2(0,1)$ with respect to the norm $\|h\|_{\frac{1}{2}} = \sqrt{\langle \mathcal{A}_\theta h, h\rangle} = \sqrt{\|h'\|^2 + \theta\|h\|^2}$. Hence, $\mathcal{D}\left(\mathcal{A}_\theta^{\frac{1}{2}}\right) = H^1(0,1)$. Let $\mathcal{H} = L^2(0,1)\times\mathbb{R}^{N+2}$ be a Hilbert space with the norm $\|\cdot\|_{\mathcal{H}}^2 := \|\cdot\|^2 + |\cdot|^2$. Let

$$\begin{aligned}&\xi(t) = \mathrm{col}\{\xi_1(t), \xi_2(t)\},\ \xi_1(t) = w(\cdot,t),\ \xi_2(t) = \hat{w}^N(t),\\ &\hat{w}^N(t) = \mathrm{col}\left\{u(t), \hat{w}_0(t),\dots,\hat{w}_N(t)\right\}\end{aligned} \tag{2.25}$$

the closed-loop system can be presented as

$$\begin{aligned}&\frac{d\xi}{dt}(t) + \mathrm{diag}\{\mathcal{A}_\theta, \mathcal{B}\}\xi(t) = \begin{bmatrix} f_1(\xi)\\ f_2(\xi)\end{bmatrix},\\ &\mathcal{D}(\mathcal{B}) = \mathbb{R}^{N+2},\ \mathcal{B}a = \begin{bmatrix} -\tilde{A}_0 + \tilde{B}_0K_0 + \tilde{L}_0[0\ C_0] & \tilde{L}_0C_1\\ B_1K_0 & -A_1\end{bmatrix}a\end{aligned} \tag{2.26}$$

where $-\mathcal{B}$ generates an analytic semigroup on $\mathcal{H}$ and

$$\begin{aligned}&f_1(t,\xi) = \theta w(\cdot,t) + g(t,\cdot,w(\cdot,t)+\psi(\cdot)u(t))\\ &\qquad + \psi(\cdot)K_0\hat{w}^{N_0}(t),\\ &f_2(t,\xi) = \mathrm{col}\left\{\hat{G}^{N_0}(t) + \tilde{L}_0 w(x_*,t), \hat{G}^{N-N_0}(t)\right\},\\ &\hat{G}^{N_0}(t) = \mathrm{col}\left\{0, \hat{g}_n(t)\right\}_{n=0}^{N_0},\\ &\hat{G}^{N-N_0}(t) = \mathrm{col}\left\{\hat{g}_n(t)\right\}_{n=N_0+1}^{N},\ \tilde{L}_0 = \mathrm{col}\{0, l_n\}_{n=0}^{N_0},\\ &A_1 = \mathrm{diag}\{-\lambda_n\}_{n=N_0+1}^N,\ B_1 = \mathrm{col}\{b_n\}_{n=N_0+1}^N.\end{aligned} \tag{2.27}$$

Let $\mathcal{G} = H^1(0,1)\times\mathbb{R}^{N+2}$ be a Hilbert space with the norm $\|\cdot\|_{\mathcal{G}}^2 := \|\cdot\|_{H^1}^2 + |\cdot|^2$. Fix $(t,\xi) \in [0,\infty)\times\mathcal{G}$. Let $\mathcal{Q} = J \times B_{\mathcal{G}}(\xi, R)$ be a neighborhood of (t,ξ), where J is an interval and $B_{\mathcal{G}}(\xi,R)$ is a ball of radius $R > 0$ around ξ. Let $(t_j, \varphi^{(j)}) \in \mathcal{Q}$, $j \in \{1,2\}$. Fixing $\Gamma = 1$, by Lemma 1.2, for any $j \in \{1,2\}$ we have

$$\begin{aligned}&\max_{x\in[0,1]}\left|\varphi_1^{(j)}(x)\right|^2 \overset{(1.5)}{\le} 2\left\|\varphi_1^{(j)}\right\|_{H^1}^2 \le 2\left(R + \|\xi_1\|_{H^1}\right)^2,\\ &\max_{x\in[0,1]}\left|[\psi(x)\ 0]\varphi_2^{(j)}\right|^2 \le \|\psi(x)\|_\infty^2\left(R + |\xi_2|\right)^2.\end{aligned} \tag{2.28}$$

Hence, for some $R_1(\xi) > 0$ we have for $j \in \{1,2\}$ that $\max_{x\in[0,1]}\left|\varphi_1^{(j)}(x) - [\psi(x)\ 0]\varphi_2^{(j)}\right| \le R_1(\xi)$. Let $\mathcal{S} = \mathrm{cl}(J)\times[0,1]\times[-R_1(\xi), R_1(\xi)] \subseteq \mathbb{R}^3$. By assumption, g is locally Lipschitz. Denote by $L_{\mathcal{S}}$ its Lipschitz constant on $\mathcal{S}$. Then, we obtain

$$\begin{aligned}&\left\|g(t_1,\cdot,\varphi_1^{(1)}(\cdot) + [\psi(\cdot)\ 0]\varphi_2^{(1)})\right.\\ &\qquad\left.- g(t_2,\cdot,\varphi_1^{(2)}(\cdot) + [\psi(\cdot)\ 0]\varphi_2^{(2)})\right\|^2\\ &\le 2L_{\mathcal{S}}^2\left(|t_1 - t_2|^2 + \left\|\varphi^{(1)} - \varphi^{(2)}\right\|_{\mathcal{G}}^2\right)\end{aligned} \tag{2.29}$$

From (1.5), (2.26) and (2.29) it easily follows that $f_1(t,\xi)$ and $f_2(t,\xi)$ satisfy assumption (F) in Theorem 6.3.1 in [34]. Furthermore, by (2.4), $f_1(t,\xi)$ and $f_2(t,\xi)$ also satisfy the conditions of Theorem 6.3.3 in [34]. Hence, given $w(\cdot,0) \in H^1(0,1)$, the system (2.26) has a unique classical solution satisfying

$$\xi \in C([0,\infty);\mathcal{H}) \cap C^1((0,\infty);\mathcal{H}) \tag{2.30}$$

such that

$$\xi(t) \in \mathcal{D}(\mathrm{diag}\{\mathcal{A}_\theta, \mathcal{B}\}) = \mathcal{D}(\mathcal{A})\times\mathbb{R}^{N+2} \quad \forall t > 0. \tag{2.31}$$

2.3. H^1-stability of the closed-loop system

Introduce the estimation error $e_n(t) = w_n(t) - \hat{w}_n(t)$, $0 \le n \le N_0$. Using the estimation error and $\{c_n\}_{n=0}^N$ in (2.21), the innovation term in (2.18) can be presented as

$$\begin{aligned}&\hat{w}(x_*,t)+\psi(x_*)u(t)-y(t)=\hat{w}(x_*,t)-w(x_*,t)\\&=-\sum_{n=0}^{N}\left[w_n(t)-\hat{w}_n(t)\right]\phi_n(x_*)-\zeta(t)\\&=-\sum_{n=0}^{N}c_ne_n(t)-\zeta(t),\\&\zeta(t)=w(x_*,t)-\sum_{n=0}^{N}w_n(t)\phi_n(x_*).\end{aligned}\tag{2.32}$$

Let $\Gamma>0$. By Lemmas 1.1 and 1.2 we have

$$\begin{aligned}\zeta^2(t)&\leq\max_{x\in[0,1]}\left|w(x,t)-\sum_{n=0}^{N}w_n(t)\phi_n(x)\right|^2\\&\overset{(1.5)}{\leq}(1+\Gamma)\left\|w(\cdot,t)-\sum_{n=0}^{N}w_n(t)\phi_n(\cdot)\right\|^2\\&\quad+\Gamma^{-1}\left\|w_x(\cdot,t)-\sum_{n=0}^{N}w_n(t)\phi_n'(\cdot)\right\|^2\\&\overset{(1.4)}{=}\sum_{n=N+1}^{\infty}\kappa_nw_n^2(t),\ \ \kappa_n=1+\Gamma+\Gamma^{-1}\lambda_n.\end{aligned}\tag{2.33}$$

Taking into account (2.13), (2.18), (2.21) and (2.32), the estimation error satisfies the following ODEs

$$\begin{aligned}\dot{e}_n(t)&=-\lambda_ne_n(t)+h_n(t)\\&\quad-l_n\sum_{n=0}^{N}c_ne_n(t)-l_n\zeta(t),\ \ 0\leq n\leq N_0,\\\dot{e}_n(t)&=-\lambda_ne_n(t)+h_n(t),\ \ N_0+1\leq n\leq N.\end{aligned}\tag{2.34}$$

where we define

$$h_n(t)=g_n(t)-\hat{g}_n(t),\quad n\geq 0.\tag{2.35}$$

Recall (2.21), (2.27) and denote

$$\begin{aligned}&\hat{w}^{N-N_0}(t)=\mathrm{col}\left\{\hat{w}_n(t)\right\}_{n=N_0+1}^{N},\\&e^{N_0}(t)=\mathrm{col}\left\{e_n(t)\right\}_{n=0}^{N_0},\\&e^{N-N_0}(t)=\mathrm{col}\left\{e_n(t)\right\}_{n=N_0+1}^{N},\\&H^{N_0}(t)=\mathrm{col}\left\{h_n(t)\right\}_{n=0}^{N_0},\\&H^{N-N_0}(t)=\mathrm{col}\left\{h_n(t)\right\}_{n=N_0+1}^{N},\\&X(t)=\mathrm{col}\left\{\hat{w}^{N_0}(t),e^{N_0}(t),\hat{w}^{N-N_0}(t),e^{N-N_0}(t)\right\},\\&L_\zeta=\mathrm{col}\left\{\tilde{L}_0,-L_0,0,0\right\}\in\mathbb{R}^{2N+3},\\&\hat{G}(t)=\mathrm{col}\left\{\hat{G}^{N_0}(t),0,\hat{G}^{N-N_0}(t),0\right\},\\&H(t)=\mathrm{col}\left\{0,H^{N_0}(t),0,H^{N-N_0}(t)\right\},\\&K_X=[K_0,0,0,0]\in\mathbb{R}^{1\times(2N+3)}.\end{aligned}\tag{2.36}$$

Then, using (2.13), (2.18)–(2.21), (2.24), (2.32), (2.34) and (2.36), the closed-loop system for $t\geq 0$ can be presented as

$$\begin{aligned}\dot{X}(t)&=F_XX(t)+L_\zeta\zeta(t)+\hat{G}(t)+H(t),\\\dot{w}_n(t)&=-\lambda_nw_n(t)+\hat{g}_n(t)+h_n(t)\\&\quad-b_nK_XX(t),\ \ n>N\end{aligned}\tag{2.37}$$

where

$$F_X=\begin{bmatrix}\tilde{A}_0-\tilde{B}_0K_0&\tilde{L}_0C_0&0&\tilde{L}_0C_1\\0&A_0-L_0C_0&0&-L_0C_1\\-B_1K_0&0&A_1&0\\0&0&0&A_1\end{bmatrix}.$$

The main stability result of this section is given in the following theorem:

Theorem 2.1. *Consider the system (2.11) with boundary conditions (2.9), point measurement (2.10) and control law (2.24). Assume that $g(t,x,z)$ is a locally Lipschitz function satisfying $g(t,x,0)\equiv 0$ and (2.4) for a given $\sigma>0$. Let $\delta>0$, $N_0\in\mathbb{N}$ satisfy (2.16) and $N\in\mathbb{N}$ satisfy $N_0\leq N$. Let L_0 and K_0 be obtained using (2.22)* and *(2.23), respectively. Given $\Gamma>0$, let there exist $0<P\in\mathbb{R}^{(2N+3)\times(2N+3)}$ and scalars $\alpha_1,\alpha_2,\alpha_3>0$ such that*

$$\begin{aligned}&\left[\begin{array}{cc|cc|c}\psi_0&P_XL_\zeta&P_X&P_X&\Pi_1\\ *&2\bar{\rho}_{N+1}&0&0&\\\hline \multicolumn{2}{c|}{*}&\multicolumn{2}{c|}{\mathrm{diag}\{-\alpha_1I,-\alpha_2I\}}&0\\\hline \multicolumn{2}{c|}{*}&\multicolumn{2}{c|}{*}&\Pi_2\end{array}\right]<0,\\&\Pi_1=\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1&1&1\end{bmatrix},\\&\Pi_2=-\frac{2\kappa_{N+1}}{\lambda_{N+1}}\mathrm{diag}\left\{\frac{\alpha_1}{\lambda_{N+1}},\frac{\alpha_2}{\lambda_{N+1}},\alpha_3\right\},\\&\bar{\rho}_{N+1}=2\kappa_{N+1}^{-1}\left(-\lambda_{N+1}^2+\delta\lambda_{N+1}+\frac{\alpha_2\sigma^2}{2}\right)\end{aligned}\tag{2.38}$$

holds with ψ_0 given in (A.11)

$$\begin{aligned}\psi_0&=P_XF_X+F_X^TP_X+2\delta P_X+\frac{2\alpha_3\xi_{N+1}}{\pi^2}K_X^TK_X\\&\quad+2\alpha_1\sigma^2\Xi_X+\alpha_2\sigma^2\Xi_E.\end{aligned}\tag{2.39}$$

Then, given $w(\cdot,0)\in H^1(0,1)$, the solution $u(t)$, $w(x,t)$ of (2.11) subject to the control law (2.24) and the observer $\hat{w}(x,t)$ defined by (2.17)–(2.19), satisfy

$$u^2(t)+\|w(\cdot,t)\|_{H^1}^2+\left\|\hat{w}(\cdot,t)\right\|_{H^1}^2\leq De^{-2\delta t}\|w(\cdot,0)\|_{H^1}^2\tag{2.40}$$

for $t\geq 0$ and some $D\geq 1$. Moreover, the LMI (2.38) is always feasible for N large enough and $\sigma>0$ small enough.

Proof. The proof is given in Appendix A. □

3. Finite-dimensional sequential sub-predictors for semilinear heat equation

3.1. Problem formulation

In this section we consider stabilization of (2.1) under the point measurement (2.3) and subject to delayed Neumann actuation

$$z_x(0,t)=0,\quad z_x(1,t)=u(t-r),\quad t\geq 0.\tag{3.1}$$

Here $r>0$ is a known constant input delay and $u(t)=0$ for $t\leq 0$. As in the previous section, $g(t,x,z)$ is a locally Lipschitz function satisfying $g(t,x,0)\equiv 0$ and (2.4) for some $\sigma>0$. We aim to achieve H^1-stabilization of (2.1) in the presence of the input delay $r>0$ in (3.1).

Let $\psi(x)=-\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)$ satisfy (2.5) and (2.6). To obtain homogeneous boundary conditions we employ the delayed change of variables

$$w(x,t)=z(x,t)-\psi(x)u(t-r),\tag{3.2}$$

that leads to the following PDE

$$\begin{aligned}w_t(x,t)&=w_{xx}(x,t)+g\left(t,x,w(x,t)+\psi(x)u(t-r)\right)\\&\quad-\psi(x)\left[\mu u(t-r)+\dot{u}(t-r)\right]\end{aligned}\tag{3.3}$$

As in the non-delayed case, we will construct an integral control law. In order to satisfy $u(t)=0,\ t\leq 0$ and to guarantee that $u(t)$ is continuously differentiable in $t\in\mathbb{R}$, we consider

$$u(t)=\int_0^te^{-\mu(t-s)}v(s)ds,\quad t\in\mathbb{R}\tag{3.4}$$

where $v(t)$ will be constructed below as continuous and satisfying to $v(t)=0$ for $t\leq 0$. Then, $u(t)$ satisfies

$$\dot{u}(t)=-\mu u(t)+v(t),\ \ t\in\mathbb{R}.\tag{3.5}$$

For our sub-predictor construction below, we would like the ODE for u and the PDE for w to contain the control input evaluated at the same time $t-r$ (see $w^{N_0}(t)$ and $w^{N-N_0}(t)$ in (3.11) below).

Hence, replacing t by $t-r$ in (3.5) and substituting into (3.3) we obtain the following ODE–PDE system for $t \geq 0$

$$\begin{aligned} &\dot{u}(t-r) = -\mu u(t-r) + v(t-r), \\ &w_t(x,t) = w_{xx}(x,t) + g\left(t, x, w(x,t) + \psi(x)u(t-r)\right) \\ &\qquad - \psi(x)v(t-r) \end{aligned} \tag{3.6}$$

with the boundary conditions (2.9) and measurement

$$y(t) = w(x_*, t) + \psi(x_*)u(t-r). \tag{3.7}$$

We will treat $u(t-r)$ as the additional state variable and $v(t-r)$ as the new control input.

We present the solution to (3.6) as (2.12), with $\{\phi_n\}_{n=0}^{\infty}$ defined in (1.3). Similar to (2.13), we obtain for $t \geq 0$

$$\begin{aligned} &\dot{w}_n(t) = -\lambda_n w_n(t) + g_n(t) + b_n v(t-r), \\ &w_n(0) = \langle w(\cdot,0), \phi_n \rangle, \ n \in \mathbb{Z}_+ \end{aligned} \tag{3.8}$$

where $\{b_n\}_{n=0}^{\infty}$ are given in (2.14) and

$$g_n(t) = \langle g\left(t, \cdot, w(\cdot,t) + \psi(\cdot)u(t-r)\right), \phi_n \rangle. \tag{3.9}$$

Let $\delta > 0$ be a desired decay rate and let $N_0 \in \mathbb{Z}_+$ subject to (2.16) define the number of modes in the controller. Let $N \in \mathbb{Z}_+, \ N \geq N_0$ and introduce

$$\begin{aligned} &w^{N_0}(t) = \mathrm{col}\left\{u(t-r), w_1(t), \dots, w_{N_0}(t)\right\}, \\ &w^{N-N_0}(t) = \mathrm{col}\left\{w_{N_0+1}(t), \dots, w_N(t)\right\}, \\ &G^{N_0}(t) = \mathrm{col}\left\{0, g_n(t)\right\}_{n=1}^{N_0}, \\ &G^{N-N_0}(t) = \mathrm{col}\left\{g_n(t)\right\}_{n=N_0+1}^{N}. \end{aligned} \tag{3.10}$$

Then, recalling A_1 and B_1 in (2.36) and using (3.8) we find that for $t \geq 0$ $w^{N_0}(t)$ and $w^{N-N_0}(t)$ satisfy

$$\begin{aligned} &\dot{w}^{N_0}(t) = \tilde{A}_0 w^{N_0}(t) + \tilde{B}_0 v(t-r) + G^{N_0}(t), \\ &\dot{w}^{N-N_0}(t) = A_1 w^{N-N_0}(t) + B_1 v(t-r) + G^{N-N_0}(t). \end{aligned} \tag{3.11}$$

3.2. Finite-dimensional observer-based controller design

Consider the ODEs satisfied by $w^{N_0}(t)$, given in (3.11). In order to deal with the input delay $r > 0$ therein, we fix $M \in \mathbb{N}$ and subdivide r into M parts of equal size $\frac{r}{M}$. We first consider $M \geq 2$ and design a chain of sub-predictors (observers of future state)

$$\begin{aligned} &\hat{w}_1^j(t-r) \mapsto \cdots \mapsto \hat{w}_i^j\left(t - \tfrac{M-i+1}{M}r\right) \mapsto \cdots \\ &\mapsto \hat{w}_M^j\left(t - \tfrac{1}{M}r\right) \mapsto w^j(t), \quad j \in \{N_0, N-N_0\}. \end{aligned} \tag{3.12}$$

Here $\hat{w}_i^j\left(t - \frac{M-i+1}{M}r\right) \mapsto \hat{w}_{i+1}^j\left(t - \frac{M-i}{M}r\right)$ means that $\hat{w}_i^j(t)$ predicts the value of $\hat{w}_{i+1}^j(t+\frac{r}{M})$. Similarly, $\hat{w}_M^j(t)$ predicts the value of $w^j(t+\frac{r}{M})$.

Remark 3.1. Differently from the linear case [30], here the sub-predictors are constructed for both $w^{N_0}(t)$ and $w^{N-N_0}(t)$. This is due to the semilinearity in (2.1), which leads to coupling between all modes of the solution.

We assume the following:

Assumption 2. The point $x_* \in [0,1]$ satisfies (2.20) and $\psi(x_*) \neq 0$.

Note that Assumption 2 holds for the particular case $x_* = 0$ of non-collocated measurement. Recall the notations in (2.21) and let

$$\tilde{C}_0 = [\psi(x_*), C_0]. \tag{3.13}$$

Under Assumption 2, the pair $(\tilde{A}_0, \tilde{C}_0)$ is observable by the Hautus lemma. Let $L_0 \in \mathbb{R}^{N_0+2}$ satisfy the Lyapunov inequality (2.22) with $0 < P_o \in \mathbb{R}^{(N_0+2)\times(N_0+2)}$ and A_0, C_0 replaced by $\tilde{A}_0, \tilde{C}_0$, respectively. We further choose the remaining gains as $l_n = 0$, $N_0+1 \leq n \leq N$. Similarly, by the Hautus lemma, the pair $(\tilde{A}_0, \tilde{B}_0)$ is controllable. Let $K_0 \in \mathbb{R}^{1\times(N_0+2)}$ satisfy (2.23) with $0 < P_c \in \mathbb{R}^{(N_0+2)\times(N_0+2)}$.

For $0 \leq n \leq N$ and $1 \leq i \leq M$ denote

$$\begin{aligned} &\hat{g}_n^{(i)}(t) = \\ &\left\langle g\left(t + \tfrac{(M+1-i)r}{M}, \cdot, Q(\cdot)\,\mathrm{col}\left\{\hat{w}_i^{N_0}(t), \hat{w}_i^{N-N_0}(t)\right\}\right), \phi_n \right\rangle, \\ &Q^T(x) = \mathrm{col}\left\{\psi(x), \phi_0(x), \dots, \phi_N(x)\right\}, \\ &\hat{G}_i^{N_0}(t) = \mathrm{col}\left\{0, \hat{g}_n^{(i)}(t)\right\}_{n=0}^{N_0}, \\ &\hat{G}_i^{N-N_0}(t) = \mathrm{col}\left\{\hat{g}_n^{(i)}(t)\right\}_{n=N_0+1}^{N}. \end{aligned} \tag{3.14}$$

The sub-predictors satisfy the following ODEs for $t \geq 0$

$$\begin{aligned} &\dot{\hat{w}}_M^{N_0}(t) = \tilde{A}_0 \hat{w}_M^{N_0}(t) + \tilde{B}_0 v\left(t - \tfrac{M-1}{M}r\right) + \hat{G}_M^{N_0}(t) \\ &\quad - L_0\left[\tilde{C}_0 \hat{w}_M^{N_0}(t - \tfrac{r}{M}) + C_1 \hat{w}_M^{N-N_0}(t - \tfrac{r}{M}) - y(t)\right] \\ &\dot{\hat{w}}_M^{N-N_0}(t) = A_1 \hat{w}_M^{N-N_0}(t) + B_1 v\left(t - \tfrac{M-1}{M}r\right) \\ &\qquad + \hat{G}_M^{N-N_0}(t), \\ &\dot{\hat{w}}_i^{N_0}(t) = \tilde{A}_0 \hat{w}_i^{N_0}(t) + \tilde{B}_0 v\left(t - \tfrac{i-1}{M}r\right) + \hat{G}_i^{N_0}(t) \\ &\quad - L_0\left[\tilde{C}_0 \hat{w}_i^{N_0}(t - \tfrac{r}{M}) + C_1 \hat{w}_i^{N-N_0}(t - \tfrac{r}{M})\right. \\ &\qquad \left. - \tilde{C}_0 \hat{w}_{i+1}^{N_0}(t) - C_1 \hat{w}_{i+1}^{N-N_0}(t)\right], \\ &\dot{\hat{w}}_i^{N-N_0}(t) = A_1 \hat{w}_i^{N-N_0}(t) + B_1 v\left(t - \tfrac{i-1}{M}r\right) \\ &\qquad + \hat{G}_i^{N-N_0}(t), \quad 1 \leq i \leq M-1 \end{aligned} \tag{3.15}$$

subject to

$$\hat{w}_i^{N_0}(t) = 0, \ \hat{w}_i^{N-N_0}(t) = 0, \quad 1 \leq i \leq M, \ t \leq 0. \tag{3.16}$$

Note that as i decreases, the input delay on the right-hand-side of the ODEs in (3.15) decreases by $\frac{r}{M}$. For the case $M = 1$, the ODEs have the following form

$$\begin{aligned} &\dot{\hat{w}}_1^{N_0}(t) = \tilde{A}_0 \hat{w}_1^{N_0}(t) + \tilde{B}_0 v(t-r) + \hat{G}_1^{N_0}(t) \\ &\quad - L_0\left[\tilde{C}_0 \hat{w}_1^{N_0}(t-r) + C_1 \hat{w}_1^{N-N_0}(t-r) - y(t)\right] \\ &\dot{\hat{w}}_1^{N-N_0}(t) = A_1 \hat{w}_1^{N-N_0}(t) + B_1 v(t-r) + \hat{G}_1^{N-N_0}(t). \end{aligned} \tag{3.17}$$

The finite-dimensional observer $\hat{w}(x,t)$ of the state $w(x,t)$, based on the $M \times (N+2)$ dimensional system of ODEs (3.15) is then given by

$$\begin{aligned} \hat{w}(x,t) = &\ \hat{w}_1^{N_0}(t-r) \cdot \mathrm{col}\left\{0, \phi_n(x)\right\}_{n=0}^{N_0} \\ &+ \hat{w}_1^{N-N_0}(t-r) \cdot \mathrm{col}\left\{\phi_n(x)\right\}_{n=N_0+1}^{N}. \end{aligned} \tag{3.18}$$

The controller is further chosen as

$$v(t) = -K_0 \hat{w}_1^{N_0}(t). \tag{3.19}$$

In particular, (3.15) and (3.16) imply continuity of $v(t)$ and $v(t) = 0$ for $t \leq 0$.

Well-posedness of the closed-loop system (3.6) and (3.15) subject to the control law (3.19) follows from arguments similar to (2.25)–(2.31) combined with the step method, meaning proof of well-posedness step by step on the intervals $[\frac{jr}{M}, \frac{(j+1)r}{M})$, $j = 0, 1, \dots$ (see Section A of [30], where such arguments have been used for sub-predictors). In particular, given $w(\cdot,0) \in H^1(0,1)$ we obtain a unique classical solution satisfying $w(\cdot,t) \in C\left([0,\infty); L^2(0,1)\right) \cap C^1\left((0,\infty); L^2(0,1) \setminus \mathcal{J}\right)$ with $\mathcal{J} = \left\{\frac{jr}{M}\right\}_{j=0}^{\infty}$. Furthermore, $w(\cdot,t) \in \mathcal{D}(\mathcal{A})$ for all $t > 0$. We omit the details due to space constraints.

3.3. H^1-stability of the closed-loop system

We define the estimation errors as follows

$$
\begin{aligned}
&e_M^{N_0}(t) = w^{N_0}(t) - \hat{w}_M^{N_0}(t - \tfrac{r}{M}),\\
&e_M^{N-N_0}(t) = w^{N-N_0}(t) - \hat{w}_M^{N-N_0}(t - \tfrac{r}{M}),\\
&e_i^{N_0}(t) = \hat{w}_{i+1}^{N_0}(t - \tfrac{M-i}{M}r) - \hat{w}_i^{N_0}(t - \tfrac{M-i+1}{M}r),\\
&e_i^{N-N_0}(t) = \hat{w}_{i+1}^{N-N_0}(t - \tfrac{M-i}{M}r)\\
&\qquad - \hat{w}_i^{N-N_0}(t - \tfrac{M-i+1}{M}r), \ \ 1 \le i \le M-1.
\end{aligned} \tag{3.20}
$$

Then, the innovation term on the right-hand-side of the ODEs for $\hat{w}_M^{N_0}(t)$ given in (3.15) can be presented as

$$
\begin{aligned}
&\tilde{C}_0 \hat{w}_M^{N_0}(t - \tfrac{r}{M}) + C_1 \hat{w}_M^{N-N_0}(t - \tfrac{r}{M}) - y(t)\\
&\overset{(3.7)}{=} -\tilde{C}_0 e_M^{N_0}(t) - C_1 e_M^{N-N_0}(t) - \zeta(t).
\end{aligned} \tag{3.21}
$$

Here, $\zeta(t)$ is given in (2.32) and satisfies the estimate (2.33) with $\Gamma > 0$. Furthermore, by (3.20), we have

$$
\hat{w}_1^{N_0}(t - r) + \textstyle\sum_{i=1}^{M} e_i^{N_0}(t) = w^{N_0}(t). \tag{3.22}
$$

If the errors $e_i^{N_0}(t)$, $1 \le i \le M$ converge to zero, we have $\hat{w}_1^{N_0}(t) \mapsto w^{N_0}(t + r)$, meaning that $\hat{w}_1^{N_0}(t)$ predicts the future system state $w^{N_0}(t + r)$.

Using (3.11), (3.15) and (3.21) we obtain

$$
\begin{aligned}
&\dot{e}_M^{N_0}(t) = \left(\tilde{A}_0 - L_0\tilde{C}_0\right) e_M^{N_0}(t) - L_0 C_1 e_M^{N-N_0}(t) + L_0\tilde{C}_0\\
&\times \Upsilon_{M,r}^{N_0}(t) + L_0 C_1 \Upsilon_{M,r}^{N-N_0}(t) - L_0\zeta(t - \tfrac{r}{M}) + H_M^{N_0}(t)\\
&\dot{e}_M^{N-N_0}(t) = A_1 e_M^{N-N_0}(t) + H_M^{N-N_0}(t),\\
&\dot{e}_{M-1}^{N_0}(t) = \left(\tilde{A}_0 - L_0\tilde{C}_0\right) e_{M-1}^{N_0}(t) - L_0 C_1 e_{M-1}^{N-N_0}(t)\\
&+ L_0\tilde{C}_0 \Upsilon_{M-1,r}^{N_0}(t) + L_0 C_1 \Upsilon_{M-1,r}^{N-N_0}(t) + L_0\tilde{C}_0 e_M^{N_0}(t)\\
&- L_0\tilde{C}_0 \Upsilon_{M,r}^{N_0}(t) + L_0 C_1 e_M^{N-N_0}(t) - L_0 C_1 \Upsilon_{M,r}^{N-N_0}(t)\\
&+ L_0\zeta(t - \tfrac{r}{M}) + H_{M-1}^{N_0}(t),\\
&\dot{e}_{M-1}^{N-N_0}(t) = A_1 e_{M-1}^{N-N_0}(t) + H_{M-1}^{N-N_0}(t), \ \ t \ge 0,
\end{aligned} \tag{3.23}
$$

whereas for $1 \le i \le M-2$

$$
\begin{aligned}
&\dot{e}_i^{N_0}(t) = (\tilde{A}_0 - L_0\tilde{C}_0) e_i^{N_0}(t) - L_0 C_1 e_i^{N-N_0}(t)\\
&+ L_0\tilde{C}_0 e_{i+1}^{N_0}(t) + L_0 C_1 e_{i+1}^{N-N_0}(t) + L_0\tilde{C}_0 \Upsilon_{i,r}^{N_0}(t)\\
&+ L_0 C_1 \Upsilon_{i,r}^{N-N_0}(t) - L_0\tilde{C}_0 \Upsilon_{i+1,r}^{N_0}(t)\\
&- L_0 C_1 \Upsilon_{i+1,r}^{N-N_0}(t) + H_i^{N_0}(t),\\
&\dot{e}_i^{N-N_0}(t) = A_1 e_i^{N-N_0}(t) + H_i^{N-N_0}(t), \ \ t \ge 0.
\end{aligned} \tag{3.24}
$$

Here

$$
\begin{aligned}
&\Upsilon_{i,r}^{N_0}(t) = e_i^{N_0}(t) - e_i^{N_0}(t - \tfrac{r}{M}),\\
&\Upsilon_{i,r}^{N-N_0}(t) = e_i^{N-N_0}(t) - e_i^{N-N_0}(t - \tfrac{r}{M}),\\
&H_M^{N_0}(t) = G^{N_0}(t) - \hat{G}_M^{N_0}(t - \tfrac{r}{M}),\\
&H_M^{N-N_0}(t) = G^{N-N_0}(t) - \hat{G}_M^{N-N_0}(t - \tfrac{r}{M}),\\
&H_i^{N_0}(t) = \hat{G}_{i+1}^{N_0}(t - \tfrac{M-i}{M}r) - \hat{G}_i^{N_0}(t - \tfrac{M-i+1}{M}r),\\
&H_i^{N-N_0}(t) = \hat{G}_{i+1}^{N-N_0}(t - \tfrac{M-i}{M}r) - \hat{G}_i^{N-N_0}(t - \tfrac{M-i+1}{M}r).
\end{aligned} \tag{3.25}
$$

From (3.11), (3.19) and (3.22) we further have

$$
\begin{aligned}
&\dot{w}^{N_0}(t) = \left(\tilde{A}_0 - \tilde{B}_0 K_0\right) w^{N_0}(t) + \tilde{B}_0 K_0 \textstyle\sum_{i=1}^{M} e_i^{N_0}(t)\\
&\qquad + G^{N_0}(t),\\
&\dot{w}^{N-N_0}(t) = A_1 w^{N-N_0}(t) + B_1 K_0 \textstyle\sum_{i=1}^{M} e_i^{N_0}(t)\\
&\qquad + G^{N-N_0}(t).
\end{aligned} \tag{3.26}
$$

We introduce the notations

$$
\begin{aligned}
&X(t) = \mathrm{col}\left\{w^{N_0}(t), w^{N-N_0}(t)\right\},\\
&X_e(t) = \mathrm{col}\left\{e_1^{N_0}(t), e_1^{N-N_0}(t), \dots, e_M^{N_0}(t), e_M^{N-N_0}(t)\right\},\\
&\Upsilon_{e,r}(t) = X_e(t) - X_e\left(t - \tfrac{r}{M}\right),\\
&H(t) = \mathrm{col}\left\{H_1^{N_0}(t), H_1^{N-N_0}(t), \dots, H_M^{N_0}(t), H_M^{N-N_0}(t)\right\}
\end{aligned} \tag{3.27}
$$

and

$$
\begin{aligned}
&G(t) = \mathrm{col}\left\{G^{N_0}(t), G^{N-N_0}(t)\right\},\\
&F_X = \begin{bmatrix} \tilde{A}_0 - \tilde{B}_0 K_0 & 0 \\ -B_1 K_0 & A_1 \end{bmatrix}, \ B_X = \mathrm{col}\left\{\tilde{B}_0, B_1\right\},\\
&\mathcal{I} = [I_{N_0+2}\ 0\ I_{N_0+2}\ 0 \dots I_{N_0+2}\ 0] \in \mathbb{R}^{1\times M(N+2)},\\
&F_0 = \begin{bmatrix} \tilde{A}_0 - L_0\tilde{C}_0 & -L_0 C_1 \\ 0 & A_1 \end{bmatrix}, \ \mathcal{L}_0 = \begin{bmatrix} L_0 \\ 0 \end{bmatrix}, \ \mathcal{C} = [\tilde{C}_0\ C_1],\\
&F_e = I_M \otimes F_0 + J_{0,M} \otimes \mathcal{L}_0\mathcal{C}, \ \ \tilde{K}_0 = [K_0, 0_{1\times(N-N_0)}]\\
&\Lambda_e = I_M \otimes \mathcal{L}_0\mathcal{C} - J_{0,M} \otimes \mathcal{L}_0\mathcal{C},\\
&\mathcal{L}_\zeta = \mathrm{col}\{0, 0, \dots, 0, \mathcal{L}_0, -\mathcal{L}_0\} \in \mathbb{R}^{M(N+2)}.
\end{aligned} \tag{3.28}
$$

Here $J_{0,M}$ is an upper triangular Jordan block of order M with zero diagonal and $\otimes$ is the Kronecker product. Then, from (3.8), (3.24), (3.26) and (3.28) we obtain the following closed-loop system for $t \ge 0$

$$
\begin{aligned}
&\dot{X}(t) = F_X X(t) + B_X K_0 \mathcal{I} X_e(t) + G(t),\\
&\dot{X}_e(t) = F_e X_e(t) + \Lambda_e \Upsilon_{e,r}(t) + \mathcal{L}_\zeta \zeta(t - \tfrac{r}{M}) + H(t),\\
&\dot{w}_n(t) = -\lambda_n w_n(t) + g_n(t) - b_n \tilde{K}_0 X(t)\\
&\qquad + b_n K_0 \mathcal{I} X_e(t), \quad n > N.
\end{aligned} \tag{3.29}
$$

Differently from the existing finite-dimensional controllers [8,35], where the closed-loop systems are written in terms of the observer and the tail $w_n(t)$ ($n > N$), here (3.29) is presented in terms of the state $X(t)$, the estimation errors $X_e(t)$ and the tail. This allows to eliminate the delay r from the ODEs of $X(t)$ and $w_n(t)$, $n > N$ while decreasing it to $\frac{r}{M}$ (which is small for large M) in the ODEs of $X_e(t)$.

Remark 3.2. Consider the ODEs satisfied by the subpredictor errors $X_e(t)$ in (3.29). For $\zeta(t) \equiv 0$ and $\sigma = 0$ (i.e. $g \equiv 0$), stability of the ODE for $X_e(t)$ was demonstrated in [30, Theorem 1], by recursively constructing a Lyapunov functional. For $\zeta(t) \equiv 0$, it can be easily verified that $|H(t)|^2 \le \sigma^2 |X_e(t)|^2$. Hence, the same Lyapunov functional can be used to show stability of $X_e(t)$ for $\zeta(t) \equiv 0$ and small $\sigma > 0$. The coupling of $X_e(t)$ with the tail ODEs through $\zeta\left(t - \frac{r}{M}\right)$ is treated in the H^1-stability analysis in Appendix B, via the Lyapunov functional defined by (B.1) and (B.2).

The main stability result of this section is given in the following theorem:

Theorem 3.1. *Consider the system (3.6) with boundary conditions (2.9), point measurement (3.7) and control law (3.19). Assume that $g(t, x, z)$ is a locally Lipschitz function satisfying $g(t, x, 0) \equiv 0$ and (2.4) for a given $\sigma > 0$. Let $\delta > 0$, $N_0 \in \mathbb{N}$ satisfy (2.16) and $N \in \mathbb{N}$ satisfy $N_0 \le N$. Let L_0 and K_0 be obtained using (2.22) (with A_0, C_0 replaced by $\tilde{A}_0, \tilde{C}_0$) and (2.23), respectively. Given $M \in \mathbb{N}$ and $\Gamma > 0$, let there exist positive definite matrices P_X, P_e, S_e, R_e and scalars $q, \alpha_1, \alpha_2, \alpha_3, \beta > 0$ such that*

$$
\begin{aligned}
&\Psi_1 < 0,\\
&\left[\begin{array}{c|c} \varphi_3 & 1 \quad 1 \quad 1 \\ \hline * & -\frac{2}{\lambda_{N+1}} \mathrm{diag}\left\{\frac{\alpha_1}{\lambda_{N+1}}, \alpha_2, \alpha_3\right\} \end{array}\right] < 0,\\
&\varphi_3 = -\lambda_{N+1}^2 + \left(\delta + \tfrac{q\Gamma}{2}\right)\lambda_{N+1}\\
&\qquad + \sigma^2(\alpha_1 + \beta) + \tfrac{q}{2}(1 + \Gamma)
\end{aligned} \tag{3.30}
$$

Table 1
Theorem 2.1: Feasibility of LMI.

N	3	4	5	6	7	8
$\sigma_{\max}$	0.39	0.47	0.59	0.64	0.76	0.83

where

$$\Psi_1 = \left[\begin{array}{c|ccc|c} \Phi_1 & \begin{matrix} P_X B_X K_0 \mathcal{I} \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline * & & \Phi_2 & & \begin{matrix} P_e \\ 0 \\ 0 \end{matrix} \\ \hline * & & * & & -\beta I \end{array}\right] + \left(\tfrac{r}{M}\right)^2 \Theta^T R_e \Theta,$$
$$\Phi_1 = \begin{bmatrix} \varphi_1 & P_X \\ & -\alpha_1 I \end{bmatrix}, \ \Phi_2 = \begin{bmatrix} \varphi_2 & P_e \mathcal{L}_\zeta & P_e \Lambda_e - \varepsilon_{r,M} S_e \\ * & -q\varepsilon_{r,M} & 0 \\ * & * & -\varepsilon_{r,M}(S_e + R_e) \end{bmatrix},$$
$$\Theta = [0, 0, F_e, \mathcal{L}_\zeta, \Lambda_e, I],$$
$$\varphi_1 = P_X F_X + F_X^T P_X + 2\delta P_X + 2\alpha_1 \sigma^2 \Xi_X + \tfrac{2\alpha_2 \xi_{N+1}}{\pi^2} \tilde{K}_0^T \tilde{K}_0$$
$$\varphi_2 = P_e F_e + F_e^T P_e + 2\delta P_e + \tfrac{2\alpha_3 \xi_{N+1}}{\pi^2} \mathcal{I}^T K_0^T K_0 \mathcal{I} + 2\beta\sigma^2 \Xi_E + \left(1 - \varepsilon_{r,M}\right) S_e. \tag{3.31}$$

Then, given $w(\cdot,0) \in H^1(0,1)$, the solution $u(t-r)$, $w(x,t)$ of (3.6) subject to the control law (3.19) and the observer $\hat{w}(x,t)$, defined by (3.15) (with notations (3.14)) and (3.18), satisfy

$$u^2(t-r) + \|w(\cdot,t)\|_{H^1}^2 + \left\|\hat{w}(\cdot,t)\right\|_{H^1}^2 \le De^{-2\delta t} \|w(\cdot,0)\|_{H^1}^2 \tag{3.32}$$

for $t \ge 0$ and some $D \ge 1$. Given $r > 0$, (3.30) are always feasible for M, N large enough and $\sigma > 0$ small enough.

Proof. The proof is given in Appendix B. □

4. Numerical example

Consider first (2.1) under Neumann actuation (2.2) and boundary measurement (2.3), where $x_* = 0$. Recall that $g(t,x,z)$ is a locally Lipschitz function satisfying $g(t,x,0) \equiv 0$ and (2.4) for a given $\sigma > 0$. Let $\delta = 0.001$ be the desired decay rate and $N_0 = 0$. This value of δ is chosen to minimize the observer dimension which preserves feasibility of the LMIs. Let the gains L_0 and K_0 satisfy (2.22) and (2.23), respectively. The gains are given by

$$L_0 = 2.75, \ K_0 = \left[-5.468 \quad 32.19\right].$$

Given $N \in \{4,5,\dots,9\}$, the LMI of Theorem 2.1 was verified using Matlab to obtain the largest value of σ which preserves feasibility of the LMI. The results are presented in Table 1. In this example, simulations show that increasing the observer dimension N allows to obtain larger σ_{max}.

Next, consider (2.1) under Neumann actuation with constant input delay (2.2) and boundary measurement (2.3), where $x_* = 0$. Let $\delta = 0.001$ be the desired decay rate, $\sigma = 0.5$ and $N_0 = 0$. This value of δ is chosen to minimize the observer dimension and to maximize the input delay which preserve feasibility of the LMIs. Let the gains L_0 and K_0 be obtained using (2.22) (with C_0 replaced by $\tilde{C}_0$ in (3.13)) and (2.23), respectively. The gains are given by

$$L_0 = \left[7.33 \quad 1.01\right]^T, \ K_0 = \left[1.95 \quad 0.55\right]. \tag{4.1}$$

Given $M = 2$ and $N \in \{4,5,6\}$, the LMIs of Theorem 3.1 were verified to obtain the largest value of the input delay $r > 0$ which preserves feasibility of the LMIs. The results are presented in Table 2.

For simulations of the closed-loop system, consider (2.1) under Neumann actuation with constant input delay (2.2), boundary measurement (2.3) at $x_* = 0$ and

$$g(t,x,z) = \sigma \sin(t + 3x + z).$$

Table 2
Theorem 3.1: Feasibility of LMIs ($\sigma = 0.5$, $M = 2$).

N	4	5	6
$r_{\max}$	0.32	0.45	0.56

We fix $\sigma = 0.5$, delay $r = 0.32$, $N = 4$ and $M = 2$ subpredictors. Let the gains be given by (4.1). The ODE–PDE system (3.6) and subpredictor ODEs (3.15) were simulated using the FTCS (Forward Time Centered Space) and Forward Euler finite-difference schemes, where the initial condition was chosen as

$$w(x,0) = 8.5x(1-x), \quad x \in [0,1].$$

The simulation results are given in Fig. 1 and confirm our theoretical analysis. Stability of the closed-loop system in simulation was preserved for $r = 0.63$, which implies that our approach is somewhat conservative in this example.

5. Conclusions

In this paper we studied global boundary stabilization of a semilinear heat equation under point measurement. For the non-delayed case, we suggested a finite-dimensional nonlinear observer-based controller. To compensate a constant input delay, we constructed nonlinear sequential sub-predictors. A numerical example demonstrated the efficiency of the approach. Our method in the future can be extended to other semilinear PDEs.

CRediT authorship contribution statement

Rami Katz: Writing – original draft, Writing – review & editing, Methodology, Validation, Investigation. **Emilia Fridman:** Supervision, Investigation, Methodology.

Declaration of competing interest

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Appendix A. Proof of Theorem 2.1

For H^1-stability analysis of the closed-loop system (2.37) we consider the Lyapunov function

$$V(t) = X^T(t) P_X X(t) + \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t) \tag{A.1}$$

where $0 < P_X \in \mathbb{R}^{(2N+3)\times(2N+3)}$ to be obtained from LMIs. Differentiating $V(t)$ along the solution to the closed-loop system (2.37) we have

$$\begin{aligned} &\dot{V} + 2\delta V = 2X^T(t)\left[P_X F_X + F_X^T P_X + 2\delta P_X\right] X(t) \\ &+ 2X^T(t) P_X L_\zeta \zeta(t) + 2X^T(t) P_X \hat{G}(t) + 2X^T(t) P_X H(t) \\ &+ 2\textstyle\sum_{n=N+1}^{\infty} \left(-\lambda_n^2 + \delta\lambda_n\right) w_n^2(t) \\ &+ 2\textstyle\sum_{n=N+1}^{\infty} \lambda_n w_n(t) \left[\hat{g}_n(t) + h_n(t) - b_n K_X X(t)\right]. \end{aligned} \tag{A.2}$$

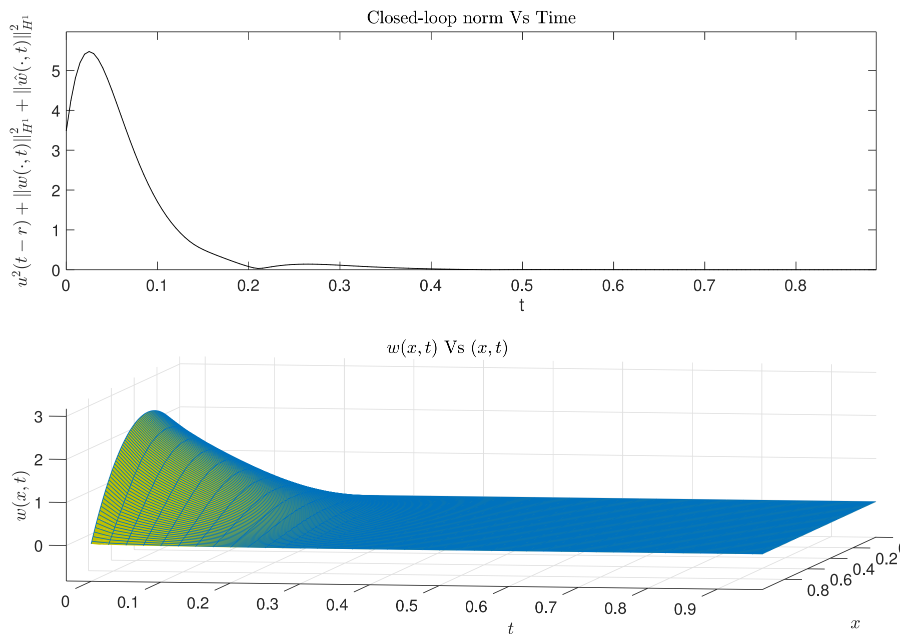

Fig. 1. Closed-loop system simulation.

Let $\alpha_1>0$, we compensate the series with $\left\{\hat{g}_n(t)\right\}_{n=N+1}^{\infty}$ by using the Young inequality

$$\begin{aligned}&2\sum_{n=N+1}^{\infty}\lambda_n w_n(t)\hat{g}_n(t)\le\tfrac{1}{\alpha_1}\sum_{n=N+1}\lambda_n^2 w_n^2(t)\\&-\alpha_1\left|\hat{G}(t)\right|^2+\alpha_1\sum_{n=0}^{\infty}\hat{g}_n^2(t).\end{aligned}\tag{A.3}$$

Then, by Parseval's equality and (2.4) we obtain

$$\begin{aligned}&\alpha_1\sum_{n=0}^{\infty}\hat{g}_n^2(t)=\alpha_1\int_0^1|g(t,x,\hat{w}(x,t)+\psi(x)u(t))|^2dx\\&\overset{(2.4)}{\le}\alpha_1\sigma^2\int_0^1|\hat{w}(x,t)+\psi(x)u(t)|^2dx\\&\le 2\alpha_1\sigma^2\left\|\hat{w}(\cdot,t)\right\|^2+2\alpha_1\sigma^2u^2(t)\left\|\psi\right\|^2\\&=2\alpha_1\sigma^2X^T(t)\varXi_X X(t),\\&\varXi_X\overset{(2.5)}{=}\mathrm{diag}\left\{\tfrac{2}{\pi^2},I_{N_0+1},0,I_{N-N_0},0\right\}.\end{aligned}\tag{A.4}$$

Similarly, introducing $\alpha_2>0$ we have

$$\begin{aligned}&2\sum_{n=N+1}^{\infty}\lambda_n w_n(t)h_n(t)\le\tfrac{1}{\alpha_2}\sum_{n=N+1}\lambda_n^2w_n^2(t)\\&-\alpha_2\left|H(t)\right|^2+\alpha_2\sum_{n=0}^{\infty}h_n^2(t).\end{aligned}\tag{A.5}$$

Recall that

$$\begin{aligned}h_n=&\left\langle g\left(t,\cdot,w(\cdot,t)+\psi(\cdot)u(t)\right),\phi_n\right\rangle\\&-\left\langle g\left(t,\cdot,\hat{w}(\cdot,t)+\psi(\cdot)u(t)\right),\phi_n\right\rangle,\quad n\ge 0.\end{aligned}\tag{A.6}$$

Then, by Parseval's equality we obtain

$$\begin{aligned}&\alpha_2\sum_{n=0}^{\infty}h_n^2(t)\overset{(2.4)}{\le}\alpha_2\sigma^2\int_0^1|\hat{w}(x,t)-w(x,t)|^2dx\\&=\alpha_2\sigma^2X^T(t)\varXi_E X(t)+\alpha_2\sigma^2\sum_{n=N+1}w_n^2(t),\\&\varXi_E=\mathrm{diag}\left\{0,I_{N_0},0,I_{N-N_0}\right\}\in\mathbb{R}^{(2N+3)\times(2N+3)}.\end{aligned}\tag{A.7}$$

We bound the last term in (A.2) by using Young's inequality with some $\alpha_3>0$:

$$\begin{aligned}&2\sum_{n=N+1}^{\infty}\lambda_n w_n(t)\left(-b_nK_XX(t)\right)\\&\le\tfrac{1}{\alpha_3}\sum_{n=N+1}^{\infty}\lambda_n w_n^2(t)+\alpha_3\left(\sum_{n=N+1}^{\infty}\lambda_n b_n^2\right)|K_XX(t)|^2\\&\overset{(2.15)}{\le}\tfrac{1}{\alpha_3}\sum_{n=N+1}^{\infty}\lambda_n w_n^2(t)+\tfrac{2\alpha_3\xi_{N+1}}{\pi^2}|K_XX(t)|^2.\end{aligned}\tag{A.8}$$

Finally, denoting for $n\ge N$

$$\rho_n=\kappa_n^{-1}\left(-\lambda_n^2+\delta\lambda_n+\frac{\lambda_n}{2\alpha_3}+\frac{\lambda_n^2}{2\alpha_2}+\frac{\lambda_n^2}{2\alpha_1}+\frac{\alpha_2\sigma^2}{2}\right)$$

and assuming that $\rho_{N+1}<0$, it can be seen that ρ_n is monotonically decreasing. The latter follows from monotonicity of λ_n. Then for the series terms in (A.2) we have

$$\begin{aligned}&\sum_{n=N+1}^{\infty}\left(-\lambda_n^2+\delta\lambda_n+\tfrac{\lambda_n}{2\alpha_3}+\tfrac{\lambda_n^2}{2\alpha_1}+\tfrac{\lambda_n^2}{2\alpha_2}+\tfrac{\alpha_2\sigma^2}{2}\right)w_n^2(t)\\&=\sum_{n=N+1}^{\infty}\rho_n\kappa_n w_n^2(t)\overset{(2.33)}{\le}\rho_{N+1}\zeta^2(t).\end{aligned}\tag{A.9}$$

Let $\eta(t)=\mathrm{col}\left\{X(t),\zeta(t),\hat{G}(t),H(t)\right\}$. From (A.2)–(A.9) we have

$$\dot{V}+2\delta V\le\eta^T(t)\Psi_0\eta(t)\le 0\tag{A.10}$$

provided

$$\Psi_0=\left[\begin{array}{cc|cc}\psi_0 & P_XL_\zeta & P_X & P_X\\ * & 2\rho_{N+1} & 0 & 0\\\hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c}{\mathrm{diag}\left\{-\alpha_1I,-\alpha_2I\right\}}\end{array}\right]<0,\tag{A.11}$$

with ψ_0 given in (2.39). By Schur complement, it can be seen that $\Psi_0<0$ is equivalent to (2.38).

Next, feasibility of (2.38) implies, by the comparison principle, that $V(t)\le e^{-2\delta t}V(0),\ t\ge 0$. Since $u(0)=0$ (see (2.9)) we have

$$\begin{aligned}&V(0)\le\sigma_{\max}(P_X)\left[w_0^2(0)+\sum_{n=1}^{N}w_n^2(0)\right]\\&+\sum_{n=N+1}^{\infty}\lambda_n w_n^2(0)\overset{(1.4)}{\le}\max\left\{\sigma_{\max}(P_X),1\right\}\|w(\cdot,0)\|_{H^1}^2.\end{aligned}\tag{A.12}$$

Similarly for $t\ge 0$

$$V(t)\overset{(1.4)}{\ge}\frac{1}{2}\min\left\{\frac{\sigma_{\min}(P_X)}{\lambda_{N+1}},1\right\}\|w(\cdot,t)\|_{H^1}^2.\tag{A.13}$$

The estimate (2.40) now follows from (A.12) and (A.13). We now consider feasibility of (2.38) for large enough N and small enough $\sigma>0$. First, note that for $\sigma=0$ (i.e. when $g\equiv 0$ in (2.1)) arguments similar to proof of Theorem 3.1 in [8] show feasibility of (2.38) for large enough N. Fixing such N and using continuity of the eigenvalues of the matrix in (2.38) we find that (2.38) is feasible for small enough $\sigma>0$.

Appendix B. Proof of Theorem 3.1

For H^1-stability analysis of (3.29) we define the Lyapunov functional

$$\begin{aligned} &V(t) := V_X(t) + V_e(t) + V_q(t),\\ &V_X(t) = |X(t)|^2_{P_X} + \textstyle\sum_{n=N+1}^{\infty} \lambda_n w_n^2(t),\\ &V_q(t) = q\int_{t-\frac{r}{M}}^{t} e^{-2\delta(t-s)}\zeta^2(s)ds,\\ &V_e(t) = |X_e(t)|^2_{P_e} + V_{S_e}(t) + V_{R_e}(t) \end{aligned} \tag{B.1}$$

Here $0 < P_X$ and $0 < P_e$ are matrices of appropriate dimensions, whereas $0 < q$ is a scalar. Furthermore, $V_{S_e}(t)$ and $V_{R_e}(t)$ are given by

$$\begin{aligned} &V_{S_e}(t) := \int_{t-\frac{r}{M}}^{t} e^{-2\delta(t-s)}\,|X_e(s)|^2_{S_e}\,ds,\\ &V_{R_e}(t) := \frac{r}{M}\int_{-\frac{r}{M}}^{0}\int_{t+\theta}^{t} e^{-2\delta(t-s)}\left|\dot{X}_e(s)\right|^2_{R_e} ds d\theta \end{aligned} \tag{B.2}$$

where $0 < S_e$ and $0 < R_e$ are matrices of appropriate dimension. Note that $V_X(t)$ allows to compensate $\zeta(t)$ using (2.33), $V_q(t)$ compensates $\zeta(t-\frac{r}{M})$, whereas $V_e(t)$ compensate the delay $\frac{r}{M}$ appearing in the ODEs of $X_e(t)$.

Differentiating $V_q(t)$ gives

$$\dot{V}_q + 2\delta V_q = q\zeta^2(t) - q\varepsilon_{r,M}\zeta^2\left(t-\frac{r}{M}\right),\ \ \varepsilon_{r,M} = e^{-\frac{2\delta r}{M}}. \tag{B.3}$$

Differentiating $V_X(t)$ along the solution to (3.29) gives

$$\begin{aligned} &\dot{V}_X + 2\delta V_X = X^T(t)\left[P_XF_X + F_X^TP_X + 2\delta P_X\right]X(t)\\ &+ 2X^T(t)P_XB_XK_0\mathcal{I}X_e(t) + 2X^T(t)P_XG(t)\\ &+ 2\textstyle\sum_{n=N+1}^{\infty}\left(-\lambda_n^2+\delta\lambda_n\right)w_n^2(t)\\ &+ 2\textstyle\sum_{n=N+1}^{\infty}\lambda_n w_n(t)\left[g_n(t) - b_n\left(\tilde{K}_0X(t) - K_0\mathcal{I}X_e(t)\right)\right]. \end{aligned} \tag{B.4}$$

Let $\alpha_1 > 0$. By the Young inequality we have

$$\begin{aligned} &2\textstyle\sum_{n=N+1}^{\infty}\lambda_n w_n(t) g_n(t)\\ &\le \frac{1}{\alpha_1}\textstyle\sum_{n=N+1}^{\infty}\lambda_n^2 w_n^2(t) - \alpha_1|G(t)|^2 + \alpha_1\textstyle\sum_{n=0}^{N} g_n^2(t). \end{aligned} \tag{B.5}$$

By Parseval's equality we have

$$\begin{aligned} &\textstyle\sum_{n=0}^{\infty} g_n^2(t) \overset{(3.9)}{=} \int_0^1 |g\left(t,s,w(s,t)+\psi(s)u(t-r)\right)|^2 ds\\ &\overset{(2.4)}{\le} \sigma^2\int_0^1\left[w(s,t)+\psi(s)u(t-r)\right]^2 ds\\ &\le 2\sigma^2X^T(t)\Xi_XX(t) + 2\sigma^2\textstyle\sum_{n=N+1}^{\infty} w_n^2(t),\\ &\Xi_X \overset{(2.5)}{=} \operatorname{diag}\left\{\frac{2}{\pi^2}, I_{N+1}\right\}. \end{aligned} \tag{B.6}$$

Similarly, we have for $\alpha_2, \alpha_3 > 0$

$$\begin{aligned} &-2\textstyle\sum_{n=N+1}^{\infty}\lambda_n w_n(t) b_n\tilde{K}_0X(t)\\ &\overset{(2.15)}{\le} \frac{1}{\alpha_2}\textstyle\sum_{n=N+1}^{\infty}\lambda_n w_n^2(t) + \frac{2\alpha_2\xi_{N+1}}{\pi^2}\left|\tilde{K}_0X(t)\right|^2, \end{aligned}$$

and

$$\begin{aligned} &2\textstyle\sum_{n=N+1}^{\infty}\lambda_n w_n(t) b_nK_0\mathcal{I}X_e(t)\\ &\overset{(2.15)}{\le} \frac{1}{\alpha_3}\textstyle\sum_{n=N+1}^{\infty}\lambda_n w_n^2(t) + \frac{2\alpha_3\xi_{N+1}}{\pi^2}\left|K_0\mathcal{I}X_e(t)\right|^2 \end{aligned} \tag{B.7}$$

Differentiation of $V_e(t)$ and Jensen's inequality lead to

$$\begin{aligned} &\dot{V}_e + 2\delta V_e \le X_e^T(t)\left[P_eF_e + F_e^TP_e + 2\delta P_e\right]X_e(t)\\ &+ 2X_e^T(t)P_e\Lambda_e\Upsilon_{e,r}(t) + 2X_e^T(t)P_e\mathcal{L}_\zeta\zeta(t-\tfrac{r}{M})\\ &+ 2X_e^T(t)P_eH(t) + |X_e(t)|^2_{S_e} - \varepsilon_{r,M}\times\\ &\left[\left|X_e(t)-\Upsilon_{e,r}(t)\right|^2_{S_e} + \left|\Upsilon_{e,r}(t)\right|^2_{R_e}\right] + \left(\tfrac{r}{M}\right)^2\left|\dot{X}_e(t)\right|^2_{R_e}. \end{aligned} \tag{B.8}$$

Recall $G^{N_0}(t)$, $G^{N-N_0}(t)$ in (3.10), $\left\{\hat{G}_i^{N_0}(t), \hat{G}_i^{N-N_0}(t)\right\}_{i=1}^{M}$ in (3.14), the estimation errors in (3.20) and $H(t)$ defined in (3.25) and (3.27). By Parseval's equality we have

$$\begin{aligned} &\left|H_M^{N_0}(t)\right|^2 + \left|H_M^{N-N_0}(t)\right|^2\\ &= \textstyle\sum_{n=0}^{N}\left[g_n(t) - \hat{g}_n^{(M)}(t-\tfrac{r}{M})\right]^2\\ &\overset{(3.10),\,(3.14)}{\le} \int_0^1 |g\left(t,s,w(s,t)+\psi(s)u(t-r)\right)\\ &- g\left(t,s,Q_1(s)\hat{w}_M^{N_0}(t-\tfrac{r}{M}) + Q_2(s)\hat{w}_M^{N-N_0}(t-\tfrac{r}{M})\right)\Big|^2 ds\\ &\overset{(2.4)}{\le} \sigma^2\int_0^1\Big[w(s,t)+\psi(s)u(t-r) - Q_1(s)\hat{w}_M^{N_0}(t-\tfrac{r}{M})\\ &\qquad - Q_2(s)\hat{w}_M^{N-N_0}(t-\tfrac{r}{M})\Big]^2 ds\\ &\le 2\sigma^2 e_M^{N_0,T}(t)\Xi_1 e_M^{N_0}(t) + 2\sigma^2\left|e_M^{N-N_0}(t)\right|^2\\ &\quad + 2\sigma^2\textstyle\sum_{n=N+1}^{\infty} w_n^2(t),\\ &\left|H_i^{N_0}(t)\right|^2 + \left|H_i^{N-N_0}(t)\right|^2 \le 2\sigma^2 e_i^{N_0,T}(t)\Xi_1 e_i^{N_0}(t)\\ &+ 2\sigma^2\left|e_i^{N-N_0}(t)\right|^2,\\ &\Xi_1 \overset{(2.5)}{=} \left\{\frac{2}{\pi^2}, I_{N_0+1}\right\},\quad 1\le i\le M-1 \end{aligned} \tag{B.9}$$

By (3.27) and (3.28), the latter implies

$$\begin{aligned} &|H(t)|^2 \le 2\sigma^2X_e^T(t)\Xi_EX_e(t) + 2\sigma^2\textstyle\sum_{n=N+1}^{\infty} w_n^2(t),\\ &\Xi_E = \operatorname{diag}\left\{\Xi_1, I_{N-N_0}, \dots, \Xi_1, I_{N-N_0}\right\}. \end{aligned} \tag{B.10}$$

Let $\eta(t) = \operatorname{col}\left\{X(t), G(t), X_e(t), \zeta(t-\frac{r}{M}), \Upsilon_{e,r}(t), H(t)\right\}$. By (B.3)–(B.10) and the S-procedure [36, Sec 3.2.3], we have for $\beta > 0$

$$\begin{aligned} &\dot{V} + 2\delta V + \beta\left\{2\sigma^2X_e^T(t)\Xi_EX_e(t)\right.\\ &\qquad\left. + 2\sigma^2\textstyle\sum_{n=N+1}^{\infty} w_n^2(t) - |H(t)|^2\right\}\\ &\le \eta^T(t)\Psi_1\eta(t) + q\zeta^2(t) + 2\textstyle\sum_{n=N+1}^{\infty}\varpi_n w_n^2(t) \end{aligned} \tag{B.11}$$

where

$$\begin{aligned} \varpi_n = &\left(-1+\tfrac{1}{2\alpha_1}\right)\lambda_n^2 + \left(\delta + \tfrac{1}{2\alpha_2} + \tfrac{1}{2\alpha_3}\right)\lambda_n\\ &+ \sigma^2(\alpha_1+\beta),\ \ n > N \end{aligned}$$

and Ψ_1 is given in (3.31). To compensate $\zeta^2(t)$ in (B.11) we use (2.33) and monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$ as follows

$$\begin{aligned} &q\zeta^2(t) + 2\textstyle\sum_{n=N+1}^{\infty}\varpi_n w_n^2(t)\\ &\overset{(2.33)}{\le} \textstyle\sum_{n=N+1}^{\infty}\left(2\varpi_n + q\kappa_n\right)w_n^2(t) \le 0 \end{aligned} \tag{B.12}$$

provided $\varpi_{N+1} + \frac{q\kappa_{N+1}}{2} < 0$. From (B.11)–(B.12) we have

$$\begin{aligned} &\dot{V} + 2\delta V + \beta\left\{2\sigma^2X_e^T(t)\Xi_2X_e(t)\right.\\ &\qquad\left. + 2\sigma^2\textstyle\sum_{n=N+1}^{\infty} w_n^2(t) - |H(t)|^2\right\} \le 0 \end{aligned} \tag{B.13}$$

provided $\Psi_1 < 0$ and $\varpi_{N+1} + \frac{q\kappa_{N+1}}{2} < 0$ hold. By Schur complement, these are satisfied iff (3.30) hold.

The upper bound (3.32) follows from arguments similar to (A.12) and (A.13) in Theorem 2.1. Next, we fix $r > 0$ and treat feasibility of (3.30) for M, N large enough and $\sigma > 0$ small enough. For $\sigma = 0$ (i.e. when $g \equiv 0$ in (2.1)), feasibility for large enough M and N follows from Theorem 1 in [30]. Fixing such M and N and using continuity of eigenvalues, we have that (3.30) are feasible provided $\sigma > 0$ is small enough.

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