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Brief paper

Stabilization by switching of parabolic PDEs with spatially scheduled actuators and sensors☆

Wen Kang[a,*], Emilia Fridman[b], Chuan-Xin Liu[c,d]

[a] School of Mathematics and Statistics, MIIT Key Laboratory of Mathematical Theory and Computation in Information Security, Beijing Institute of Technology, China
[b] Department of Electrical Engineering-Systems, Tel Aviv University, Israel
[c] School of Automation and Electrical Engineering, University of Science and Technology Beijing, China
[d] Shunde Graduate School of University of Science and Technology Beijing, China

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ABSTRACT

This paper addresses a switched sampled-data control design for stabilization of Kuramoto-Sivashinsky equation under the Dirichlet/periodic boundary conditions with spatially scheduled actuators. It is supposed that discrete-time point-like or averaged measurements are available. The system is known to be stabilizable by static output-feedback employing several distributed in space actuators and sensors, but is not stabilizable by only one of the actuator–sensor pairs. Does there exist a switching stabilizing static output-feedback such that at all times, only one actuator–sensor pair is active? We give a positive answer and find the appropriate switching sampled-data control law. The proposed switching controller can be implemented either by N actuators and sensors placed in each subdomain (here switching control may reduce the energy that the system spends) or by using one actuator–sensor pair that can move to the active subdomain. For implementation of the control law by moving actuators and sensors, we take into account a moving time by treating it as an additional switching between the open-loop system and the closed-loop switched system. The guidance of active (or mobile) actuators and sensors is provided by using output-dependent switching. Constructive conditions are derived to ensure that the resulting closed-loop system is regionally stable by means of the Lyapunov approach. Numerical example illustrates the efficiency of the method.

1. Introduction

In recent years, substantial efforts have been taken to develop switched control of partial differential equations (PDEs) (see e.g. Iftime and Demetriou (2009) and Zuazua (2010)). In Iftime and Demetriou (2009), optimal and switching policies of spatially scheduled actuators were suggested that were based on finite horizon Linear Quadratic Regulator problem. In Zuazua (2010) the following problem was formulated: assuming that one can control a system using two or more actuators, does there exist a control strategy such that at all times, only one actuator is active? The positive answer for the controllability of some classes of PDEs along with the corresponding switching laws was given in Zuazua (2010).

The switching control laws can be implemented either by stationary actuators or by one moving actuator that can move to the active subdomain in the negligible time (Iftime & Demetriou, 2009). In Butkovskiy and Pustyl'nikov (1987) and Demetriou (2010, 2012), moving actuators and sensors guiding laws were suggested for PDEs. Intermittent control of reaction–diffusion equation by time-dependent switching between all working pairs of collocated mobile actuators and sensors and the rest (all not working) has been studied in Wu and Zhang (2019). Furthermore, Wu and Zhang (2020) has dealt with the stabilization problem of linear reaction–diffusion equation with time-varying delay via the projection modification algorithm using collocated mobile actuators and sensors. In Zhao, Lin, and Xue (2012), switching control of closed quantum systems governed by Schrödinger equation has been proposed for the degenerate case via multiple Lyapunov function technique. Note that the mentioned above methods for switching control or for control by mobile actuators and sensors may be inefficient for the unstable open-loop systems, which motivates our study.

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* Corresponding author.
E-mail addresses: kangwen@amss.ac.cn (W. Kang), emilia@tauex.tau.ac.il (E. Fridman), s20200565@xs.ustb.edu (C.-X. Liu).

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Control of Kuramoto-Sivashinsky equation (KSE) has attracted extensive attention owing to their wide range of applications (Armaou & Christofides, 2000; Christofides & Armaou, 2000; El-Farra, Lou, & Christofides, 2003). KSE models a variety of physical–chemical systems including falling liquid films, chemical reactors and interfacial instabilities in viscous flows (El-Farra et al., 2003). Active control of fluid flow which is modeled by KSE can be achieved by injection of polymers, mass transport through porous walls (e.g. blowing/ suction) and application of electro-magnetic forcing. In these cases actuators and sensors are in-domain.

In-domain control of KSE with a large number of in-domain actuators and sensors has received significant research attention (Armaou & Christofides, 2000; Christofides & Armaou, 2000; El-Farra et al., 2003). Particularly, El-Farra et al. (2003) studied the motion of a liquid film falling down on a vertical wall, where in-domain control actuators were introduced to suppress the unstable behavior of the falling liquid films. There are also many fruitful works on the boundary control of KSE (see e.g. Cerpa (2010) and Cerpa and Mercado (2011)). In El-Farra et al. (2003) and Ghantasala and El-Farra (2012), for KSE, a switching law reconfigured the control actuators following fault detection. It should be noticed that stabilization by switching of open-loop unstable PDE with several actuators, where the system is not stabilizable by using only one actuator, has not been achieved yet. Thus, the design of a switching controller for open-loop unstable parabolic PDEs is a challenging topic.

Stabilization of unstable systems by switching is a classical, challenging and well-studied problem for ordinary differential equations (ODEs) (Liberzon, 2003). Thus, for linear switched systems with a stable convex combination, stabilization can be achieved by state or output-dependent switching (see e.g. Deaecto (2016) and Hetel and Fridman (2013) and the references therein). The key idea of switching control design for PDEs is to schedule the position of the actuator and sensor in order to achieve the control goal.

This paper addresses a switched sampled-data control design for stabilization of KSE under the Dirichlet/periodic boundary conditions with spatially scheduled actuators. It is supposed that discrete-time point-like or averaged measurements are available. The system is known to be stabilizable by static output-feedback employing several distributed in space actuators and sensors, but is not stabilizable by only one of the actuator–sensor pairs. Does there exist a stabilizing switching static output-feedback such that at all times, only one actuator–sensor pair is active? We give a positive answer and find the appropriate switching sampled-data control law. The proposed switching static output-feedback can be implemented either by N actuators and sensors placed in each subdomain (here the switching control may reduce the energy that the system spends) or by using one actuator–sensor pair that can move to the active subdomain. The guidance of active (or mobile) actuators and sensors is provided by using output-dependent switching. Constructive conditions are derived to ensure that the resulting closed-loop system is regionally stable by means of the Lyapunov approach. For implementation of the control law by moving actuators and sensors, we take into account a moving time by treating it as an additional switching between the open-loop system and the closed-loop switched system. Consistent simulation results that support the proposed theoretical statements are provided. Preliminary results on global stabilization by switching of 1-D semilinear heat equation under averaged measurements were presented in Kang, Fridman, and Liu (2021).

This work is organized as follows. Section 2 presents some preliminaries. In Sections 3 and 4, the switching control strategy for KSE is proposed under the point-like measurements and main theoretical results are presented, whereas extensions to the case of periodic boundary conditions and the case of averaged state measurements are presented in Section 5. Section 6 provides simulation results and Section 7 gives conclusions.

2. Mathematical preliminaries

Notation. Throughout the paper the support of a function g is denoted by supp$g$, and conv(supp$g$) represents the convex hull of supp$g$. $L^2(0,L)$ stands for the Hilbert space of square integrable scalar functions $f(x)$ on $(0,L)$ with the corresponding norm $\|f\|_{L^2(0,L)} = [\int_0^L f^2(x)dx]^{\frac{1}{2}}$. $L^\infty(0,L)$ denotes the space of essentially bounded function $f(x)$ on $(0,L)$ with the corresponding norm $\|f\|_{L^\infty(0,L)} = \text{esssup}_{x\in[0,L]}|f(x)|$. The Sobolev space $H^k(0,L)$ with $k\in\mathbb{Z}$ is defined as $H^k(0,L) = \{f : f^{(\alpha)} \in L^2(0,L),\ \forall\ 0 \le |\alpha| \le k\}$ with norm $\|f\|_{H^k(0,L)} = \{\sum_{0\le|\alpha|\le k}\|f^{(\alpha)}\|^2_{L^2(0,L)}\}^{\frac{1}{2}}$. Moreover, $H_0^k(0,L) = \{f \in H^k(0,L)|f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0, f(L) = f'(L) = \cdots = f^{(k-1)}(L) = 0\}$. $\mathbb{Z}_+$ denotes the set of nonnegative numbers.

Lemma 2.1 (*Poincaré's Inequality Fridman & Bar Am, 2013; Hardy, Littlewood, & Pólya, 1988*)**.** *For $a < b$, let $g \in H^1(a,b)$ be a scalar function with $\int_a^b g(x)dx = 0$. Then $\|g\|^2_{L^2(a,b)} \le \frac{(b-a)^2}{\pi^2}\|g'\|^2_{L^2(a,b)}$.*

Cauchy-Schwartz's inequality leads to the next lemma:

Lemma 2.2 (*Jensen's Inequality Fridman, 2014*)**.** *For $a < b$, let $c : [a,b] \to [0,\infty)$ and $g : [a,b] \to \mathbb{R}$ be such that the integration concerned is well defined. Then $\left[\int_a^b c(x)g(x)dx\right]^2 \le \int_a^b c(x)dx \int_a^b c(x)g^2(x)dx$.*

Lemma 2.3 (*Sobolev's Embedding and Inequality*)**.** *The embedding $H^1(0,L) \subset C([0,L])$ is compact and for any $z \in H_0^1(0,L)$, it holds $\|z\|_{L^\infty(0,L)} \le \sqrt{L}\|z'\|_{L^2(0,L)}$.*

3. Problem formulation

Let $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \ldots,\ \lim_{k\to\infty} t_k = \infty$ be a sequence of sampling instants. Consider the following plant governed by KSE:

$$\begin{cases} z_t(x,t) + z_{xx}(x,t) + \nu z_{xxxx}(x,t) + z(x,t)z_x(x,t) \\ = b_{\sigma_k}(x)u_{\sigma_k}(t),\ x\in(0,L),\ t\in[t_k,t_{k+1}), \\ z(x,0) = z_0(x), \end{cases} \tag{3.1}$$

where $k \in \mathbb{Z}_+$, under the Dirichlet boundary conditions:

$$z(0,t) = z(L,t) = 0,\ z_x(0,t) = z_x(L,t) = 0,\ t > 0. \tag{3.2}$$

Here $z(x,t)$ is the state of KSE, $z_0(x)$ is the initial state, and $u_{\sigma_k}(t)$ is the control input. The switching function $\sigma_k : k \in \mathbb{Z}_+ \to \{1,\ldots,N\}$ selects at each sampling time t_k one of the N available actuators corresponding to the shape function $b_{\sigma_k}(x)$ that will be shortly defined. We make the following assumptions:

- Spatial sampling: Similar to Azouani and Titi (2014), Fridman and Bar Am (2013), Fridman and Blighovsky (2012) and Lunasin and Titi (2017), we assume that the points $0 = x_0 < x_1 < \cdots < x_N = L$ divide $[0,L]$ into N equal-length subintervals $\Omega_j = [x_{j-1}, x_j)$ such that $\cup_j\Omega_j = [0,L]$ and $|\Omega_j| = \frac{L}{N}$ meaning that all the subintervals have the same length. The shape functions $b_j(x)$ are chosen to be characteristic functions $b_j(x)$ of Ω_j as follows:

$$\begin{cases} b_j(x) = 0,\ x \notin \Omega_j, \\ b_j(x) = 1,\ otherwise, \end{cases} \quad j = 1,\ldots,N. \tag{3.3}$$

- Time sampling: The length of sampling subintervals in time is supposed to be uniformly bounded:

$$0 < h_0 \le t_{k+1} - t_k \le h, \quad \forall k \in \mathbb{Z}_+. \tag{3.4}$$

- Moving time: The moving time $\delta \in (0,h_0)$ for sensors and actuators to the appropriate domain Ω_{σ_k} is taken into account.

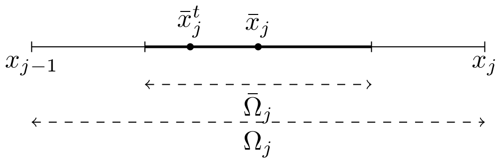

Fig. 1. Subdomains $\bar{\Omega}_j$ of point-like measurements.

We first consider sensors that provide the discrete-time point-like measurements:

$$y_j(t_k) = \int_{\Omega_j} c_j(x) z(x, t_k) dx, \quad k \in \mathbb{Z}_+ \tag{3.5}$$

with

$$0 \le c_j \in L^2(\Omega_j), \ \int_{\Omega_j} c_j(x) dx = 1, \tag{3.6}$$

$$c_j(x) = \begin{cases} \frac{1}{\varepsilon}, \ \text{if } x \in \bar{\Omega}_j, \\ 0, \ otherwise, \end{cases} \quad j = 1, \ldots, N, \tag{3.7}$$

where $\bar{\Omega}_j$ is subinterval of Ω_j with the length ε independent of j (see Fig. 1, where $\bar{x}_j$ is the midpoint of Ω_j).

We will also consider the averaged state measurements

$$y_j(t_k) = \frac{\int_{\Omega_j} z(x,t_k)dx}{|\Omega_j|} \quad = \frac{N}{L} \int_{\Omega_j} z(x, t_k) dx, \quad j = 1, \ldots, N, k \in \mathbb{Z}_+. \tag{3.8}$$

Differently from point-like measurements, the sensors in the case of averaged measurements cover the whole subdomain. However, our method under the averaged measurements leads to a smaller number of actuators and sensors or allow larger sampling in time (see comparison in Table 1 in Section 6). Note that the presented method under the averaged measurements can be extended to $N - D$ PDEs for any N (on the basis of static output-feedback without switching presented in Bar and Fridman (2014)), whereas such extension under the point-like measurements is questionable (see Selivanov and Fridman (2019), where non-switched static output-feedback for heat equation under point-like measurements is confined to $N \le 2$).

For both measurements, our aim is to find a sampled-data switching law and a sampled-data regionally exponentially stabilizing controller for KSE (3.1) implemented by zero-order hold device. As already mentioned, in this paper we will take into account the moving time δ for sensors and actuators. For the actuators moving time, we consider additional switching between the open-loop system (when the actuator is moving) during the part of the sampling interval and the closed-loop switched system during the remaining part of the interval, where

$$u_{\sigma_k}(t) = \begin{cases} 0, & t \in [t_k, t_k + \delta), \\ -K y_{\sigma_k}(t_k), & t \in [t_k + \delta, t_{k+1}) \end{cases} \tag{3.9}$$

with some $K > 0$. The switching signal σ_k is calculated at time t_k, whereas it takes δ seconds for actuators and sensors to move to the domain Ω_{σ_k}.

Our main objective is to find an appropriate output-depending switching law. Denote the characteristic function of the time interval $[t_k, t_k+\delta]$ by $\chi_{[t_k,t_k+\delta]}(t)$. Consider first the case of averaged state measurements (3.8), where the closed-loop system (3.1), (3.9) has the form

$$\begin{aligned} & z_t(x, t) + z_{xx}(x, t) + \nu z_{xxxx}(x, t) + z(x, t) z_x(x, t) \\ & = -\tfrac{KN}{L}(1 - \chi_{[t_k,t_k+\delta]}(t)) b_{\sigma_k}(x) \int_{\Omega_{\sigma_k}} z(x, t_k) dx, \\ & \qquad x \in (0, L), \ t \in [t_k, t_{k+1}) \end{aligned} \tag{3.10}$$

subject to (3.2). Note that if $b_{\sigma_k}(x) u_{\sigma_k}(t)$ in (3.1) is changed by $\sum_{j=1}^{N} b_j(x) u_j(t)$, then there exists $K > 0$ that regionally exponentially stabilizes the system by $u_j(t) = -K y_j(t)$ (Kang & Fridman, 2018). The latter means that the average of systems (3.1) with $b_{\sigma_k}(x) u_{\sigma_k}(t)$ changed by $b_j(x) u_j(t)$ is stabilizable by the static output-feedback (3.9). Similar to state-dependent switching for ODEs in the case of stable convex combination of systems (Hetel & Fridman, 2013), we will define a min-type switching function by using the corresponding Lyapunov function $V(t)$ according to

$$\sigma_k \approx \arg\min \ \dot{V}(t)$$

for $t \in [t_k + \delta, t_{k+1})$ along the closed-loop system. Thus, for $V(t) = \int_0^L z^2(x, t) dx$ we have

$$\begin{aligned} \dot{V}(t) \ & = -\tfrac{KN}{L} \int_{\Omega_j} z(x, t) dx \int_{\Omega_j} z(x, t_k) dx \\ & \quad - \int_0^L z(x, t) [z_{xx}(x, t) + \nu z_{xxxx}(x, t) + z(x, t) z_x(x, t)] dx \end{aligned}$$

that leads (for small enough h) to

$$\begin{aligned} \arg\min \dot{V}(t) \ & = \arg\min_j \left[- \int_{\Omega_j} z(x, t) dx \int_{\Omega_j} z(x, t_k) dx \right] \\ & \approx \arg\max_j \left[\int_{\Omega_j} z(x, t_k) dx \right]^2 \end{aligned}$$

i.e. to the following discrete-time switching law:

$$\sigma_k = \arg\max_j \left[\int_{\Omega_j} z(x, t_k) dx \right]^2. \tag{3.11}$$

Similarly to (3.11) for the point-like measurements we choose

$$\sigma_k = \arg\max_j \left[\int_{\Omega_j} c_j(x) z(x, t_k) dx \right]^2. \tag{3.12}$$

Our sampled-data switching law (3.12) with (3.4) and $\lim_{k=\infty} t_k = \infty$ rules out the possibility of Zeno behavior. Note that (3.12) is calculated at time t_k. The law (3.12) means that the σ_k-th mode is active if

$$\left[\int_{\Omega_j} c_j(x) z(x, t_k) dx \right]^2 \le \left[\int_{\Omega_{\sigma_k}} c_{\sigma_k}(x) z(x, t_k) dx \right]^2, \quad \forall j = 1, \ldots, N. \tag{3.13}$$

4. Main results

In this section, we will analyze the well-posedness and regional exponential stability of the system (3.1) under the static output-feedback (3.9) and the switching law (3.9) (where $c_j = 1$ in the case of averaged measurements).

4.1. Well-posedness of the controlled system

We establish the existence, uniqueness and regularity of the system (3.1) under the switching control law (3.9), (3.12) and Dirichlet boundary conditions (3.2) by using the step method (see e.g. Section 1.2 in Fridman (2014)). For the switching law (3.9), (3.12), we assume that the σ_k-th mode is active. We first consider $t \in [0, \delta]$. Then (3.1), (3.2) becomes

$$\begin{cases} z_t(x, t) + z_{xx}(x, t) + \nu z_{xxxx}(x, t) + z(x, t) z_x(x, t) = 0, \\ \qquad x \in (0, L), \ t \in [0, \delta], \\ z(0, t) = z(L, t) = 0, \ z_x(0, t) = z_x(L, t) = 0, \\ z(x, 0) = z_0(x). \end{cases} \tag{4.1}$$

Define the system operator $A : D(A) \subset L^2(0, L) \to L^2(0, L)$ as follows:

$$\begin{cases} Af = -\nu \left[\frac{\partial^4 f}{\partial x^4} \right], \\ D(A) = H^4(0, L) \cap H_0^2(0, L). \end{cases}$$

It is well-known that A is a dissipative operator, and A generates an analytic semigroup. Operator $-A$ is positive implying that its square root $(-A)^{\frac{1}{2}}$ is also positive. Moreover, $D((-A)^{\frac{1}{2}}) = H_0^2(0, L)$ with the norm $\|f\|_{D((-A)^{\frac{1}{2}})} = \nu^{\frac{1}{2}}\|f''\|_{L^2(0,L)}$. Then the system (4.1) can be represented as an evolution equation:

$$\begin{cases} \frac{d}{dt}z(\cdot,t) = Az(\cdot,t) + F(z(\cdot,t)), \\ z(\cdot,0) = z_0(\cdot), \end{cases} \tag{4.2}$$

where the nonlinear term F is defined on function $z(\cdot,t)$ according to

$$F(z(\cdot,t)) = -z(x,t)z_x(x,t) - z_{xx}(x,t), \ \ t \in [0,\delta].$$

It should be noticed that the nonlinear term F is locally Lipschitz continuous, that is, there exists a positive constant $l(M)$ such that

$$\|F(z_1) - F(z_2)\|_{L^2(0,L)} \le l(M)\|z_1 - z_2\|_{H_0^2(0,L)}$$

holds for $z_1, z_2 \in H_0^2(0,L)$ with $\|z_1\|_{H_0^2(0,L)} \le M$, $\|z_2\|_{H_0^2(0,L)} \le M$. Therefore, Theorem 3.3.3 of Henry (1981) is applicable to (4.2). For any initial condition $z_0 \in H_0^2(0,L)$, there exists a unique local strong solution of (4.2) on some interval $[0,T] \subset [0,\delta]$, where $T = T(z_0) > 0$:

$$z \in C([0,T]; H_0^2(0,L)) \cap L^2([0,T]; D(A)),$$
$$\dot z \in L^2([0,T]; L^2(0,L)).$$

From Theorem 6.23.5 of Krasnoselskii, Zabreiko, Pustylii, and Sobolevskii (1976), we obtain that if the solution admits a priori estimate (i.e. bounded), then it exists on the entire interval $[0,\delta]$. The a priori estimate on the solutions will be guaranteed by the conditions that we will provide (see Theorem 1).

For $t \in [\delta, t_1]$, the system (3.1) under the switching control law (3.9), (3.12) can be also written in the form of (4.2) with the following nonlinearity

$$F(z(\cdot,t)) = -z(x,t)z_x(x,t) - z_{xx}(x,t)$$
$$-Kb_{\sigma_k}(x)\int_{\Omega_{\sigma_k}} c_{\sigma_k}(x)z_0(x)dx, \ \ t \in [\delta, t_1].$$

Since F is locally Lipschitz continuous, the same line of reasoning is applied to the time interval $[\delta, t_1]$. The strong solution exists on $[\delta, t_1]$ due to a priori estimate on the solutions starting from the domain of attraction, which is guaranteed by the stability conditions of Theorem 1.

4.2. Stability analysis of the switched system

By the mean-value theorem, from (3.6) it follows that there exists $\bar x_j^t \in \mathrm{conv}(\mathrm{supp} c_j)$ (see Fig. 1) such that

$$\int_{\Omega_j} c_j(x)z(x,t)dx = z(\bar x_j^t, t), \ \ t \in [t_k, t_{k+1}).$$

Denote

$$f_j(x,t) \triangleq z(x,t) - z(\bar x_j^t, t), \ \ t \in [t_k, t_{k+1}), \tag{4.3}$$

$$\rho_j(t) \ \ \triangleq \int_{\Omega_j}\int_{t_k}^t c_j(x)z_s(x,s)dsdx, \ \ t \in [t_k, t_{k+1}). \tag{4.4}$$

Then the switching controller (3.9) can be rewritten as

$$u_{\sigma_k}(t) = \begin{cases} 0, & t \in [t_k, t_k+\delta), \\ -K[z(x,t) - f_{\sigma_k}(x,t) - \rho_{\sigma_k}(t)], & t \in [t_k+\delta, t_{k+1}) \end{cases} \tag{4.5}$$

whereas the switching law chooses σ_k that satisfies

$$\begin{aligned} &\int_{\Omega_j}\left[z(x,t) - f_j(x,t) - \rho_j(t)\right]^2 dx \\ &\le \int_{\Omega_{\sigma_k}}\left[z(x,t) - f_{\sigma_k}(x,t) - \rho_{\sigma_k}(t)\right]^2 dx, \ \ j = 1,2,\dots,N. \end{aligned} \tag{4.6}$$

Thus, under the controller (4.5), the closed-loop system becomes

$$\begin{aligned} &z_t(x,t) + z_{xx}(x,t) + \nu z_{xxxx}(x,t) + z(x,t)z_x(x,t) \\ &= -Kb_{\sigma_k}(x)(1 - \chi_{[t_k,t_k+\delta]}(t))[z(x,t) - f_{\sigma_k}(x,t) - \rho_{\sigma_k}(t)], \\ &\qquad x \in (0,L), \ \ t \in [t_k, t_{k+1}), \end{aligned} \tag{4.7}$$

subject to (3.2), (3.12).

Note that (3.1) may be not stabilizable with a desired decay rate by the non-switched control. The challenge in the stability analysis is to take efficiently into account the switching condition (3.12) in order to derive feasible stability conditions (see (4.24) below and the resulting expression in (4.25)).

Now we focus on the stability of the closed-loop system that switches at times t_k and $t_k + \delta$. Consider the following Lyapunov-Krasovskii functional:

$$V(t) = V_{P_1}(t) + V_{P_2}(t) + V_R(t), \ \ t \in [t_k, t_{k+1}) \tag{4.8}$$

where

$$\begin{aligned} V_{P_1}(t) = \ & P_1\int_0^L z^2(x,t)dx, \\ V_{P_2}(t) = \ & P_2\nu\int_0^L z_{xx}^2(x,t)dx, \\ V_R(t) = \ & R\frac{4h^2}{\pi^2}\sum_{j=1}^N \int_{\Omega_j}\int_{t_k}^t e^{-2\alpha(t-s)}[\rho_{js}(s)]^2dsdx \\ & - Re^{-2\alpha h}\sum_{j=1}^N \int_{\Omega_j}\int_{t_k}^t e^{-2\alpha(t-s)}[\rho_j(s)]^2dsdx \end{aligned}$$

with $P_1 > 0$, $P_2 > 0$, $R > 0$. Here $\rho_{js}(s)$ is the derivative of $\rho_j(s)$ with respect to s. By Wirtinger's inequality, $V_R(t)$ is non-negative (see Lemma 1 in Selivanov and Fridman (2013)), it does not grow in the switching times t_k and it is continuous in the switching times $t_k + \delta$. Moreover, V_R extends the corresponding terms in Selivanov and Fridman (2019) to Wirtinger-based Lyapunov functional.

For $z(\cdot,t) \in H_0^2(0,L)$ we define

$$\|z(\cdot,t)\|_V^2 = P_1\|z(\cdot,t)\|_{L^2(0,L)}^2 + P_2\nu\|z_{xx}(\cdot,t)\|_{L^2(0,L)}^2$$

with $P_1 > 0$, $P_2 > 0$.

Remark 4.1. In order to find a bound on the domain of attraction for closed-loop system (4.7) subject to (3.2), we use positive invariance principle in Theorem 1: we show that if $\Psi_0 < 0$, $\Psi_1 < 0$ and $\Psi_2 < 0$, where Ψ_0, Ψ_1, Ψ_2 are given by (4.12)–(4.14), then $V(t) \le V(0)$ for all $t \ge 0$. Matrices Ψ_0, Ψ_1, Ψ_2 are affine in z_x. Let $C > 0$ be the upper bound of z_x, i.e., $\max_{x\in[0,L]}|z_x(x,t)| \le C$ for all $t \ge 0$. Then it is sufficient to verify the matrix inequalities $\Psi_0 < 0$, $\Psi_1 < 0$ and $\Psi_2 < 0$ in the vertices $z_x = \pm C$ (see (4.9)–(4.11)).

The following result provides sufficient stability conditions in the form of linear matrix inequalities (LMIs) for the closed-loop system (4.7), (3.2), (3.12):

Theorem 1. *Consider the closed-loop system (4.7) subject to (3.2) and the switching law (3.12). Given positive scalars h, α, K and tuning parameter $C > 0$, $\alpha_0 > 0$ such that $\alpha h_0 > (\alpha_0 + \alpha)\delta$, let there exist scalars $R > 0$, $P_n > 0$, $\lambda_n \ge 0$ ($n = 1,2,3$) that satisfy the inequalities:*

$$\Psi_1|_{z_x=\pm C} < 0, \tag{4.9}$$

$$\Psi_2|_{z_x=\pm C} < 0, \tag{4.10}$$

$$\Psi_0|_{z_x=\pm C} < 0, \tag{4.11}$$

where

$$\Psi_1 = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \frac{\lambda_1}{N-1} & \frac{\lambda_1}{N-1} \\ * & \psi_{22} & -P_2 & 0 & 0 \\ * & * & \psi_{33} & 0 & 0 \\ * & * & * & -\lambda_2 - \frac{\lambda_1}{N-1} & -\frac{\lambda_1}{N-1} \\ * & * & * & * & -Re^{-2\alpha h} - \frac{\lambda_1}{N-1} \end{bmatrix}, \tag{4.12}$$

$$\Psi_2 = \begin{bmatrix} \tilde{\psi}_{11} & \tilde{\psi}_{12} & \psi_{13} & P_3K-\lambda_1 & P_3K-\lambda_1 \\ * & \psi_{22} & -P_2 & P_2K & P_2K \\ * & * & \psi_{33} & 0 & 0 \\ * & * & * & \lambda_1-\lambda_2 & \lambda_1 \\ * & * & * & * & \lambda_1 - Re^{-2\alpha h} \end{bmatrix}, \quad (4.13)$$

$$\Psi_0 = \begin{bmatrix} -2\alpha_0 P_1 & P_1-P_4-P_2z_x & -P_4 \\ * & -2P_2 + R\frac{4h^2}{\pi^2}\frac{L}{N\varepsilon} & -P_2 \\ * & * & -2\alpha_0P_2\nu - 2P_4\nu \end{bmatrix}, \quad (4.14)$$

$\psi_{11} = 2\alpha P_1 - \lambda_3\frac{\pi^4}{L^4} - \frac{\lambda_1}{N-1}$,
$\psi_{12} = P_1 - P_3 - P_2z_x$,
$\psi_{13} = -P_3 - \frac{\lambda_2(\frac{L}{N}+\varepsilon)^2}{2\pi^2}$,
$\psi_{22} = R\frac{4h^2}{\pi^2}\frac{L}{N\varepsilon} - 2P_2$,
$\psi_{33} = 2\alpha P_2\nu - 2P_3\nu + \lambda_3$,
$\tilde{\psi}_{11} = 2\alpha P_1 - \lambda_3\frac{\pi^4}{L^4} - 2P_3K + \lambda_1$,
$\tilde{\psi}_{12} = P_1 - P_3 - P_2z_x - P_2K$.
Let α_1 be subject to

$$0 < \alpha_1 h_0 \le \alpha h_0 - (\alpha_0+\alpha)\delta. \quad (4.15)$$

Then for any initial function $z_0 \in H_0^2(0,L)$ that satisfies the bound $\|z_0\|_V < \sqrt{\frac{P_2\nu}{L}}C$, the closed-loop system (4.7) subject to (3.2) and (3.12) is exponentially stable with a decay rate α_1, i.e. the following holds

$$\|z(\cdot,t)\|_V^2 \le V(t) \le e^{-2\alpha_1(t-h)+2\alpha_0\delta}V(0).$$

Proof. Step 1: Let us only highlight that there exists a unique local strong solution of (4.1) on some interval $[0,T] \subset [0,\delta]$, where $T = T(z_0)$. Due to Theorem 6.23.5 of Krasnoselskii et al. (1976), the solution exists on the entire interval $[0,\delta]$ provided it is bounded. Therefore, by applying the same arguments at $[\delta, t_1]$ and any step $k \in \mathbb{N}$, one can conclude that the strong solution exists for all $t \ge 0$.

Step 2: Assume formally that strong solution of (4.7) subject to (3.2) starting from $\|z_0\|_V < \sqrt{\frac{P_2\nu}{L}}C$ exists for all $t \ge 0$. We first derive sufficient LMI-based conditions to guarantee that $\dot{V} + 2\alpha V \le 0$ for $[t_k+\delta, t_{k+1})$. Differentiating $V(t)$ along the solution of the closed-loop system and integrating by parts, we obtain

$$\begin{aligned} &\dot{V}(t) + 2\alpha V(t) = 2P_1\int_0^L z(x,t)z_t(x,t)dx \\ &+2\alpha P_1\int_0^L z^2(x,t)dx + 2\alpha P_2\nu\int_0^L z_{xx}^2(x,t)dx \\ &+2P_2\nu\int_0^L z_{xx}(x,t)z_{xxt}(x,t)dx \\ &+R\frac{4h^2}{\pi^2}\sum_{j=1}^N\int_{\Omega_j}[\rho_{jt}(t)]^2dx - Re^{-2\alpha h}\sum_{j=1}^N\int_{\Omega_j}[\rho_j(t)]^2dx. \end{aligned} \quad (4.16)$$

Jensen's inequality leads to

$$\begin{aligned} &\int_{\Omega_j}[\rho_{jt}(t)]^2dx = \tfrac{L}{N}\left(\int_{\Omega_j}c_j(x)z_t(x,t)dx\right)^2 \\ &\le \tfrac{L}{N}\int_{\Omega_j}c_j(x)dx\int_{\Omega_j}c_j(x)z_t^2(x,t)dx \le \tfrac{L}{N\varepsilon}\int_{\Omega_j}z_t^2(x,t)dx. \end{aligned} \quad (4.17)$$

Note that $f_j(x,t) \triangleq z(x,t) - z(\bar{x}_j^t,t)$ and $f_{jx}(x,t) = z_x(x,t)$. Then, application of Wirtinger's inequality yields

$$\begin{aligned} &\int_{\Omega_j}f_j^2(x,t)dx \\ &= \int_{x_{j-1}}^{\bar{x}_j^t}[z(x,t)-z(\bar{x}_j^t,t)]^2dx + \int_{\bar{x}_j^t}^{x_j}[z(x,t)-z(\bar{x}_j^t,t)]^2dx \\ &\le \tfrac{(\frac{L}{N}+\varepsilon)^2}{\pi^2}\int_{\Omega_j}z_x^2(x,t)dx. \end{aligned} \quad (4.18)$$

Moreover, we obtain

$$\|z(\cdot,t)\|_{L^2(0,L)}^2 \le \left(\tfrac{L^2}{\pi^2}\right)^2\|z_{xx}(\cdot,t)\|_{L^2(0,L)}^2. \quad (4.19)$$

Therefore, (4.6), (4.18) and (4.19) lead to

$$\begin{aligned} &-\tfrac{\lambda_1}{N-1}\textstyle\sum_{j\ne\sigma_k}^N\int_{\Omega_j}[z(x,t)-f_j(x,t)-\rho_j(t)]^2dx \\ &+\lambda_1\int_{\Omega_{\sigma_k}}[z(x,t)-f_{\sigma_k}(x,t)-\rho_{\sigma_k}(t)]^2dx \ge 0, \end{aligned} \quad (4.20)$$

$$\begin{aligned} &\lambda_2\left[\tfrac{(\frac{L}{N}+\varepsilon)^2}{\pi^2}\|z_x(\cdot,t)\|_{L^2(0,L)}^2 - \textstyle\sum_{j=1}^N\|f_j(\cdot,t)\|_{L^2(\Omega_j)}^2\right] \\ &= \lambda_2\left[\tfrac{(\frac{L}{N}+\varepsilon)^2}{\pi^2}\textstyle\sum_{j\ne\sigma_k}^N\int_{\Omega_j}z_x^2(x,t)dx - \sum_{j\ne\sigma_k}^N\int_{\Omega_j}f_j^2(x,t)dx\right] \\ &+\lambda_2\left[\tfrac{(\frac{L}{N}+\varepsilon)^2}{\pi^2}\int_{\Omega_{\sigma_k}}z_x^2(x,t)dx - \int_{\Omega_{\sigma_k}}f_{\sigma_k}^2(x,t)dx\right] \ge 0, \end{aligned} \quad (4.21)$$

$$\begin{aligned} &\lambda_3\left[\|z_{xx}(\cdot,t)\|_{L^2(0,L)}^2 - \left(\tfrac{\pi^2}{L^2}\right)^2\|z(\cdot,t)\|_{L^2(0,L)}^2\right] \\ &= \lambda_3\left[\textstyle\sum_{j\ne\sigma_k}^N\int_{\Omega_j}z_{xx}^2(x,t)dx - \left(\tfrac{\pi^2}{L^2}\right)^2\sum_{j\ne\sigma_k}^N\int_{\Omega_j}z^2(x,t)dx\right] \\ &+\lambda_3\left[\int_{\Omega_{\sigma_k}}z_{xx}^2(x,t)dx - \left(\tfrac{\pi^2}{L^2}\right)^2\int_{\Omega_{\sigma_k}}z^2(x,t)dx\right] \ge 0. \end{aligned} \quad (4.22)$$

We further apply the descriptor method (see Section 3.5 in Fridman (2014)), where the left-hand side of the following equation

$$\begin{aligned} &2\int_0^L[P_3z(x,t)+P_2z_t(x,t)]\{-z_t(x,t)-z_{xx}(x,t) \\ &-z(x,t)z_x(x,t) - \nu z_{xxxx}(x,t) \\ &-Kb_{\sigma_k}(x)[z(x,t)-f_{\sigma_k}(x,t)-\rho_{\sigma_k}(t)]\}dx = 0 \end{aligned} \quad (4.23)$$

with some $P_3 > 0$ is added to $\dot{V}$. Then adding the left-hand sides of (4.20)–(4.22) to (4.16) and taking into account (4.17), we obtain

$$\begin{aligned} &\dot{V}(t) + 2\alpha V(t) \le (2P_1-2P_3)\sum_{j\ne\sigma_k}^N\int_{\Omega_j}z(x,t)z_t(x,t)dx \\ &+\left(2\alpha P_1 - \lambda_3\frac{\pi^4}{L^4}\right)\sum_{j\ne\sigma_k}^N\int_{\Omega_j}z^2(x,t)dx \\ &-\left[2P_3 + \frac{\lambda_2(\frac{L}{N}+\varepsilon)^2}{\pi^2}\right]\sum_{j\ne\sigma_k}^N\int_{\Omega_j}z(x,t)z_{xx}(x,t)dx \\ &+(2\alpha P_2\nu - 2P_3\nu + \lambda_3)\sum_{j\ne\sigma_k}^N\int_{\Omega_j}z_{xx}^2(x,t)dx \\ &+\left(R\frac{4h^2}{\pi^2}\frac{L}{N\varepsilon} - 2P_2\right)\sum_{j\ne\sigma_k}^N\int_{\Omega_j}z_t^2(x,t)dx \\ &-2P_2\sum_{j\ne\sigma_k}^N\int_{\Omega_j}z_t(x,t)[z(x,t)z_x(x,t)+z_{xx}(x,t)]dx \\ &+(2P_1-2P_3)\int_{\Omega_{\sigma_k}}z(x,t)z_t(x,t)dx \\ &+\left(2\alpha P_1 - \lambda_3\frac{\pi^4}{L^4}\right)\int_{\Omega_{\sigma_k}}z^2(x,t)dx \\ &-\left[2P_3 + \frac{\lambda_2(\frac{L}{N}+\varepsilon)^2}{\pi^2}\right]\int_{\Omega_{\sigma_k}}z(x,t)z_{xx}(x,t)dx \\ &+(2\alpha P_2\nu - 2P_3\nu + \lambda_3)\int_{\Omega_{\sigma_k}}z_{xx}^2(x,t)dx \\ &+\left(R\frac{4h^2}{\pi^2}\frac{L}{N\varepsilon} - 2P_2\right)\int_{\Omega_{\sigma_k}}z_t^2(x,t)dx \end{aligned} \quad (4.24)$$

$$
\begin{aligned}
&- 2P_2 \int_{\Omega_{\sigma_k}} z_t(x,t)[z(x,t)z_x(x,t) + z_{xx}(x,t)]dx \\
&- 2P_3K \int_{\Omega_{\sigma_k}} z(x,t)[z(x,t) - f_{\sigma_k}(x,t) - \rho_{\sigma_k}(t)]dx \\
&- 2P_2K \int_{\Omega_{\sigma_k}} z_t(x,t)[z(x,t) - f_{\sigma_k}(x,t) - \rho_{\sigma_k}(t)]dx \\
&- Re^{-2\alpha h} \int_{\Omega_{\sigma_k}} \rho^2_{\sigma_k}(t)dx - \lambda_2 \int_{\Omega_{\sigma_k}} f^2_{\sigma_k}(x,t)dx \\
&- \frac{\lambda_1}{N-1} \sum_{j\neq\sigma_k}^{N} \int_{\Omega_j} [z(x,t) - f_j(x,t) - \rho_j(t)]^2 dx \\
&+ \lambda_1 \int_{\Omega_{\sigma_k}} [z(x,t) - f_{\sigma_k}(x,t) - \rho_{\sigma_k}(t)]^2 dx.
\end{aligned}
$$

From (4.24), we have

$$
\dot V(t) + 2\alpha V(t) \le \sum_{j\neq\sigma_k}^{N} \int_{\Omega_j} \eta_1^T \Psi_1 \eta_1 dx + \int_{\Omega_{\sigma_k}} \eta_2^T \Psi_2 \eta_2 dx, \quad t \in [t_k+\delta, t_{k+1}), \tag{4.25}
$$

where $\eta_1 = \mathrm{col}\{z(x,t), z_t(x,t), z_{xx}(x,t), f_j(x,t), \rho_j(x,t)\}$, $\eta_2 = \mathrm{col}\{z(x,t), z_t(x,t), z_{xx}(x,t), f_{\sigma_k}(x,t), \rho_{\sigma_k}(x,t)\}$, Ψ_i $(i = 1, 2)$ are given by (4.12), (4.13) respectively.

Let us first assume that

$$
\max_{x\in[0,L]} |z_x(x,t)| < C, \ \forall t \ge 0. \tag{4.26}
$$

Under the assumption (4.26), we have

$$
\dot V(t) + 2\alpha V(t) \le 0, \tag{4.27}
$$

if $\Psi_1 < 0$ and $\Psi_2 < 0$ hold for all $-C \le z_x \le C$.

Matrices Ψ_i $(i = 1, 2)$ given by (4.12) and (4.13) are affine in z_x. Hence, $\Psi_1 < 0$ and $\Psi_2 < 0$ for all $-C \le z_x \le C$ if these inequalities hold in the vertices $z_x = \pm C$, i.e., if LMIs (4.9) and (4.10) are feasible.

Step 3: Now we derive sufficient LMI-based conditions to guarantee that $\dot V(t) - 2\alpha_0 V(t) \le 0$ for $[t_k, t_k+\delta)$.

Differentiating $V(t)$ along (4.7) subject to (3.2), we have

$$
\begin{aligned}
&\dot V(t) - 2\alpha_0 V(t) = 2P_1 \int_0^L z(x,t)z_t(x,t)dx \\
&-2\alpha_0 P_1 \int_0^L z^2(x,t)dx - 2\alpha_0 P_2\nu \int_0^L z^2_{xx}(x,t)dx \\
&+2P_2\nu \int_0^L z_{xx}(x,t)z_{xxt}(x,t)dx + R\tfrac{4h^2}{\pi^2}\sum_{j=1}^{N}\int_{\Omega_j}[\rho_{jt}(t)]^2dx \\
&-Re^{-2\alpha h}\sum_{j=1}^{N}\int_{\Omega_j}[\rho_j(t)]^2dx - 2(\alpha+\alpha_0)V_R(t).
\end{aligned}
$$

We further apply the descriptor method, where the left-hand side of the following equation

$$
\begin{aligned}
&2\int_0^L [P_4 z(x,t) + P_2 z_t(x,t)][-z_t(x,t) - z_{xx}(x,t) \\
&-z(x,t)z_x(x,t) - \nu z_{xxxx}(x,t)]dx = 0
\end{aligned}
$$

with some $P_4 > 0$ is added to $\dot V$.

Using (4.17), we obtain

$$
\dot V(t) - 2\alpha_0 V(t) \le \int_0^L \eta_0^T \Psi_0 \eta_0 dx, \ t \in [t_k, t_k+\delta),
$$

where $\eta_0 = \mathrm{col}\{z(x,t), z_t(x,t), z_{xx}(x,t)\}$.

Step 4: From Step 1-Step 3, we obtain if $\|z_0\|_V < \sqrt{\frac{P_2\nu}{L}}C$, then the feasibility of LMIs (4.9)–(4.11) implies that any strong solution of (4.7), (3.2) initialized with z_0 admits a priori estimate

$$
\begin{aligned}
&V(t) \le e^{2\alpha_0(t-t_k)}V(t_k), \ \forall t \in [t_k, t_k+\delta), \\
&V(t) \le e^{-2\alpha(t-t_k-\delta)}V(t_k+\delta), \ \forall t \in [t_k+\delta, t_{k+1}).
\end{aligned} \tag{4.28}
$$

Since $\alpha_1 < \alpha$ and $t_{k+1} - t_k \ge h_0$, (4.15) implies

$$
(\alpha_1 - \alpha)(t_{k+1} - t_k) \le (\alpha_1 - \alpha)h_0 \le -(\alpha_0 + \alpha)\delta,
$$

which together with (4.28) leads to

$$
\begin{aligned}
&V(t_{k+1}) \le e^{2\alpha_0\delta}e^{-2\alpha(t_{k+1}-t_k-\delta)}V(t_k) \\
&\le e^{-2\alpha_1(t_{k+1}-t_k)}V(t_k).
\end{aligned} \tag{4.29}
$$

From (4.28) it follows

$$
\begin{aligned}
&V(t) \le e^{2\alpha_0\delta}V(t_k), \ t \in [t_k, t_k+\delta); \\
&V(t) \le V(t_k+\delta) \le e^{2\alpha_0\delta}V(t_k), \ t \in [t_k+\delta, t_{k+1}).
\end{aligned}
$$

Therefore, for $t \in [t_k, t_{k+1})$

$$
\begin{aligned}
&V(t) \le e^{2\alpha_0\delta}V(t_k) \le e^{2\alpha_0\delta - 2\alpha_1(t_k - t_{k-1})}V(t_{k-1}) \\
&\le e^{2\alpha_0\delta - 2\alpha_1(t-t_{k-1}-h)}V(t_{k-1}) \\
&\le e^{2\alpha_0\delta - 2\alpha_1(t-t_{k-2}-h)}V(t_{k-2}) \\
&\le \cdots \le e^{2\alpha_0\delta - 2\alpha_1(t-h)}V(0).
\end{aligned}
$$

Hence,

$$
V(t) \le e^{-2\alpha_1(t-h)+2\alpha_0\delta}V(0), \ \forall t \ge 0.
$$

The latter bound guarantees the existence of these strong solutions for all $t \in [0, t_1]$. Then using step method (Fridman, 2014), we conclude that the strong solution exists for all $t \ge 0$.

We will prove next that (4.26) holds. On one hand, for $t = 0$, inequality (4.26) holds by hypothesis in Theorem 1. On the other hand, let (4.26) be false for some $t > 0$ and let t^* be the smallest instant such that $V(t^*) \ge \frac{P_2\nu}{L}C^2$. Since V is continuous in time, we have $V(t^*) = \frac{P_2\nu}{L}C^2$ and $V(t) < \frac{P_2\nu}{L}C^2$ for $t \in [0, t^*)$. Since $z_x(0,t) = 0$, the Sobolev inequality (Lemma 2.3) implies: $\max_{x\in[0,L]} |z_x(x,t)|^2 \le L\|z_{xx}(\cdot,t)\|^2_{L^2} \le \frac{L}{P_2\nu}V(t) \le \frac{L}{P_2\nu}V(0) = \frac{L}{P_2\nu}\|z_0\|^2_V < C^2$ for $t \in [0, t^*)$. Thus, the feasibility of LMIs (4.9)–(4.11) guarantees that (4.27) is true for all $t \in [0, t^*)$. Hence, by continuity, $V(t) \le V(0) < \frac{P_2\nu}{L}C^2$ for all $t \in [0, t^*]$, which contradicts the definition of t^*. Therefore, (4.26) holds.

Remark 4.2. The LMIs in Theorem 1 are always feasible for appropriate decision variables and small enough α, α_0, h, δ, C, ε, P_2 and large enough N such that P_2K and $\frac{K}{N-1}$ are small. Indeed, consider Ψ_1, Ψ_2 and Ψ_0 given by (4.12), (4.13) and (4.14) respectively. We will show that strict inequalities (4.9)–(4.11) hold with $\alpha = h = C = 0$. Then LMIs (4.9)–(4.11) hold with small enough $\alpha > 0$, $h > 0$ and $C > 0$. Set $R = \lambda_2 > 2K$, $P_1 = P_3 = P_4 = 1$, $\lambda_1 = K$ such that $\psi_{12} = P_1 - P_3 = 0$ and $P_3K - \lambda_1 = 0$. By applying Schur complement, for small enough h and large enough N, we obtain

$$
\Psi_1|_{z_x=0} < 0 \iff \begin{bmatrix} \psi_{11} + \frac{2(\frac{K}{N-1})^2}{\lambda_2 + 2\frac{K}{N-1}} & \psi_{13} \\ * & \psi_{33} + \frac{P_2}{2} \end{bmatrix} < 0,
$$

$$
\Psi_2 < 0 \iff \lambda_2 - 2K > P_2K^2,
$$

$$
\begin{bmatrix} \tilde\psi_{11} + \dfrac{P_2K^2}{2 - \frac{2P_2K^2}{\lambda_2 - 2K}} & \psi_{13} + \dfrac{P_2K}{2 - \frac{2P_2K^2}{\lambda_2-2K}} \\ * & \psi_{33} + \dfrac{P_2}{2 - \frac{2P_2K^2}{\lambda_2-2K}} \end{bmatrix} < 0,
$$

$$
\Psi_0 < 0 \iff \begin{bmatrix} -2\alpha_0 & 0 & -1 \\ * & -2P_2 & -P_2 \\ * & * & -2\alpha_0P_2\nu - 2\nu \end{bmatrix} < 0,
$$

where

$$
\psi_{11} = -\lambda_3\frac{\pi^4}{L^4} - \frac{K}{N-1}, \ \psi_{13} = -1 - \frac{\lambda_2(\frac{L}{N}+\varepsilon)^2}{2\pi^2},
$$

$$
\psi_{33} = -2\nu + \lambda_3, \ \tilde\psi_{11} = -\lambda_3\frac{\pi^4}{L^4} - K.
$$

Then the latter LMIs are feasible with appropriate $\nu-\sqrt{\nu^2-\frac{L^4}{\pi^4}}<\lambda_3<\nu+\sqrt{\nu^2-\frac{L^4}{\pi^4}}$, small enough α_0, ε, P_2 and large enough N such that P_2K^2, $\frac{K}{N-1}$ are small.

Remark 4.3. Technically our Lyapunov-based approach employs the following novel tools compared to Kang and Fridman (2018):
(a) The inequality (3.13) (that results from the switching law) and (4.20) that allow to derive feasible LMIs.
(b) For the moving time we have derived a new LMI that guarantees a bound on V in step 3 of Theorem 1 proof (see p.7) and the bound on the dwelling time δ in step 4.
(c) For point-like measurements that were not considered in Kang and Fridman (2018), we use a mean value theorem ((4.3)) and further Wirtinger's inequality in (4.18), which is different from the direct application of the Poincaré inequality in Kang and Fridman (2018) for the averaged measurements. Moreover, we employ the Wirtinger-based Lyapunov functional for sampled-data control with ρ_j which includes c_j, which leads to simpler LMIs comparatively to Lyapunov functional of Kang and Fridman (2018) with $c_j=1$.

Under the averaged state measurements (3.8) we have the following result:

Theorem 2. *Consider the closed-loop system (3.10) subject to (3.2) and the switching law (3.12) with $c_j=1$. Given positive scalars h, α, K and tuning parameter $C>0$, $\alpha_0>0$ such that $\alpha h_0>(\alpha_0+\alpha)\delta$, let there exist scalars $R>0$, $P_n>0$, $\lambda_n\geq 0$ $(n=1,2,3)$ that satisfy the LMIs:*

$$\Theta_1|_{z_x=\pm C}<0, \tag{4.30}$$

$$\Theta_2|_{z_x=\pm C}<0, \tag{4.31}$$

$$\Psi_0|_{z_x=\pm C}<0, \tag{4.32}$$

where

$$\Theta_1=\Psi_1+\Pi, \tag{4.33}$$

$$\Theta_2=\Psi_2+\Pi, \tag{4.34}$$

where Ψ_1,Ψ_2,Ψ_0 are given by (4.12),(4.13), (4.14) respectively, and

$$\Pi=\left[\begin{array}{ccc|c} 0 & 0 & \frac{\lambda_2}{2\pi^2}\left(\frac{2L\varepsilon}{N}+\varepsilon^2\right) & \\ 0 & R\frac{4h^2}{\pi^2}\left(1-\frac{L}{N\varepsilon}\right) & 0 & 0_{3\times 2} \\ 0 & 0 & 0 & \\ \hline & * & & 0_{2\times 2}\end{array}\right] \tag{4.35}$$

Then for any initial function $z_0\in H_0^2(0,L)$ subject to $\|z_0\|_V<\sqrt{\frac{P_2\nu}{L}}C$, the closed-loop system (3.10) subject to (3.2) is exponentially stable with a decay rate $\alpha_1>0$ which is subject to (4.15).

Proof. See the Appendix.

Remark 4.4. For the negligible moving time with $\delta\to 0$, the LMI conditions of Theorem 1 are reduced to (4.9) and (4.10), and the LMI conditions of Theorem 2 are reduced to (4.30) and (4.31).

5. Extensions

The proposed method is efficient for various classes of PDEs. In this section we will discuss its extension to KSE under the periodic boundary conditions and to the reaction–diffusion equation.

Consider (3.10) under the periodic boundary conditions:

$$\frac{\partial^m z}{\partial x^m}(0,t)=\frac{\partial^m z}{\partial x^m}(L,t),\ t>0,\ m=0,1,2,3 \tag{5.1}$$

and the switching law (3.12). The well-posedness under the periodic boundary conditions can be established similar to the case of Dirichlet boundary conditions. Denote

$$H^2_{per}(0,L)\triangleq\{g\in H^2(0,L)|g(0)=g(L),g'(0)=g'(L)\}.$$

We employ the extension of Sobolev's inequality (Kang & Fridman, 2018, 2019):

Lemma 5.1. *Let $z\in H^1(0,L)$ be a scalar function. Then*

$$\max_{x\in[0,L]}|z(x,t)|^2\leq(1+\frac{1}{L})\|z(\cdot,t)\|^2_{L^2(0,L)}+\|z_x(\cdot,t)\|^2_{L^2(0,L)}.$$

Note that Wirtinger's inequality cannot be applied for the case of the periodic boundary conditions (5.1). Therefore, from Lemma 5.1 we directly obtain the following result:

- If the LMI conditions of Theorem 1/Theorem 2 with $\lambda_3=0$ hold, then for the initial function $z_0\in H^2_{per}(0,L)$ satisfying $\|z_0\|_V<\sqrt{\frac{L}{(L+1)M}}C$, there exists a unique strong solution of the corresponding closed-loop system and the closed-loop system is stable, where $M\triangleq\max\left\{\frac{1}{2P_1},\frac{3}{2P_2\nu}\right\}$.

Remark 5.1. In the companion conference paper (Hetel & Fridman, 2013), the presented switching control sampled-data control was presented for the semilinear heat equation

$$z_t(x,t)=\frac{\partial}{\partial x}[a(x)z_x(x,t)]+\varphi(z(x,t))z(x,t)+b_{\sigma_k}(x)u_{\sigma_k}(t),$$

under the averaged measurements and the Dirichlet boundary conditions $z(0,t)=z(1,t)=0,\ \ t>0$. The functions a and φ are of class C^1 and may be unknown. Global sampled-data stabilization by switching was achieved under assumption that the inequalities $a(x)\geq a_0>0,\ \varphi_m\leq\varphi\leq\varphi_M$ hold with known bounds a_0, φ_m and φ_M.

6. Numerical example

In this section, we present a numerical example which verifies the effectiveness of the proposed method for KSE. For the heat equation see example in Kang et al. (2021).

6.1. Case of the Dirichlet boundary conditions

Consider KSE (3.1) under the Dirichlet boundary conditions (3.2) with $L=2\pi$ and instability parameter $\nu=0.8<1$. The initial function is chosen as follows:

$$z(x,0)=z_0(x)=0.015(1-\cos x)\sin x.$$

Fig. 2 demonstrates the profile of the open-loop system initialized by $z_0(x)$. It is shown that the unforced system is unstable.

We choose $K=25,\alpha_0=0.32,\alpha=0.07$, $C=1$ and verify LMIs of Theorems 1 and 2 under point-like and averaged measurements. The results are given in Table 1, which shows that as N increases, the maximum value of h increases. Since the averaged measurements contain more information than the point-like measurements, the corresponding switched closed-loop system under the averaged measurements preserves the stability for larger h. Note that e.g. for $N=15$ and $\varepsilon=\pi/30$ the closed-loop system preserves the exponential stability within a given domain of initial conditions $\|z_0\|_V<0.743$ for $0.0059\leq t_{k+1}-t_k\leq h\leq 0.006$. Note also that for the choice of K we used trials and errors

Table 1
Maximal values of h, δ under the Dirichlet boundary conditions.

	Averaged measurements		Point-like measurements		
N	h	δ	ε	h	δ
10	0.020	0.003	$\pi/90$ $\pi/30$	0.0033 0.0038	0.0005 0.0006
13	0.027	0.004	$\pi/90$ $\pi/30$	0.0045 0.0054	0.0007 0.0009
15	0.030	0.005	$\pi/90$ $\pi/30$	0.0050 0.0060	0.0008 0.0010

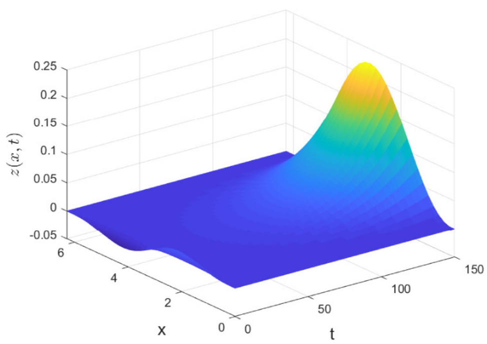

Fig. 2. State of unforced system: Dirichlet boundary conditions.

to minimize N for $h = 0$ and $C = 1$. Then, we increased h and the corresponding value of N.

We further provide numerical simulations of the solutions to the corresponding closed-loop systems with $N = 15$, $K = 25$, $\alpha = 0.07$:
(a) A finite difference method was utilized for numerical simulations of the solution to the closed-loop system (4.7) subject to (3.2) under the output-feedback

$$u_{\sigma_k}(t) = \begin{cases} 0, & t \in [t_k, t_k + \delta), \\ -25 \int_{\Omega_{\sigma_k}} c_{\sigma_k}(x) z(x, t_k) dx, & t \in [t_k + \delta, t_{k+1}) \end{cases}$$

via the switching law (3.9), (3.12) with $t_{k+1} - t_k = 0.006$, $\varepsilon = \pi/30$, $|\Omega_{\sigma_k}| = \frac{2\pi}{15}$, and the moving time for sensors and actuators $\delta = 0.001$. The steps in time and space were set as $dx = \pi/30$ and $dt = 10^{-5}$, respectively. Fig. 3 shows the time evolution of $\|z(\cdot, t)\|_{H^2}$ of the closed-loop and open-loop systems. Note that the closed-loop system with switching controller is stable, whereas the closed-loop system under only one stationary actuator is unstable.
(b) Next, a finite difference method is utilized to compute the numerical solution of the closed-loop system (3.10) subject to (3.2) under the output-feedback

$$u_{\sigma_k}(t) = \begin{cases} 0, & t \in [t_k, t_k + \delta), \\ -\frac{375}{2\pi} \int_{\Omega_{\sigma_k}} z(x, t_k) dx, & t \in [t_k + \delta, t_{k+1}) \end{cases}$$

via the switching law (3.9), with $t_{k+1} - t_k = 0.030$, $|\Omega_{\sigma_k}| = \frac{2\pi}{15}$, and the moving time for sensors and actuators $\delta = 0.005$. Set the steps in time and space as $dx = \pi/30$ and $dt = 10^{-5}$, respectively. Fig. 4 shows the profile of the closed-loop system. The locations of sensor/actuator under the switching control law are given in Fig. 5.

6.2. Case of the periodic boundary conditions

We proceed further with the case of periodic boundary conditions (5.1) under the same initial conditions. For the switching controller (3.9) via (3.12), by using the Yalmip, we verify LMI conditions of Theorem 1 with $\lambda_3 = 0$, $N = 15$, $K = 25$, $C = 1$,

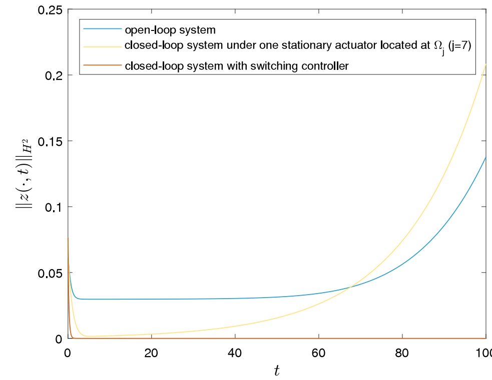

Fig. 3. H^2-norm $\|z(\cdot, t)\|_{H^2}$ of the open-loop system, closed-loop system under one stationary actuator located at Ω_7 and switched closed-loop system: Dirichlet boundary conditions.

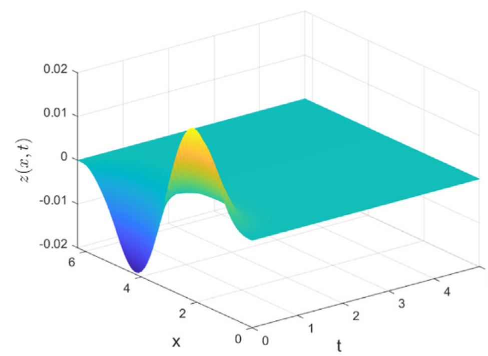

Fig. 4. State response of closed-loop system under the averaged measurements and Dirichlet boundary conditions.

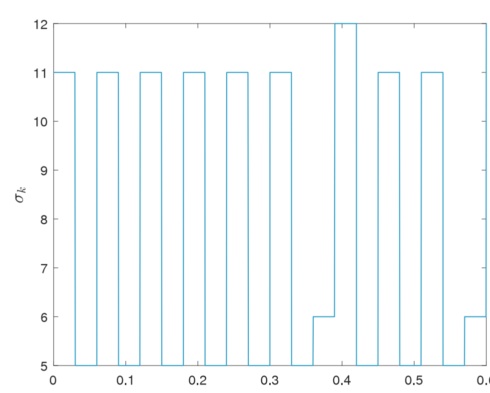

Fig. 5. Sensor/actuator locations for $N = 15$, $t_{k+1} - t_k = 0.030$.

$\alpha = 0.07$, $\alpha_0 = 0.32$, $\delta = 0.001$, and $\varepsilon = \pi/30$. We find that the closed-loop system (4.7), (5.1) preserves exponential stability within a given domain of initial conditions $\|z_0\|_V < 0.773$ for $0.0059 \le t_{k+1} - t_k \le h \le 0.006$. In the numerical simulation, a finite difference method is utilized to compute the numerical solution of the closed-loop system (4.7), (5.1) under the output-feedback (3.9) via the switching law (3.12) with $|\Omega_{\sigma_k}| = \frac{2\pi}{15}$ and $t_{k+1} - t_k = 0.006$. Numerical simulations of the solutions confirm the theoretical results that follow from LMIs.

7. Conclusions

The present paper introduced stabilization by switched sampled-data controller of parabolic PDEs via the employment of either stationary or moving actuating devices that can move to the active subdomain. Extension of the results to other PDEs may be a topic for future research.

Appendix. Proof of Theorem 2

For the case of switched controller under the averaged measurements, by arguments of Theorem 1, the well-posedness of (3.10) subject to (3.2) can be established via the step method.

Denote $\tilde{f}_j(x,t) \triangleq z(x,t) - \frac{\int_{\Omega_j} z(x,t)dx}{|\Omega_j|}$, $\tilde{\rho}_j(t) \triangleq \frac{\int_{\Omega_j}\int_{t_k}^{t} z_s(x,s)dsdx}{|\Omega_j|}$, where $|\Omega_j| = \frac{L}{N}$.

Then the switching controller (3.9) via the switching law (3.12) with $c_j = 1$ can be rewritten as

$$u_{\sigma_k}(t) = -K[z(x,t) - \tilde{f}_{\sigma_k}(x,t) - \tilde{\rho}_{\sigma_k}(t)]. \tag{A.1}$$

We choose the Lyapunov function V with $\tilde{\rho}_j$ instead of ρ_j. Differentiating V along the solution of the closed-loop system (3.10) subject to (3.2), we get (4.16) with $\tilde{\rho}_j$ instead of ρ_j. The substitution $f_j \to \tilde{f}_j$ and $\rho_j \to \tilde{\rho}_j$ in Theorem leads to the following changes:

$$\int_{\Omega_j}[\tilde{\rho}_{jt}(t)]^2dx = \frac{1}{|\Omega_j|}\left(\int_{\Omega_j} z_t(x,t)dx\right)^2 \leq \int_{\Omega_j} z_t^2(x,t)dx,$$

$$\begin{aligned}&-\frac{\lambda_1}{N-1}\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}[z(x,t)-\tilde{f}_j(x,t)-\tilde{\rho}_j(t)]^2dx\\&+\lambda_1\int_{\Omega_{\sigma_k}}[z(x,t)-\tilde{f}_{\sigma_k}(x,t)-\tilde{\rho}_{\sigma_k}(t)]^2dx \geq 0,\end{aligned} \tag{A.2}$$

$$\frac{\lambda_2 L^2}{N^2\pi^2}\|z_x(\cdot,t)\|^2_{L^2(0,L)} - \lambda_2\sum_{j=1}^{N}\|\tilde{f}_j(\cdot,t)\|^2_{L^2(\Omega_j)} \geq 0 \tag{A.3}$$

for any $\lambda_1 \geq 0$, $\lambda_2 \geq 0$.

Set $\tilde{\eta}_1 = \mathrm{col}\{z(x,t), z_t(x,t), z_{xx}(x,t), \tilde{f}_j(x,t), \tilde{\rho}_j(x,t)\}$, $\tilde{\eta}_2 = \mathrm{col}\{z(x,t), z_t(x,t), z_{xx}(x,t), \tilde{f}_{\sigma_k}(x,t), \tilde{\rho}_{\sigma_k}(x,t)\}$, $\eta_0 = \mathrm{col}\{z(x,t), z_t(x,t), z_{xx}(x,t)\}$. Applying descriptor method and adding the left-hand sides of (A.2)–(4.22) and (A.3) to $\dot{V}$, we finally obtain

$$\begin{aligned}&\dot{V}(t) + 2\alpha V(t) \leq \sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}\tilde{\eta}_1^T\Theta_1\tilde{\eta}_1dx + \int_{\Omega_{\sigma_k}}\tilde{\eta}_2^T\Theta_2\tilde{\eta}_2dx,\\&\qquad\qquad t\in[t_k+\delta, t_{k+1}),\\&\dot{V}(t) - 2\alpha_0 V(t) \leq \int_0^L \eta_0^T\Psi_0\eta_0dx, \qquad t\in[t_k, t_k+\delta),\end{aligned}$$

where Θ_l $(l = 1,2)$ and Ψ_0 are given by (4.33), (4.34) and (4.14), respectively.

Hence, $\dot{V}(t) + 2\alpha V(t) \leq 0$, $\dot{V}_0(t) - 2\alpha V_0(t) \leq 0$, if $\Theta_l < 0 (l = 1,2)$ and $\Psi_0 < 0$ hold for all $-C \leq z_x \leq C$. Matrices $\Theta_l (l = 1,2)$ and Ψ_0 given by (4.33), (4.34) and (4.14) are affine in z_x. Hence, $\Theta_l < 0 (l = 1,2)$ and $\Psi_0 < 0$ for all $-C \leq z_x \leq C$ if these inequalities hold in the vertices $z_x = \pm C$, i.e. if LMIs (4.30)–(4.32) are feasible.

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Wen Kang received the B.S. degree in mathematics from Wuhan University, Wuhan, China, in 2009, and the Ph.D. degree in mathematics from Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, in 2014. She was a Lecturer with the Harbin Institute of Technology, Harbin, China, from 2014 to 2017. From 2015 to 2017, she was a Postdoctoral Researcher with the School of Electrical Engineering, Tel Aviv University, Tel Aviv-Yafo, Israel. She was an Associate Professor in University of Science and Technology Beijing, China. In 2022, she jointed Beijing Institute of Technology as a Research Professor. Her research interests include distributed parameter systems, time-delay systems, and event-triggered control. Dr. Kang serves as Associate Editor for IMA Journal of Mathematical Control and Information.

Emilia Fridman received the M.Sc. degree from Kuibyshev State University, USSR, in 1981 and the Ph.D. degree from Voronezh State University, USSR, in 1986, all in mathematics. From 1986 to 1992 she was an Assistant and Associate Professor in the Department of Mathematics at Kuibyshev Institute of Railway Engineers, USSR. Since 1993 she has been at Tel Aviv University, where she is currently Professor of Electrical Engineering-Systems. She has held visiting positions at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin (Germany), INRIA in Rocquencourt (France), Ecole Centrale de Lille (France), Valenciennes University (France), Leicester University (UK), Kent University (UK), CINVESTAV (Mexico), Zhejiang University (China), St. Petersburg IPM (Russia), Melbourne University (Australia), Supelec (France), KTH (Sweden). Her research interests include time-delay systems, networked control systems, distributed parameter systems, robust control, singular perturbations and nonlinear control. She has published 200 articles in international scientific journals and the author of two monographs. She serves/served as Associate Editor in Automatica, SIAM Journal on Control and Optimization and IMA Journal of Mathematical Control and Information. In 2014 she was Nominated as a Highly Cited Researcher by Thomson ISI. Since 2018, she has been the incumbent for Chana and Heinrich Manderman Chair on System Control at Tel Aviv University. She is IEEE Fellow since 2019. She is currently a member of the IFAC Council. In 2021 she was recipient of IFAC Delay Systems Life Time Achievement Award and of Kadar Award for outstanding research at Tel Aviv University.

Chuan-Xin Liu received his B.S. degree from the College of Information Engineering, Henan University of Science and Technology in 2020. He is currently pursuing the M.S. degree with the School of Automation and Electrical Engineering, University of Science and Technology Beijing. His research interests include event-triggered control, switching control and their applications.