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Sampled-data control of 2D Kuramoto-Sivashinsky equation

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***Abstract*—This paper addresses sampled-data control of 2D Kuramoto-Sivashinsky equation over a rectangular domain Ω. We suggest to divide the 2D rectangular Ω into N subdomains, where sensors provide spatially averaged or point state measurements to be transmitted through communication network to the controller. Note that differently from 2D heat equation, here we manage with sampled-data control under point measurements. We design a regionally stabilizing controller applied through distributed in space characteristic functions. Sufficient conditions ensuring regional stability of the closed-loop system are established in terms of linear matrix inequalities (LMIs). By solving these LMIs, we find an estimate on the set of initial conditions starting from which the state trajectories of the system are exponentially converging to zero. A numerical example demonstrates the efficiency of the results.**

I. Introduction

In recent decades, Kuramoto-Sivashinsky equation (KSE) has drawn a lot of attention as a nonlinear model of pattern formations on unstable flame fronts and thin hydrodynamic films (see e.g. [1], [2]). KSE arises in the study of thin liquid films, exhibiting a wide range of dynamics in different parameter regimes, including unbounded growth and full spatiotemporal chaos. For 1D KSE, distributed control (see e.g. [3]–[6]) has been considered. Boundary stabilization of KSE has been studied in [7]–[10].

Sampled-data control of PDEs became recently an active research area (see e.g. [11]–[14]) for practical application of finite-dimensional controllers for PDEs, where LMI conditions for the exponential/regional stability of the closed-loop systems were derived in the framework of time-delay approach by employing appropriate Lyapunov functionals. Most of the existing results deal with 1D PDEs system. Sampled-data observers for ND and 2D heat equations with globally Lipschitz nonlinearities have been suggested in [15] and [16]. However, the above results were confined to diffusion equations. Sampled-data control of various classes of high dimensional PDEs is an interesting and challenging problem.

In our recent paper [14], we have suggested sampled-data control of 1D KSE, where both point and averaged state measurements were studied. In this paper we aim to extend results of [14] to 2D case. Extension from 1D ([14], [30]) to 2D is far from being straightforward. Thus, in the case of heat equation, sampled data extension under the point measurements seems to be not possible (see [16]). This is due to the fact that stability analysis of the closed-loop system is based on the bound of the L^2-norm of the difference between the state and its point value. However, according to Friedrich's inequality [16], [17], this bound depends on the L^2-norm of the second-order spatial derivatives of the state. Differently from the heat equation, stability analysis for KSE in H^2 allows to compensate such terms. We establish stability analysis of the closed-loop sampled-data system by constructing an appropriate Lyapunov-Krasovskii functional. Some preliminary results under averaged measurements were presented in [29], where the sampled-data case is limited to averaged measurements and there is no detailed proof of the well-posedness.

In the present paper, we design a sampled-data controller for 2D KSE under averaged/point measurements based on LMIs. In comparison to the existing known results, new special challenges of this work are the following:

1) The present paper gives the first extension to 2D PDE in the case of sampled-data point measurements. The results from [16] cannot be extended to the case of point measurements. This is due to the second order spatial derivative in Lemma 3 that cannot be compensated in Lyapunov analysis.

2) Here we have the nonlinear term "zz_{x_1}" which is locally Lipschitz in $D((-A)^{\frac{1}{2}})$ (defined in Section III below). Due to the nonlinear term, the challenge is to find a bound on the domain of attraction. This bound is based on the new 2D Sobolev inequality (Lemma 4) that we have derived. Lemma 4 bounds a function in the C^0-norm using L^2-norms of its first and second spatial derivatives. In [15] and [16], the nonlinear term is subject to sector bound inequality which holds globally and leads to global results.

3) The well-posedness is challenging. We have provided more detailed proof for this, and shown that A generates an analytic semigroup even for rectangular domain Ω (non C^1 boundary).

The remainder of this work is organized as follows. Useful lemmas are introduced and the problem setting is reported in Section II. Sections III-IV are devoted to construction of continuous static output-feedback/sampled-data controllers under the averaged or point measurements. In Section V, a numerical example is carried out to illustrate the efficiency of the main results. Finally, some concluding remarks and possible future research lines are presented in Section VI.

Notation The superscript "T" stands for matrix transposi-

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tion, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the norm $|\cdot|$. $\Omega \subset \mathbb{R}^2$ denotes a computational domain, $L^2(\Omega)$ denotes the space of measurable squared-integrable functions over Ω with the corresponding norm $\|z\|^2_{L^2(\Omega)} = \int_\Omega |z(x)|^2 dx$. Let $\partial\Omega$ be the boundary of Ω. The Sobolev space $H^k(\Omega)$ is defined as $H^k(\Omega) = \{z : D^\alpha z \in L^2(\Omega),\ \forall\ 0 \le |\alpha| \le k\}$ with norm $\|z\|_{H^k(\Omega)} = \left(\sum_{0\le|\alpha|\le k} \|D^\alpha z\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$. The space $H^k_0(\Omega)$ is the closure of $C^\infty_c(\Omega)$ in the space $H^k(\Omega)$ with the norm $\|z\|_{H^k_0(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha z\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$.

II. Problem formulation and useful lemmas

Denote by Ω the two dimensional (2D) unit square

$$\Omega = [0,1]\times[0,1] \subset \mathbb{R}^2.$$

Consider the biharmonic operator:

$$\Delta^2 = \frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^4}.$$

As in [18], we consider the following 2D Kuramoto-Sivashinsky equation (KSE) over Ω under the Dirichlet boundary conditions:

$$\begin{cases} z_t + z z_{x_1} + (1-\kappa) z_{x_1x_1} - \kappa z_{x_2x_2} + \Delta^2 z = \sum_{j=1}^{N} \chi_j(x) U_j(t), \\ \qquad (x,t) \in \Omega \times [0,\infty), \\ z|_{\partial\Omega} = 0,\ \frac{\partial z}{\partial n}|_{\partial\Omega} = 0, \\ z(x,0) = z_0(x), \end{cases} \tag{1}$$

where $x = (x_1, x_2) \in \Omega$, $z \in \mathbb{R}$ is the state of KSE, $\frac{\partial z}{\partial n}$ is the normal derivative, and $U_j(t) \in \mathbb{R}$, $j = 1, 2, \cdots, N$ are the control inputs. Here the parameter κ denotes the angle of the substrate to the horizontal:
for $\kappa > 0$ we have overlying film flows, a vertical film flow for $\kappa = 0$, and hanging flows when $\kappa < 0$.

Motivated by [6], [12], [13], [15], [19] we suggest to divide Ω into N square sub-domains Ω_j covering the whole region $\cup_{j=1}^N \Omega_j = \Omega$ (see Fig. 1) with an actuator and a sensor placed in each Ω_j. Here

$$\Omega_j = \{x = (x_1,x_2)^T \in \Omega | x_i \in [x_i^{\min}(j), x_i^{\max}(j)], i = 1,2\}, \quad j = 1,2,\cdots,N.$$

The measure of their intersections is zero. Let

$$0 = t_0 < t_1 < \cdots < t_k \cdots, \quad \lim_{k\to\infty} t_k = \infty$$

be sampling time instants. The sampling sub-domains in time and in space may be bounded,

$$\begin{array}{l} 0 \le t_{k+1} - t_k \le h, \\ 0 < x_i^{\max}(j) - x_i^{\min}(j) = \Delta_j \le \bar{\Delta}, \\ \qquad i = 1,2;\ \ j = 1,2,\cdots,N, \end{array}$$

where h and $\bar{\Delta}$ are the corresponding upper bounds.

Remark 1. *For simplicity, we consider each sub-domain Ω_j is a square (i.e. $x_1^{\max}(j) - x_1^{\min}(j)) = x_2^{\max}(j) - x_2^{\min}(j)$, $j = 1, 2, \cdots, N$). Indeed, Ω_j can be a rectangular. For the case that $x_1^{\max}(j) - x_1^{\min}(j)) \neq x_2^{\max}(j) - x_2^{\min}(j)$ for some j (see Fig. 1 of [16]), Δ_j can be chosen as follows*

$$\Delta_j = \max\{x_1^{\max}(j) - x_1^{\min}(j), x_2^{\max}(j) - x_2^{\min}(j)\}.$$

Thus, the results of this work are applicable to the case of nonsquare sub-domains.

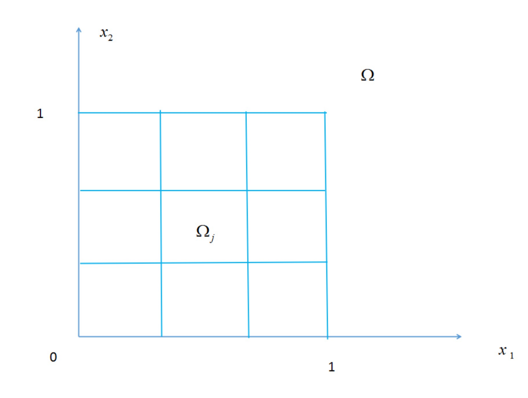

Fig. 1. Unit square Ω and sub-domains Ω_j

The spatial characteristic functions are taken as

$$\begin{cases} \chi_j(x) = 1,\ x \in \Omega_j, \\ \chi_j(x) = 0,\ x \notin \Omega_j, \end{cases} \quad j = 1,\cdots,N. \tag{2}$$

We assume that sensors provide the following averaged measurements

$$y_{jk} = \frac{\int_{\Omega_j} z(x,t_k)dx}{|\Omega_j|}, \quad j = 1,\ldots,N;\ k = 0,1,2\ldots \tag{3}$$

or point measurements

$$y_{jk} = z(\bar{x}_j, t_k), \quad j = 1,\ldots,N;\ k = 0,1,2\ldots \tag{4}$$

where $\bar{x}_j$ locates in the center of the square sub-domain Ω_j, and $|\Omega_j|$ stands for the Lebesgue measure of the domain Ω_j.

We aim to design for (1) an exponentially stabilizing sampled-data controller that can be implemented by zero-order hold devices:

$$\begin{array}{l} U_j(t) = -\mu y_{jk},\quad j = 1,\ldots,N; \\ \qquad t \in [t_k, t_{k+1}),\ k = 0,1,2\ldots, \end{array} \tag{5}$$

where μ is a positive controller gain and y_{jk} is given by (3) or (4).

We present below some useful lemmas:

Lemma 1. *Let $\Omega = (0, L_1) \times (0, L_2)$. Assume $f : \Omega \to \mathbb{R}$ and $f \in H^1(\Omega)$.*
(i) (Poincaré's inequality) If $\int_\Omega f(x)dx = 0$, then according to [20]

$$\|f\|^2_{L^2(\Omega)} \le \frac{L_1^2 + L_2^2}{\pi^2} \|\nabla f\|^2_{L^2(\Omega)}. \tag{6}$$

(ii) (Wirtinger's inequality) [15] If $f|_{\partial\Omega} = 0$, then the following inequality holds:

$$\|f\|^2_{L^2(\Omega)} \le \frac{L_1^2 + L_2^2}{\pi^2} \|\nabla f\|^2_{L^2(\Omega)}. \tag{7}$$

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The following lemma gives a classical Friedrich's inequality (Theorem 18.1 of [33]) with tight bounds on the coefficients of terms $\left\|\frac{\partial f}{\partial x_1}\right\|^2_{L^2(\Omega)}$, $\left\|\frac{\partial f}{\partial x_2}\right\|^2_{L^2(\Omega)}$ and $\left\|\frac{\partial^2 f}{\partial x_1 \partial x_2}\right\|^2_{L^2(\Omega)}$. The inequality (8) bounds the L^2-norm of a function by the reciprocally convex combination of the L^2-norm of its derivatives.

Lemma 2. *(see (2) of [16]) Let $\Omega = (0, l)^2$, $f \in H^2(\Omega)$ with $f(0, 0) = 0$. Then the following inequality holds:*

$$\begin{aligned} \|f\|^2_{L^2(\Omega)} \leq & \frac{1}{\alpha_1}\left(\frac{2l}{\pi}\right)^2 \left\|\frac{\partial f}{\partial x_1}\right\|^2_{L^2(\Omega)} + \frac{1}{\alpha_2}\left(\frac{2l}{\pi}\right)^2 \left\|\frac{\partial f}{\partial x_2}\right\|^2_{L^2(\Omega)} \\ & + \frac{1}{\alpha_3}\left(\frac{2l}{\pi}\right)^4 \left\|\frac{\partial^2 f}{\partial x_1 \partial x_2}\right\|^2_{L^2(\Omega)} \end{aligned} \tag{8}$$

where α_1, α_2, α_3 are positive constants satisfying

$$\alpha_1 + \alpha_2 + \alpha_3 = 1.$$

Lemma 3. *[16] Let $\Omega = (0, l)^2$, $f \in H^2(\Omega)$ with $f(0, 0) = 0$, $\eta > 0$. Then*

$$\begin{aligned} \eta\|f\|^2 \leq & \beta_1 \left(\frac{2l}{\pi}\right)^2 \left\|\frac{\partial f}{\partial x_1}\right\|^2 + \beta_2 \left(\frac{2l}{\pi}\right)^2 \left\|\frac{\partial f}{\partial x_2}\right\|^2 \\ & + \beta_3 \left(\frac{2l}{\pi}\right)^4 \left\|\frac{\partial^2 f}{\partial x_1 \partial x_2}\right\|^2 \end{aligned} \tag{9}$$

for any β_1, β_2, β_3 satisfying

$$\text{diag}\{\beta_1, \beta_2, \beta_3\} \geq \eta \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{10}$$

The following version of 2D Sobolev inequality will be useful:

Lemma 4. *Let $\Omega = (0, 1)^2$ and $w = w(x_1, x_2) \in H^2(\Omega) \cap H^1_0(\Omega)$, where $(x_1, x_2) \in \Omega$. Then*

$$\begin{aligned} \|w\|^2_{C^0(\bar{\Omega})} \leq & \frac{1}{2}(1+\Gamma)\left[\|w_{x_1}\|^2_{L^2(\Omega)} + \|w_{x_2}\|^2_{L^2(\Omega)}\right] \\ & + \frac{1}{\Gamma}\|w_{x_1 x_2}\|^2_{L^2(\Omega)}, \ \forall \Gamma > 0. \end{aligned} \tag{11}$$

Proof. Due to $w \in H^1_0(\Omega)$, application of 1D Sobolev's inequality to w in x_2 yields

$$\max_{x_2 \in [0,1]} w^2(x_1, x_2) \leq \int_0^1 w^2_{x_2}(x_1, \xi_2) d\xi_2. \tag{12}$$

Further application of Lemma 4.1 of [14] to w_{x_2} in x_1 leads to

$$\begin{aligned} & \max_{x_1 \in [0,1]} w^2_{x_2}(x_1, \xi_2) \\ & \leq (1+\Gamma)\int_0^1 w^2_{x_2}(\xi_1, \xi_2) d\xi_1 + \frac{1}{\Gamma}\int_0^1 w^2_{x_1 x_2}(\xi_1, \xi_2) d\xi_1, \\ & \qquad \forall \Gamma > 0. \end{aligned} \tag{13}$$

Substitution of (13) into (12) yields

$$\begin{aligned} & \|w\|^2_{C^0(\bar{\Omega})} = \max_{(x_1,x_2)\in\bar{\Omega}} w^2(x_1, x_2) = \max_{x_1 \in [0,1]}\left[\max_{x_2 \in [0,1]} w^2(x_1, x_2)\right] \\ & \leq \max_{x_1 \in [0,1]}\left[\int_0^1 w^2_{x_2}(x_1, \xi_2) d\xi_2\right] \leq \int_0^1 \max_{x_1 \in [0,1]} w^2_{x_2}(x_1, \xi_2) d\xi_2 \\ & \leq (1+\Gamma)\int_0^1 \int_0^1 w^2_{x_2}(\xi_1, \xi_2) d\xi_1 d\xi_2 \\ & + \frac{1}{\Gamma}\int_0^1 \int_0^1 w^2_{x_1 x_2}(\xi_1, \xi_2) d\xi_1 d\xi_2. \end{aligned} \tag{14}$$

Following the same procedure, we can obtain

$$\begin{aligned} \|w\|^2_{C^0(\bar{\Omega})} \leq & (1+\Gamma)\int_0^1 \int_0^1 w^2_{x_1}(\xi_1, \xi_2) d\xi_1 d\xi_2 \\ & + \frac{1}{\Gamma}\int_0^1 \int_0^1 w^2_{x_1 x_2}(\xi_1, \xi_2) d\xi_1 d\xi_2. \end{aligned} \tag{15}$$

From (14) and (15) it follows that (11) holds. □

Remark 2. *In Lemma 4, we give a new 2D Sobolev inequality with constants depending on a free parameter $\Gamma > 0$. Lemma 4 is very useful and plays an important role in the stability analysis leading to LMIs (with Γ as a decision parameter) that guarantee stability and give a bound on the domain of attraction.*

Lemma 5. *(Halanay's Inequality [24]) Let $V : [-h, \infty) \to [0, \infty)$ be an absolutely continuous function. If there exist $0 < \delta_1 < 2\delta$ such that for all $t \geq 0$ the following inequality holds*

$$\dot{V}(t) + 2\delta V(t) - \delta_1 \sup_{-h \leq \theta \leq 0} V(t+\theta) \leq 0, \tag{16}$$

then we have

$$V(t) \leq e^{-2\sigma t} \sup_{-h \leq \theta \leq 0} V(\theta), \ t \geq 0, \tag{17}$$

where σ is a unique solution of

$$\sigma = \delta - \frac{\delta_1}{2} e^{2\sigma h}. \tag{18}$$

III. Global stabilization: continuous static output-feedback

In this section, we will establish the well-posedness and stability analysis for the system (1) under the continuous-time averaged measurements

$$y_j(t) = \frac{\int_{\Omega_j} z(x,t) dx}{|\Omega_j|}, \quad j = 1, \ldots, N \tag{19}$$

or point measurements

$$y_j(t) = z(\bar{x}_j, t), \quad j = 1, \ldots, N \tag{20}$$

via a controller

$$U_j(t) = -\mu y_j(t), \quad j = 1, \ldots, N. \tag{21}$$

The closed-loop system can be represented in the following form:

$$\begin{cases} z_t + z z_{x_1} + (1-\kappa) z_{x_1 x_1} - \kappa z_{x_2 x_2} + \Delta^2 z \\ = -\mu \sum_{j=1}^{N} \chi_j(x)[z - f_j], \quad (x,t) \in \Omega \times [0, \infty), \\ z|_{\partial\Omega} = 0, \ \frac{\partial z}{\partial n}|_{\partial\Omega} = 0, \\ z(x, 0) = z_0(x), \end{cases} \tag{22}$$

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where for (19)

$$f_j(x,t) = z(x,t) - \frac{\int_{\Omega_j} z(\zeta,t)d\zeta}{|\Omega_j|}, \tag{23}$$

for (20)

$$f_j(x,t) = z(x,t) - z(\bar{x}_j,t). \tag{24}$$

Now we establish the well-posedness of the system (22) subject to (23) or (24). Define the spatial differential operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ as follows:

$$\begin{cases} Af = -\Delta^2 f, \ \forall f \in D(A), \\ D(A) = \{f \in H^4(\Omega) : f|_{\partial\Omega} = 0, \ \dfrac{\partial f}{\partial n}|_{\partial\Omega} = 0\}. \end{cases} \tag{25}$$

Note that $A^* = A$ and $\mathrm{Re}\langle Af, f\rangle \leq 0, \ \forall f \in D(A)$. Thus, the operator A is self-adjoint and dissipative. Moreover, the inverse A^{-1} is bounded, and hence $0 \in \rho(A)$. By the Lumer-Phillips theorem [28], A generates a C_0-semigroup. Since the resolvent of A is compact on $L^2(\Omega)$, the spectrum of A consists of isolated eigenvalues only, and a sequence of corresponding eigenfunctions of A forms an orthonormal basis of $L^2(\Omega)$. Let $\{\lambda_n\}$ be the eigenvalues of A and let $\{\phi_n\}$ be the corresponding eigenfunctions, i.e. $A\phi_n = \lambda_n\phi_n$. Since A is negative, all the eigenvalues are located on the negative real axis, i.e. $\lambda_n < 0$. For any $x_0 \in L^2(\Omega)$, it can be presented in the following form: $x_0 = \sum_{n=1}^{\infty} a_n\phi_n$ with $\sum_{n=1}^{\infty} |a_n|^2 = \|x_0\|^2_{L^2(\Omega)}$. Therefore, for any $\tau \in \mathbb{R}$,

$$\begin{aligned} \|(i\tau I - A)^{-1}x_0\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty} \frac{|a_n|^2}{|i\tau - \lambda_n|^2} = \sum_{n=1}^{\infty} \frac{|a_n|^2}{\tau^2 + \lambda_n^2} \\ &\leq \sum_{n=1}^{\infty} \frac{|a_n|^2}{\tau^2} = \frac{1}{\tau^2}\|x_0\|^2_{L^2(\Omega)}, \end{aligned}$$

which implies

$$\|(i\tau I - A)^{-1}\| \leq \frac{1}{|\tau|}, \ \forall \tau \in \mathbb{R}.$$

Hence,

$$\overline{\lim_{|\tau|\to\infty}} \|\tau(i\tau I - A)^{-1}\| < \infty.$$

Then from Theorem 1.3.3 of [36], it follows that A generates an analytic semigroup. Since $-A$ is positive, $(-A)^{\frac{1}{2}}$ is also positive and

$$D((-A)^{\frac{1}{2}}) = \{f \in H^2(\Omega) : f|_{\partial\Omega} = 0, \ \frac{\partial f}{\partial n}|_{\partial\Omega} = 0\}.$$

The norm of $D((-A)^{\frac{1}{2}})$ is given by

$$\|f\|^2_{D((-A)^{\frac{1}{2}})} = \int_\Omega |\Delta f|^2 dx.$$

Throughout the paper, we assume that $z_0 \in D((-A)^{\frac{1}{2}})$. We can rewrite the system (22) subject to (23) or (24) as the evolution equation:

$$\begin{cases} \dfrac{d}{dt}z(\cdot,t) = Az(\cdot,t) + F(z(\cdot,t)), \\ z(\cdot,0) = z_0(\cdot) \end{cases} \tag{26}$$

subject to

$$\begin{aligned} F(z(\cdot,t)) &= -z(x,t)z_{x_1}(x,t) - (1-\kappa)z_{x_1x_1}(x,t) \\ &\quad +\kappa z_{x_2x_2}(x,t) - \mu\sum_{j=1}^{N} \chi_j(x)[z(x,t) - f_j(x,t)]. \end{aligned}$$

Note that the nonlinear term F is locally Lipschitz continuous, that is, there exists a positive constant $l(C)$ such that the following inequality holds:

$$\|F(z_1) - F(z_2)\|_{L^2(\Omega)} \leq l(C)\|z_1 - z_2\|_{D((-A)^{\frac{1}{2}})}$$

for any $z_1, \ z_2 \in D((-A)^{\frac{1}{2}})$ with $\|z_1\|_{D((-A)^{\frac{1}{2}})} \leq C$, $\|z_2\|_{D((-A)^{\frac{1}{2}})} \leq C$. Here we prove the nonlinear term F is locally Lipschitz continuous for the case of point measurements. From the expression of $F(z(\cdot,t))$, using Minkowskii's inequality we have

$$\begin{aligned} \|F(z_1) - F(z_2)\|_{L^2(\Omega)} &\leq \|z_1z_{1x_1} - z_2z_{2x_1}\|_{L^2(\Omega)} \\ &\quad +|1-\kappa| \cdot \|z_{1x_1x_1} - z_{2x_1x_1}\|_{L^2(\Omega)} \\ &\quad +|\kappa| \cdot \|z_{1x_2x_2} - z_{2x_2x_2}\|_{L^2(\Omega)} + \mu\|z_1 - z_2\|_{L^2(\Omega)} \\ &\quad +\mu\sum_{j=1}^{N} \| \textstyle\int_{\bar{x}_j}^{x} z_{1\xi}(\xi,t) - z_{2\xi}(\xi,t)d\xi\|_{L^2(\Omega_j)}. \end{aligned} \tag{$*$}$$

Note that $D((-A)^{\frac{1}{2}}) = \{f \in H^2(\Omega) : f|_{\partial\Omega} = 0, \ \dfrac{\partial f}{\partial n}|_{\partial\Omega} = 0\}$. Since $z_1, \ z_2 \in D((-A)^{\frac{1}{2}})$ with $\|z_1\|_{D((-A)^{\frac{1}{2}})} \leq C$, $\|z_2\|_{D((-A)^{\frac{1}{2}})} \leq C$, we obtain that $z_1 - z_2 \in D((-A)^{\frac{1}{2}})$, and there exist some positive constants M_1 and M_2 such that

$$\begin{cases} \|z_1\|_{L^2(\Omega)} \leq M_1\left[\|z_{1x_1}\|_{L^2(\Omega)} + \|z_{1x_2}\|_{L^2(\Omega)}\right], \\ \|z_{1x_1}\|_{L^2(\Omega)} + \|z_{1x_2}\|_{L^2(\Omega)} \leq M_2\|z_1\|_{D((-A)^{\frac{1}{2}})}, \\ \|z_2\|_{L^2(\Omega)} \leq M_1\left[\|z_{2x_1}\|_{L^2(\Omega)} + \|z_{2x_2}\|_{L^2(\Omega)}\right], \\ \|z_{2x_1}\|_{L^2(\Omega)} + \|z_{2x_2}\|_{L^2(\Omega)} \leq M_2\|z_2\|_{D((-A)^{\frac{1}{2}})}, \\ \|z_1 - z_2\|_{L^2(\Omega)} \leq M_1\left[\|z_{1x_1} - z_{2x_1}\|_{L^2(\Omega)} + \|z_{1x_2} - z_{2x_2}\|_{L^2(\Omega)}\right], \\ \|z_{1x_1} - z_{2x_1}\|_{L^2(\Omega)} + \|z_{1x_2} - z_{2x_2}\|_{L^2(\Omega)} \leq M_2\|z_1 - z_2\|_{D((-A)^{\frac{1}{2}})}. \end{cases}$$

Hence, the following holds:

$$\begin{aligned} &\|z_1z_{1x_1} - z_2z_{2x_1}\|_{L^2(\Omega)} \\ &= \|z_1z_{1x_1} - z_2z_{1x_1} + z_2z_{1x_1} - z_2z_{2x_1}\|_{L^2(\Omega)} \\ &\leq \|z_1 - z_2\|_{L^2(\Omega)}\|z_{1x_1}\|_{L^2(\Omega)} + \|z_2\|_{L^2(\Omega)}\|z_{1x_1} - z_{2x_1}\|_{L^2(\Omega)} \\ &\leq M_1M_2\|z_1 - z_2\|_{D((-A)^{\frac{1}{2}}}M_2\|z_1\|_{D((-A)^{\frac{1}{2}}} \\ &\quad +M_1M_2\|z_2\|_{D((-A)^{\frac{1}{2}}}M_2\|z_1 - z_2\|_{D((-A)^{\frac{1}{2}}} \\ &\leq 2M_1M_2^2C\|z_1 - z_2\|_{D((-A)^{\frac{1}{2}}}, \end{aligned}$$

$$\begin{aligned} &|1-\kappa| \cdot \|z_{1x_1x_1} - z_{2x_1x_1}\|_{L^2(\Omega)} + |\kappa| \cdot \|z_{1x_2x_2} - z_{2x_2x_2}\|_{L^2(\Omega)} \\ &\leq \max\{|1-\kappa|, |\kappa|\}\|z_1 - z_2\|_{D((-A)^{\frac{1}{2}}}, \end{aligned}$$

$$\mu\|z_1 - z_2\|_{L^2(\Omega)} \leq \mu M_1M_2\|z_1 - z_2\|_{D((-A)^{\frac{1}{2}}}.$$

Moreover, Lemma 3 implies that there exist some positive constants M_3, M_4 and M_5 such that

$$\begin{aligned} &\mu\sum_{j=1}^{N} \| \textstyle\int_{\bar{x}_j}^{x} z_{1\xi}(\xi,t) - z_{2\xi}(\xi,t)d\xi\|_{L^2(\Omega_j)} \\ &\leq \mu\left[M_3\|z_{1x_1} - z_{2x_1}\|_{L^2(\Omega)} + M_4\|z_{1x_2} - z_{2x_2}\|_{L^2(\Omega)}\right. \\ &\qquad \left.+M_5\|z_{1x_1x_2} - z_{2x_1x_2}\|_{L^2(\Omega)}\right] \\ &\leq \mu M_3M_2\|z_1 - z_2\|_{D((-A)^{\frac{1}{2}}} + \mu M_4M_2\|z_1 - z_2\|_{D((-A)^{\frac{1}{2}}} \\ &\quad +\mu M_5\|z_1 - z_2\|_{D((-A)^{\frac{1}{2}}}. \end{aligned}$$

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Substitution of the above inequalities into the right-hand side of (*) yields

$$\|F(z_1) - F(z_2)\|_{L^2(\Omega)} \leq l(C)\|z_1 - z_2\|_{D(-A)^{\frac{1}{2}}},$$

where

$$\begin{aligned} l(C) &= 2M_1M_2^2C + \max\{|1-\kappa|, |\kappa|\} \\ &+\mu(M_1M_2 + M_3M_2 + M_4M_2 + M_5). \end{aligned}$$

From Theorem 6.3.1 of [28], it follows that the system (22) subject to (23) or (24) has a unique local classical solution $z \in C([0,T), L^2(\Omega)) \cap C^1((0,T), L^2(\Omega))$ for any initial function $z_0 \in D((-A)^{\frac{1}{2}})$.

Remark 3. *The above mentioned well-posedness result is dependent on the initial condition $z_0 \in D((-A)^{\frac{1}{2}})$. If the initial function $z_0 \in L^2(\Omega)$, the solution of the system (22) subject to (23) or (24) may become a mild solution or weak solution.*

A. Distributed controller under averaged measurements

Proposition 1. *Consider the closed-loop system (22) subject to (23). Given positive scalars $\bar{\Delta}$, if there exist $\delta > 0$, $\mu > 0$ and $\lambda_i \geq 0$ $(i = 1, 2)$ such that the following LMI holds:*

$$\Upsilon \triangleq \begin{bmatrix} -2\mu + 2\delta - \lambda_1\dfrac{\pi^2}{2} & \Upsilon_{12} & \mu \\ * & -2 & 0 \\ * & * & -\lambda_2 \end{bmatrix} \leq 0 \tag{27}$$

where

$$\Upsilon_{12} = -\frac{\lambda_1}{2} - \lambda_2\frac{\bar{\Delta}^2}{\pi^2} - (1-\kappa),$$

then the closed-loop system is globally exponentially stable in the L^2-sense:

$$\int_\Omega z^2(x,t)dx \leq e^{-2\delta t}\int_\Omega z^2(x,0)dx, \ \forall t \geq 0. \tag{28}$$

Furthermore, if the strict LMI (27) is feasible for $\delta = 0$, then the closed-loop system is exponentially stable with a small enough decay rate.

Proof. The proof is divided into three parts.
Step 1: We have shown that there exists a local clasical solution to (22) subject to (23), where $T = T(z_0)$. By Theorem 6.23.5 of [27], we obtain that the solution exists for any $T > 0$ if this solution admits a priori estimate. In Step 3, it will be shown that the feasibility of LMI (27) guarantees that the solution of (22) subject to (23) admits a priori bound, which can further guarantee the existence of the solution for all $t \geq 0$.

Step 2: Assume formally that there exists a classical solution of (22) subject to (23) for all $t \geq 0$. We consider the following Lyapunov-Krasovskii functional:

$$V(t) = \|z(\cdot,t)\|^2_{L^2(\Omega)}. \tag{29}$$

Since $z|_{\partial\Omega} = 0$ and $\dfrac{\partial z}{\partial n}|_{\partial\Omega} = 0$, integration by parts leads to

$$\int_\Omega z^2 z_{x_1}dx = 0, \tag{30}$$

$$\int_\Omega z^2_{x_1x_2}dx = \int_\Omega z_{x_1x_1}z_{x_2x_2}dx. \tag{31}$$

Furthermore, from (31) it follows that

$$\begin{aligned} &-2\int_\Omega [z^2_{x_1x_1} + z^2_{x_2x_2} + 2z^2_{x_1x_2}]dx \\ &= -2\int_\Omega [z_{x_1x_1} + z_{x_2x_2}]^2dx = -2\int_\Omega |\Delta z|^2dx. \end{aligned} \tag{32}$$

Differentiating (29) along (22), integrating by part and using (30), (32) we obtain

$$\begin{aligned} &\dot{V}(t) + 2\delta V(t) = 2\int_\Omega zz_tdx + 2\delta\int_\Omega z^2dx \\ &= 2(1-\kappa)\int_\Omega z^2_{x_1}dx - 2\kappa\int_\Omega z^2_{x_2}dx - 2\int_\Omega |\Delta z|^2dx \\ &-(2\mu - 2\delta)\int_\Omega z^2dx + 2\mu\sum_{j=1}^N\int_{\Omega_j} zf_jdx. \end{aligned} \tag{33}$$

From Lemma 1, the Wirtinger's inequality yields

$$\lambda_1\left[\int_\Omega \nabla z^T\nabla zdx - \frac{\pi^2}{2}\int_\Omega z^2dx\right] \geq 0, \tag{34}$$

where $\lambda_1 \geq 0$.
For the case of averaged measurements, f_j is given by (23). Since $\int_{\Omega_j} f_j(x,t)dx = 0$, from Lemma 1, the Poincaré inequality leads to

$$\int_{\Omega_j} f_j^2dx \leq \frac{2\bar{\Delta}^2}{\pi^2}\int_{\Omega_j} \nabla z^T\nabla zdx.$$

Hence,

$$\lambda_2\sum_{j=1}^N\left[\frac{2\bar{\Delta}^2}{\pi^2}\int_{\Omega_j}\nabla z^T\nabla zdx - \int_{\Omega_j} f_j^2dx\right] \geq 0, \tag{35}$$

where $\lambda_2 \geq 0$.
Integration by parts yields

$$-\int_\Omega \nabla z^T\nabla zdx = \int_\Omega z\Delta zdx. \tag{36}$$

Applying S-procedure [31], we add to $\dot{V}(t) + 2\delta V(t)$ the left-hand side of (34), (35) and use (36). Then it follows that

$$\begin{aligned} &\dot{V}(t) + 2\delta V(t) \\ &\leq \dot{V}(t) + 2\delta V(t) + \lambda_1\left[\int_\Omega \nabla z^T\nabla zdx - \frac{\pi^2}{2}\int_\Omega z^2dx\right] \\ &+\lambda_2\left[\frac{2\bar{\Delta}^2}{\pi^2}\int_\Omega \nabla z^T\nabla zdx - \sum_{j=1}^N\int_{\Omega_j} f_j^2dx\right] \\ &\leq 2(1-\kappa)\int_\Omega z^2_{x_1}dx - 2\kappa\int_\Omega z^2_{x_2}dx - 2\int_\Omega |\Delta z|^2dx \\ &-(2\mu - 2\delta + \lambda_1\frac{\pi^2}{2})\int_\Omega z^2dx + 2\mu\sum_{j=1}^N\int_{\Omega_j} zf_jdx \\ &-(\lambda_1 + \lambda_2\frac{2\bar{\Delta}^2}{\pi^2})\int_\Omega z\Delta zdx - \lambda_2\sum_{j=1}^N\int_{\Omega_j} f_j^2dx. \end{aligned}$$

Note that

$$\begin{aligned} &2(1-\kappa)\int_\Omega z^2_{x_1}dx - 2\kappa\int_\Omega z^2_{x_2}dx \\ &= 2(1-\kappa)\int_\Omega \nabla z^T\nabla zdx - 2\int_\Omega z^2_{x_2}dx \\ &\leq 2(1-\kappa)\int_\Omega \nabla z^T\nabla zdx. \end{aligned} \tag{37}$$

Then substitution of (36) into (37) yields

$$\begin{aligned} &2(1-\kappa)\int_\Omega z^2_{x_1}dx - 2\kappa\int_\Omega z^2_{x_2}dx \\ &\leq -2(1-\kappa)\int_\Omega z\Delta zdx. \end{aligned} \tag{38}$$

Hence,

$$\dot{V}(t) + 2\delta V(t) \leq \sum_{j=1}^N\int_{\Omega_j}\begin{bmatrix} z & \Delta z & f_j \end{bmatrix}\Upsilon\begin{bmatrix} z \\ \Delta z \\ f_j \end{bmatrix}dx \leq 0$$

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if $\Upsilon \le 0$ holds. Therefore,

$$V(t) \le e^{-2\delta t} V(0),\ \forall t \ge 0.$$

Note that the feasibility of the strict LMI (27) with $\delta = 0$ implies its feasibility with a small enough $\delta_0 > 0$. Therefore, if the strict LMI (27) holds for $\delta = 0$, then the closed-loop system is exponentially stable with a small decay rate $\delta_0 > 0$.

Step 3: The feasibility of LMI (27) yields that the solution of (22) subject to (23) admits a priori estimate $V(t) \le e^{-2\delta t} V(0)$. By Theorem 6.23.5 of [27], continuation of this solution under a priori bound to entire interval $[0, \infty)$. □

B. Distributed controller under point measurements

Proposition 2. *Consider the closed-loop system (22) subject to (24). Given positive scalars $\bar\Delta$, if there exist $\delta > 0$, $\mu > 0$, $\eta > 0$, $\lambda_1 \ge 0$, $\lambda_2 \in \mathbb{R}$ and $\beta_i > 0$ $(i = 1, 2, 3)$ such that (10) is satisfied and the following LMIs hold:*

$$2(1-\kappa) + \beta_1 \left(\frac{\bar\Delta}{\pi}\right)^2 + \lambda_1 - \lambda_2 \le 0, \tag{39}$$

$$-2\kappa + \beta_2 \left(\frac{\bar\Delta}{\pi}\right)^2 + \lambda_1 - \lambda_2 \le 0, \tag{40}$$

$$\Lambda = \begin{bmatrix} -2\mu + 2\delta - \lambda_1 \frac{\pi^2}{2} & -\frac{\lambda_2}{2} & -\frac{\lambda_2}{2} & \mu \\ * & -2 & -2 + \frac{\beta_3}{2}\left(\frac{\bar\Delta}{\pi}\right)^4 & 0 \\ * & * & -2 & 0 \\ * & * & * & -\eta \end{bmatrix} \le 0, \tag{41}$$

the closed-loop system is globally exponentially stable satisfying (28). Furthermore, if the strict LMI (41) is feasible for $\delta = 0$, then the closed-loop system is exponentially stable with a small enough decay rate.

Proof. Step 1: We have shown that there exists a local clasical solution to (22) subject to (24), where $T = T(z_0)$. By Theorem 6.23.5 of [27], we obtain that the solution exists for any $T > 0$ if this solution admits a priori estimate. In Step 3, it will be shown that the feasibility of LMIs (39)-(41) guarantees that the solution of (22) subject to (24) admits a priori bound, which can further guarantee the existence of the solution for all $t \ge 0$.

Step 2: Assume formally that there exists a classical solution of (22) subject to (24) for all $t \ge 0$. Consider V given by (29). Differentiating V along (22) and integrating by parts, we have (33).

For the case of point measurements, f_j is given by (24). From Lemma 3, we have

$$\begin{aligned}\eta\|f_j\|^2_{L^2(\Omega_j)} &\le \beta_1 \left(\tfrac{\bar\Delta}{\pi}\right)^2 \|z_{x_1}\|^2_{L^2(\Omega_j)} + \beta_2 \left(\tfrac{\bar\Delta}{\pi}\right)^2 \|z_{x_2}\|^2_{L^2(\Omega_j)} \\ &\quad + \beta_3 \left(\tfrac{\bar\Delta}{\pi}\right)^4 \|z_{x_1x_2}\|^2_{L^2(\Omega_j)}\end{aligned}$$

for any scalars β_1, β_2, β_3 such that (10) holds.

Hence,

$$\begin{aligned}\sum_{j=1}^{N} &\left[\beta_1 \left(\tfrac{\bar\Delta}{\pi}\right)^2 \|z_{x_1}\|^2_{L^2(\Omega_j)} + \beta_2 \left(\tfrac{\bar\Delta}{\pi}\right)^2 \|z_{x_2}\|^2_{L^2(\Omega_j)} \right. \\ &\left. + \beta_3 \left(\tfrac{\bar\Delta}{\pi}\right)^4 \|z_{x_1x_2}\|^2_{L^2(\Omega_j)} - \eta\|f_j\|^2_{L^2(\Omega_j)}\right] \ge 0.\end{aligned} \tag{42}$$

From (36), for any $\lambda_2 \in \mathbb{R}$ we have

$$\lambda_2 \left[\int_\Omega \nabla z^T \nabla z dx + \int_\Omega z \Delta z dx\right] = 0. \tag{43}$$

Similarly, we add to $\dot V(t) + 2\delta V(t)$ the left-hand side of (34) and (42). Then by taking into account (33), we obtain

$$\begin{aligned}&\dot V(t) + 2\delta V(t) \\ &\le \dot V(t) + 2\delta V(t) + \lambda_1 \left[\textstyle\int_\Omega \nabla z^T \nabla z dx - \frac{\pi^2}{2} \int_\Omega z^2 dx\right] \\ &+ \lambda_2 \left[-\textstyle\int_\Omega \nabla z^T \nabla z dx - \int_\Omega z \Delta z dx\right] \\ &+ \left[\beta_1 \left(\tfrac{\bar\Delta}{\pi}\right)^2 \|z_{x_1}\|^2_{L^2(\Omega_j)} + \beta_2 \left(\tfrac{\bar\Delta}{\pi}\right)^2 \|z_{x_2}\|^2_{L^2(\Omega_j)} \right. \\ &\left. + \beta_3 \left(\tfrac{\bar\Delta}{\pi}\right)^4 \|z_{x_1x_2}\|^2_{L^2(\Omega_j)} - \eta\|f_j\|^2_{L^2(\Omega_j)}\right] \\ &\le \left[2(1-\kappa) + \beta_1 \left(\tfrac{\bar\Delta}{\pi}\right)^2 + \lambda_1 - \lambda_2\right] \textstyle\int_\Omega z_{x_1}^2 dx \\ &+ \left[-2\kappa + \beta_2 \left(\tfrac{\bar\Delta}{\pi}\right)^2 + \lambda_1 - \lambda_2\right] \textstyle\int_\Omega z_{x_2}^2 dx \\ &+ \beta_3 \left(\tfrac{\bar\Delta}{\pi}\right)^4 \textstyle\int_\Omega z_{x_1x_2}^2 dx - 2\int_\Omega |\Delta z|^2 dx \\ &- (2\mu - 2\delta + \lambda_1 \tfrac{\pi^2}{2}) \textstyle\int_\Omega z^2 dx + 2\mu \sum_{j=1}^{N} \int_{\Omega_j} z f_j dx \\ &- \lambda_2 \textstyle\int_\Omega z \Delta z dx - \eta \sum_{j=1}^{N} \int_{\Omega_j} f_j^2 dx.\end{aligned}$$

By using (31) and (32), the following inequality holds for all $t \ge 0$

$$\dot V(t) + 2\delta V(t) \le \sum_{j=1}^{N} \int_{\Omega_j} \psi^T(x,t) \Lambda \psi(x,t) dx \tag{44}$$

where

$$\psi(x,t) = \mathrm{col}\{z, z_{x_1x_1}, z_{x_2x_2}, f_j\}.$$

Therefore, the LMIs (39)-(41) yield (28).

Step 3: The feasibility of LMIs (39)-(41) yields that the solution of (22) subject to (24) admits a priori estimate $V(t) \le e^{-2\delta t} V(0)$. By Theorem 6.23.5 of [27], continuation of this solution under a priori bound to entire interval $[0, \infty)$. □

IV. Sampled-data regional stabilization

A. Sampled-data control under averaged measurements

For $j = 1, \cdots, N$; $k = 0, 1, \cdots$ we consider the quantities

$$f_j(x,t) = z(x,t) - \frac{\int_{\Omega_j} z(\zeta,t) d\zeta}{|\Omega_j|}, \tag{45}$$

$$g_j(t) = \frac{1}{t - t_k} \frac{\int_{\Omega_j} \int_{t_k}^{t} z_s(\zeta,s) ds d\zeta}{|\Omega_j|}. \tag{46}$$

Then the controller (5) subject to (3) leads to the closed-loop system

$$\begin{cases} z_t + z z_{x_1} + (1-\kappa) z_{x_1x_1} - \kappa z_{x_2x_2} + \Delta^2 z \\ = -\mu \sum_{j=1}^{N} \chi_j(x) [z - f_j - (t - t_k) g_j], \\ \qquad\qquad\qquad (x,t) \in \Omega \times [t_k, t_{k+1}), \\ z|_{\partial\Omega} = 0,\ \frac{\partial z}{\partial n}|_{\partial\Omega} = 0. \end{cases} \tag{47}$$

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Now we use the step method (see e.g. [23], [25]) to establish the proof of the well-posedness for system (47). For $t \in [t_0, t_1]$, we consider the following equation:

$$\begin{cases} z_t + zz_{x_1} + (1-\kappa)z_{x_1x_1} - \kappa z_{x_2x_2} + \Delta^2 z \\ = -\mu \sum_{j=1}^{N} \chi_j(x) \frac{\int_{\Omega_j} z_0(x)dx}{|\Omega_j|}, \\ z|_{\partial\Omega} = 0, \ \frac{\partial z}{\partial n}|_{\partial\Omega} = 0. \end{cases} \quad (48)$$

Then system (48) can be represented as an evolution equation (26) subject to

$$\begin{aligned} F(z(\cdot,t)) &= -z(x,t)z_{x_1}(x,t) - (1-\kappa)z_{x_1x_1}(x,t) + \kappa z_{x_2x_2}(x,t) \\ &\quad -\mu \sum_{j=1}^{N} \chi_j(x) \frac{\int_{\Omega_j} z_0(x)dx}{|\Omega_j|}. \end{aligned}$$

Note that the nonlinearity $F(z(\cdot,t))$ is locally Lipschitz continuous. From Theorem 3.3.3 of [26], it follows that there exists a unique local strong solution $z(\cdot,t) \in C([0,T]; D((-A)^{\frac{1}{2}})) \cap C^1((0,T]; D(A))$ of (48) initialized with $z_0 \in D((-A)^{\frac{1}{2}})$ on some interval $[0,T] \subset [0,t_1]$, where $T = T(z_0) > 0$. By Theorem 6.23.5 of [27], we obtain that if this solution admits a priori estimate, then the solution exists on the entire $[0,t_1]$. The priori estimate on the solutions starting from the domain of attraction will be guaranteed by the stability conditions that we will provide (see Theorem 1). Then we apply the same line of reasoning step-by-step to the time segments $[t_1,t_2]$, $[t_2,t_3]$, $\cdots$. Following this procedure, we find that the strong solution exists for all $t \geq 0$.

In order to derive the stability conditions for (47) we employ the following Lyapunov-Krasovskii functional

$$\begin{aligned} V_1(t) &= p_1\|z\|^2_{L^2(\Omega)} + p_2\|\Delta z\|^2_{L^2(\Omega)} \\ &\quad + r(t_{k+1}-t)\int_\Omega \int_{t_k}^t e^{2\delta(s-t)} z_s^2(x,s)dsdx, \\ &\quad t \in [t_k, t_{k+1}), \ p_1 > 0, \ p_2 > 0, \ r > 0. \end{aligned} \quad (49)$$

Remark 4. *Note that without delay/sampling behavior, the energy norm is usually used. In the present work, due to the sampling terms, we need to use Lyapunov-Krasovskii functionals (see e.g. [13], [35]). Therefore, additionally to the energy norm $p_1\|z\|^2_{L^2(\Omega)} + p_2\|\Delta z\|^2_{L^2(\Omega)}$, we employ the term $r(t_{k+1}-t)\int_\Omega \int_{t_k}^t e^{2\delta(s-t)} z_s^2(x,s)dsdx$ to deal with the sampling.*

For convenience we define

$$V = D((-A)^{\frac{1}{2}})$$

with the norm

$$\|z\|^2_V = p_1\|z\|^2_{L^2(\Omega)} + p_2\|\Delta z\|^2_{L^2(\Omega)}.$$

Here p_1 and p_2 are positive constants that are related to the Lyapunov-Krasovskii functional (49). By using Lyapunov-Krasovskii functional (49), in Theorem 1 we provide LMI conditions for regional exponential stability of (47) and for a bound on the domain of attraction.

Theorem 1. *Consider the closed-loop system (47). Given positive scalars C, h, μ, $\bar{\Delta}$ and δ, let there exist scalars $r > 0$, $\Gamma > 0$, $p_1 > 0$, $p_2 > 0$, $\lambda_i \geq 0$ $(i=1,2)$ and $\lambda_3 \in \mathbb{R}$ satisfy the linear matrix inequalities:*

$$\Xi_i|_{z=C} < 0, \ \Xi_i|_{z=-C} < 0, \ i=1,2 \quad (50)$$

$$\begin{bmatrix} p_2 - (1+\Gamma)\frac{1}{\pi^2} & \sqrt{\frac{1}{2}} \\ * & \Gamma \end{bmatrix} > 0, \quad (51)$$

where

$$\Xi_1 = \left[\begin{array}{c:c} \Phi_1 & \begin{matrix} -rhz \\ 0 \\ -(1-\kappa)rh \\ \kappa rh \\ -rh \\ -\mu rh \\ \mu rh \end{matrix} \\ \hdashline * & -rh \end{array}\right], \quad (52)$$

$$\Xi_2 = \left[\begin{array}{c:c} \Phi_2 & \begin{matrix} -rhz \\ 0 \\ -(1-\kappa)rh \\ \kappa rh \\ -rh \\ -\mu rh \\ \mu rh \\ \mu rh^2 \end{matrix} \\ \hdashline * & -rh \end{array}\right], \quad (53)$$

$\Phi_1 = \{\phi_{ij}\}$ is a symmetric matrix composed from

$$\begin{aligned} &\phi_{11} = 2p_1(1-\kappa) + \lambda_1 + \frac{2\bar{\Delta}^2}{\pi^2}\lambda_2 - \lambda_3, \ \phi_{15} = -p_2 z, \\ &\phi_{22} = -2p_1\kappa + \lambda_1 + \frac{2\bar{\Delta}^2}{\pi^2}\lambda_2 - \lambda_3, \\ &\phi_{33} = -2p_1 + 2\delta p_2, \ \phi_{34} = 2\delta p_2, \\ &\phi_{35} = -p_2(1-\kappa), \phi_{36} = -\frac{\lambda_3}{2}, \\ &\phi_{44} = -2p_1 + 2\delta p_2, \ \phi_{45} = p_2\kappa, \ \phi_{46} = -\frac{\lambda_3}{2}, \\ &\phi_{55} = -2p_2, \ \phi_{56} = -p_2\mu, \ \phi_{57} = p_2\mu, \\ &\phi_{66} = -2p_1\mu + 2\delta p_1 - \frac{\pi^2}{2}\lambda_1, \ \phi_{67} = p_1\mu, \\ &\phi_{77} = -\lambda_2, \end{aligned}$$

$$\Phi_2 = \left[\begin{array}{c:c} \Phi_1 & * \\ \hdashline 0 \ 0 \ 0 \ 0 \ p_2\mu h \ p_1\mu h \ 0 & -rhe^{-2\delta h} \end{array}\right]. \quad (54)$$

Then for any initial state $z_0 \in V$ satisfying $\|z_0\|^2_V < C^2$, a unique solution of (47) exists and satisfies

$$\begin{aligned} &p_1\|z\|^2_{L^2(\Omega)} + p_2\|\Delta z\|^2_{L^2(\Omega)} \\ &\leq e^{-2\delta t}[p_1\|z_0\|^2_{L^2(\Omega)} + p_2\|\Delta z_0\|^2_{L^2(\Omega)}], \ ,t \geq 0. \end{aligned}$$

Furthermore, if the strict LMI (50) is feasible for $\delta = 0$, then the closed-loop system is exponentially stable with a small enough decay rate.

Proof. The proof is divided into three parts.
Step 1: We have shown that there exists a unique local strong solution to (47) on some interval $[0,T] \subset [0,t_1]$. By Theorem 6.23.5 of [27], we obtain that the solution exists on the entire

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interval $[0,t_1]$ if this solution admits a priori estimate. In Step 3, it will be shown that the feasibility of LMIs (50), (51) guarantees that the solution of (47) admits a priori bound, which can further guarantee the existence of the solution for all $t\geq 0$.

Step 2: Assume formally that there exists a solution of (47) for all $t\geq 0$. Differentiating V_1 along (47), we have

$$\begin{aligned}\dot V_1(t)+2\delta V_1(t) \quad &= 2p_1\int_\Omega z z_t dx+2p_2\int_\Omega \Delta z\Delta z_t dx\\ &-r\int_\Omega\int_{t_k}^t e^{2\delta(s-t)}z_s^2(x,s)dsdx\\ &+r(t_{k+1}-t)\int_\Omega z_t^2(x,t)dx\\ &+2\delta p_1\int_\Omega z^2dx+2\delta p_2\int_\Omega|\Delta z|^2dx.\end{aligned}\tag{55}$$

Substitution of z_t from (47) leads to

$$\begin{aligned}&r(t_{k+1}-t)\int_\Omega z_t^2(x,t)dx\\ &=r(t_{k+1}-t)\sum_{j=1}^N\int_{\Omega_j}[-zz_{x_1}-(1-\kappa)z_{x_1x_1}+\kappa z_{x_2x_2}\\ &-\Delta^2z-\mu z+\mu f_j+\mu(t-t_k)g_j]^2dx\\ &\leq rh\sum_{j=1}^N\int_{\Omega_j}[-zz_{x_1}-(1-\kappa)z_{x_1x_1}+\kappa z_{x_2x_2}-\Delta^2 z\\ &-\mu z+\mu f_j+\mu(t-t_k)g_j]^2dx.\end{aligned}\tag{56}$$

Integration by parts yields

$$\begin{aligned}&2p_1\int_\Omega zz_tdx\\ &=2p_1(1-\kappa)\int_\Omega z_{x_1}^2dx-2p_1\kappa\int_\Omega z_{x_2}^2dx-2p_1\int_\Omega|\Delta z|^2dx\\ &-2p_1\mu\int_\Omega z^2dx+2p_1\mu\sum_{j=1}^N\int_{\Omega_j}z[f_j+(t-t_k)g_j]dx,\end{aligned}\tag{57}$$

and

$$\begin{aligned}&2p_2\int_\Omega\Delta z\Delta z_tdx=2p_2\int_\Omega\Delta^2z\cdot z_tdx\\ &=2p_2\int_\Omega\Delta^2z[-zz_{x_1}-(1-\kappa)z_{x_1x_1}+\kappa z_{x_2x_2}-\Delta^2z]dx\\ &-2p_2\mu\int_\Omega\Delta^2z[z-f_j-(t-t_k)g_j]dx.\end{aligned}\tag{58}$$

The Jensen inequality leads to

$$\begin{aligned}&-r\int_\Omega\int_{t_k}^t e^{2\delta(s-t)}z_s^2(x,s)dsdx\\ &\leq -re^{-2\delta h}\int_\Omega\frac{1}{t-t_k}\left[\int_{t_k}^t z_s(x,s)ds\right]^2dx\\ &\leq -r(t-t_k)e^{-2\delta h}\sum_{j=1}^N\int_{\Omega_j}g_j^2dx.\end{aligned}\tag{59}$$

From (36), we obtain

$$\lambda_3\left[-\int_\Omega\nabla z^T\nabla zdx-\int_\Omega z\Delta zdx\right]=0,\tag{60}$$

where $\lambda_3\in\mathbb{R}$.
Set

$$\begin{aligned}&\eta_1=\mathrm{col}\{z_{x_1},z_{x_2},z_{x_1x_1},z_{x_2x_2},\Delta^2z,z,f_j\},\\ &\eta_2=\mathrm{col}\{\eta_1,g_j\},\\ &\beta\triangleq\begin{bmatrix}-z & 0 & -(1-\kappa) & \kappa & -1 & -\mu & \mu & \mu(t-t_k)\end{bmatrix}.\end{aligned}$$

Then

$$\begin{aligned}&[-zz_{x_1}-(1-\kappa)z_{x_1x_1}+\kappa z_{x_2x_2}-\Delta^2z-\mu z+\mu f_j\\ &+\mu(t-t_k)g_j]^2=\eta_2^T\beta^T\beta\eta_2.\end{aligned}\tag{61}$$

From (56) and (61) we have

$$r(t_{k+1}-t)\int_\Omega z_t^2(x,t)dx\leq rh\sum_{j=1}^N\int_{\Omega_j}\eta_2^T\beta^T\beta\eta_2dx.\tag{62}$$

Applying S-procedure, we add to $\dot V_1(t)+2\delta V_1(t)$ the left-hand side of (34), (35), (60). Then,

$$\begin{aligned}&\dot V_1(t)+2\delta V_1(t)\\ &\leq\dot V_1(t)+2\delta V_1(t)+\lambda_1\left[\int_\Omega\nabla z^T\nabla zdx-\frac{\pi^2}{2}\int_\Omega z^2dx\right]\\ &+\lambda_2\left[\frac{2\bar\Delta^2}{\pi^2}\int_\Omega\nabla z^T\nabla zdx-\sum_{j=1}^N\int_{\Omega_j}f_j^2dx\right]\\ &+\lambda_3\left[-\int_\Omega\nabla z^T\nabla zdx-\int_\Omega z\Delta zdx\right]\\ &\leq\sum_{j=1}^N\int_{\Omega_j}\frac{h-t+t_k}{h}\eta_1^T\Phi_1\eta_1+\frac{t-t_k}{h}\eta_2^T\Phi_2\eta_2dx\\ &+rh\sum_{j=1}^N\int_{\Omega_j}\eta_2^T\beta^T\beta\eta_2dx-(4p_1-4\delta p_2)\int_\Omega z_{x_1x_2}^2dx.\end{aligned}\tag{63}$$

Note that LMIs (50) imply that $\phi_{33}<0$, i.e. $p_1>\delta p_2$. As in [22], first we assume that

$$\|z(\cdot,t)\|_{C^0(\bar\Omega)}<C,\ \forall t\geq 0.\tag{64}$$

Note that $\frac{h-t+t_k}{h}+\frac{t-t_k}{h}=1$ and $t-t_k\leq h$. Under the assumption (64), applying Schur complement to (62), from (61)-(63) we obtain

$$\begin{aligned}&\dot V_1(t)+2\delta V_1(t)\\ &\leq\sum_{j=1}^N\int_{\Omega_j}\frac{h-t+t_k}{h}\begin{bmatrix}\eta_1^T & 1\end{bmatrix}\Xi_1\begin{bmatrix}\eta_1\\ 1\end{bmatrix}dx\\ &+\sum_{j=1}^N\int_{\Omega_j}\frac{t-t_k}{h}\begin{bmatrix}\eta_2^T & 1\end{bmatrix}\Xi_2\begin{bmatrix}\eta_2\\ 1\end{bmatrix}dx\\ &\leq 0\end{aligned}\tag{65}$$

if $\Xi_1<0$, $\Xi_2<0$ for all $z\in(-C,C)$.
Matrices Ξ_1 and Ξ_2 given by (52), (53) are affine in z. Hence, $\Xi_1<0$ and $\Xi_2<0$ for all $z\in(-C,C)$ if these inequalities hold in the vertices $z=\pm C$ hold, i.e. if LMIs (50) are feasible.

We prove next that (64) holds. Lemma 4 and Schur complement theorem lead to

$$\begin{aligned}&\|z\|^2_{C^0(\bar\Omega)}\leq\frac{1}{2}(1+\Gamma)\Big[\|z_{x_1}\|^2_{L^2(\Omega)}+\|z_{x_2}\|^2_{L^2(\Omega)}\Big]+\frac{1}{\Gamma}\|z_{x_1x_2}\|^2_{L^2(\Omega)}\\ &\leq\left[(1+\Gamma)\frac{1}{\pi^2}+\frac{1}{2\Gamma}\right]\|\Delta z\|^2_{L^2(\Omega)}\leq V_1(t).\end{aligned}\tag{66}$$

The last inequality in (66) follows from (51), and for the second inequality in (66) we use the Wirtinger's inequality, (31) and (32). Therefore, it is sufficient to show that

$$V_1(t)<C^2,\ \forall t\geq 0.\tag{67}$$

Indeed, for $t=0$, the inequality (67) holds. Let (67) be false for some t_1. Then $V_1(t_1)\geq C^2>V_1(0)$. Since V_1 is continuous in time, there must exist $t^*\in(0,t_1]$ such that

$$V_1(t)<C^2\ \forall t\in[0,t^*)\text{ and }V_1(t^*)=C^2.\tag{68}$$

The first relation of (68), together with the feasibility of (50), guarantees that $\dot V_1(t)+2\delta V_1(t)\leq 0$ on $[0,t^*)$. Therefore, $V_1(t)\leq V_1(0)<C^2$. This contradicts the second relation of (68). Thus, (67) and consequently, (65) is true, which implies provided that $\|z_0\|_V<C$.
Note that the feasibility of LMI (50) with $\delta=0$ implies its feasibility with a small enough $\delta_0>0$. Therefore, if LMI (50)

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holds for $\delta = 0$, then the closed-loop system is exponentially stable with a small decay rate.

Step 3: The feasibility of LMIs (50), (51) yields that the solution of (47) admits a priori estimate $V_1(t) \le e^{-2\delta t} V_1(0)$. By Theorem 6.23.5 of [27], this solution (under a priori bound) can be continued to entire interval $[0, \infty)$. □

B. Sampled-data control under point measurements

Under the controller (5) subject to (4), the closed-loop system becomes

$$\begin{cases} z_t + z z_{x_1} + (1-\kappa) z_{x_1 x_1} - \kappa z_{x_2 x_2} + \Delta^2 z \\ = -\mu \sum\limits_{j=1}^{N} \chi_j(x) z(\bar{x}_j, t_k),\ (x,t) \in \Omega \times [t_k, t_{k+1}), \\ z|_{\partial\Omega} = 0,\ \dfrac{\partial z}{\partial n}|_{\partial\Omega} = 0. \end{cases} \tag{69}$$

Theorem 2. *Consider the closed-loop system (69). Given positive scalars C, h, μ, $\bar{\Delta}$ and $\delta_1 < 2\delta$, let there exist scalars $r > 0$, $\Gamma > 0$, $p_1 > 0$, $p_2 > 0$, $\eta > 0$, $\lambda_i \ge 0$ $(i = 1, 2)$, $\lambda_3 \in \mathbb{R}$ and $\beta_i > 0$ $(i = 1, 2, 3)$ such that (10) is satisfied and the following LMIs hold:*

$$-2\delta_1 p_2 + \beta_3 \left(\frac{\bar{\Delta}}{\pi}\right)^4 \le 0, \tag{70}$$

$$\bar{\Theta} = \begin{bmatrix} -\delta_1 p_2 & -\frac{\beta_1}{2}\left(\frac{\bar{\Delta}}{\pi}\right)^2 & -\frac{\beta_2}{2}\left(\frac{\bar{\Delta}}{\pi}\right)^2 \\ * & -\delta_1 p_2 & 0 \\ * & * & -\delta_1 p_2 \end{bmatrix} \le 0, \tag{71}$$

$$\begin{bmatrix} p_2 - (1+\Gamma)\frac{1}{\pi^2} & \sqrt{\frac{1}{2}} \\ * & \Gamma \end{bmatrix} > 0, \tag{72}$$

$$\Lambda_i|_{z=C} < 0,\ \Lambda_i|_{z=-C} < 0,\ i = 1, 2 \tag{73}$$

where

$$\Lambda_1 = \left[\begin{array}{c:c} \Theta_0 & \Theta_1 \\ \hdashline * & -rh \end{array}\right],$$

$$\Lambda_2 = \left[\begin{array}{c:c} \Theta_2 & \Theta_3 \\ \hdashline * & -rh \end{array}\right],$$

$\Theta_0 = \{\theta_{ij}\}$ is a symmetric matrix composed from

$$\begin{aligned} &\theta_{11} = 2p_1(1-\kappa) + \lambda_1 - \lambda_2,\ \theta_{22} = -2p_1\kappa + \lambda_1 - \lambda_2, \\ &\theta_{33} = -2p_1 + 2\delta p_2,\ \theta_{35} = -p_2(1-\kappa),\ \theta_{36} = -\frac{\lambda_2}{2}, \\ &\theta_{44} = -2p_1 + 2\delta p_2,\ \theta_{45} = p_2\kappa,\ \theta_{46} = -\frac{\lambda_2}{2}, \\ &\theta_{55} = -2p_2,\ \theta_{56} = -p_2\mu,\ \theta_{57} = p_2\mu, \\ &\theta_{66} = -2p_1\mu + 2\delta p_1 - \frac{\pi^2}{2}\lambda_1,\ \theta_{67} = p_1\mu, \\ &\theta_{77} = -\eta, \end{aligned}$$

$$\Theta_1 = [\ -rhz \quad 0 \quad -(1-\kappa)rh \quad \kappa rh \quad -rh \quad -\mu rh \quad \mu rh\ ]^T \tag{74}$$

$$\Theta_2 = \left[\begin{array}{c:c} \Theta_0 & * \\ \hdashline 0\ \ 0\ \ 0\ \ 0\ \ p_2\mu h\ \ p_1\mu h\ \ 0 & -rhe^{-2\delta h} \end{array}\right], \tag{75}$$

$$\Theta_3 = [\ \Theta_1 \quad \mu r h^2\ ]^T \tag{76}$$

Then for any initial state $z_0 \in V$ satisfying $\|z_0\|_V^2 < C^2$, a unique solution of (69) exists and satisfies

$$\begin{aligned} &p_1\|z\|^2_{L^2(\Omega)} + p_2\|\Delta z\|^2_{L^2(\Omega)} \\ &\le e^{-2\sigma t}[p_1\|z_0\|^2_{L^2(\Omega)} + p_2\|\Delta z_0\|^2_{L^2(\Omega)}],\ t \ge 0, \end{aligned}$$

where σ is a unique positive solution of (18).

Proof. See Appendix. □

V. Numerical example

Consider the system (1) under the sampled-data control law (5) with the averaged measurements (3). Here we choose $\mu = 0.95$. By verifying LMI conditions of Theorem 1 with $\delta = 0.1$, $\kappa = -0.5$, $\bar{\Delta} = 1/4$, $C = 2$, we find that the closed-loop system (47) preserves the exponential stability within a given domain of initial conditions $\|z_0\|_V^2 < 4$ for $t_{k+1} - t_k \le h \le 0.39$. Note that $\bar{\Delta} = 1/4$ corresponds to $N = 16$ square subdomains with the sides length $1/4$. The feasible solutions of LMIs with $h = 0.35$ are given as follows: $p_1 = 80.6354$, $p_2 = 5.145$.

We compute the solution of the closed-loop system (47) numerically via finite element method. Let $\xi = 0.1$ and $M = 1/\xi = 10$. Define $(x_1^i, x_2^j) = (i\xi, j\xi)$, $i, j = 0, 1, 2, \ldots, M$. We divide Ω on $M^2 = 100$ squares R_{ij} defined by

$$R_{ij} \triangleq \{(x_1, x_2) \in \Omega | x_1^i \le x_1 \le x_1^{i+1}, x_2^j \le x_2 \le x_2^{j+1}\}.$$

On the node (x_1^i, x_2^j), four finite element basis functions are selected as

$$N_1^{i,j}(x) = \begin{cases} \left(1 - \dfrac{x_1 - x_1^i}{\xi}\right)\left(\dfrac{x_2 - x_2^j}{\xi}\right), & x \in R_{ij}, \\ 0, & otherwise \end{cases}$$

$$N_2^{i,j}(x) = \begin{cases} \left(1 - \dfrac{x_1 - x_1^i}{\xi}\right)\left(1 - \dfrac{x_2 - x_2^j}{\xi}\right), & x \in R_{ij}, \\ 0, & otherwise \end{cases}$$

$$N_3^{i,j}(x) = \begin{cases} \left(\dfrac{x_1 - x_1^i}{\xi}\right)\left(1 - \dfrac{x_2 - x_2^j}{\xi}\right), & x \in R_{ij}, \\ 0, & otherwise \end{cases}$$

$$N_4^{i,j}(x) = \begin{cases} \left(\dfrac{x_1 - x_1^i}{\xi}\right)\left(\dfrac{x_2 - x_2^j}{\xi}\right), & x \in R_{ij}, \\ 0, & otherwise \end{cases}$$

We consider the Galerkin approximation solution of the closed-loop system in finite dimensional space generated by these basis functions, which takes the form

$$z^M(x,t) = \sum_{k=1}^{4}\sum_{i,j=1}^{M} m_k^{i,j}(t) N_k^{i,j}(x),$$

where $m_k^{i,j}(t)$ are determined by standard finite element Galerkin method to satisfy some ODEs. Fig. 2 shows snapshots of the state $z(x,t) = z(x_1, x_2, t)$ at different times for the closed-loop system (47) with $t_{k+1} - t_k = 0.35$, $N = 16$ and initial condition $z(x_1, x_2, 0) = 0.236\sin(\pi x_1)\sin(\pi x_2)$, $(x_1, x_2) \in \Omega = (0,1)^2$. It is seen that the closed-loop system is stable. Fig. 3 demonstrates the time evolution of $V_1(t)$ via the finite difference method, where the steps of space and time are taken as $1/4$ and 0.00025, respectively.

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By verifying the LMI conditions of Theorem 1, we obtain the maximum value $h = 0.39$ that preserves the exponential stability. By simulation of the solution to the closed-loop system starting from the same initial condition, we find that stability is preserved for essentially larger values of h till approximately $h = 2.45$.

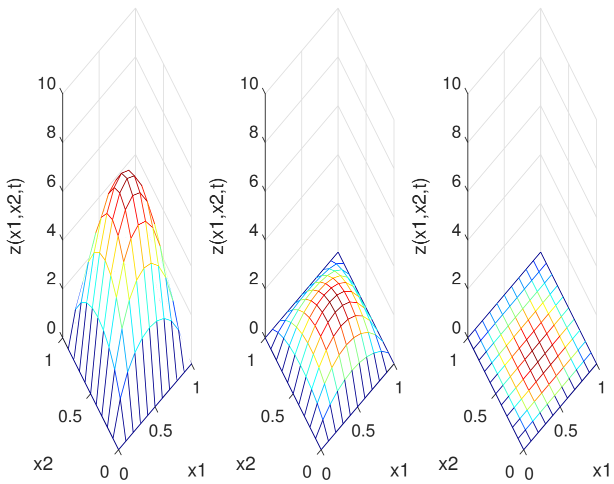

Fig. 2. Snapshots of state $z(x_1, x_2, t)$ at different time $t \in \{0, 1.4, 14\}$

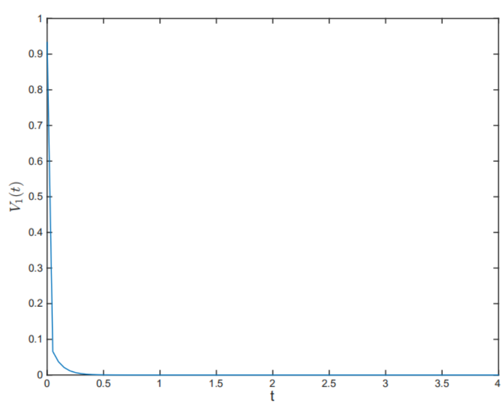

Fig. 3. Lyapunov function $V_1(t)$

For the sampled-data controller (5) under the point measurement (4), by choosing $\mu = 0.95$, and using Yalmip we verify LMI conditions of Theorem 2 with $\delta = 0.2$, $\delta_1 = 0.15$, $\kappa = -0.5$, $\bar{\Delta} = 1/4$, $C = 2$. Then we find that the resulting closed-loop system is exponentially stable for $t_{k+1} - t_k \leq h \leq 0.37$ for any initial values satisfying $\|z_0\|_{L^2(0,1)} < 1$.

Since point measurements use less information on the state, the point measurements allow smaller sampling intervals than averaged measurements. Simulations of the solutions to the closed-loop system under the point measurements confirm the theoretical results.

VI. Conclusion

The present paper discusses sampled-data control of 2D KSE under the spatially distributed averaged or point measurements. Sufficient LMI conditions have been investigated such that the regional stability of the closed-loop system is guaranteed.

Our results are applicable to sampled-data controller design of high dimensional distributed parameter systems. Our next step places its main focus on H_∞ filtering problem of high dimensional coupled ODE-PDE/PDE-PDE system.

Appendix

Proof of Theorem 2

Step 1: We have shown that there exists a unique local strong solution to (69) on some interval $[0, T] \subset [0, t_1]$. By Theorem 6.23.5 of [27], we obtain that the solution exists on the entire interval $[0, t_1]$ if this solution admits a priori estimate. In Step 3, it will be shown that the feasibility of LMIs (50), (51) guarantees that the solution of (69) admits a priori bound, which can further guarantees the existence of the solution for all $t \geq 0$.

Step 2: Assume formally that there exists a strong solution of (69) starting from $\|z_0\|_V < C$ for all $t \geq 0$. Differentiating V_1 along (69), we obtain the inequality (55). Denote

$$f_j(x,t) = z(x,t) - z(\bar{x}_j, t) = \int_{\bar{x}_j}^{x} z_\xi(\xi, t)d\xi, \tag{A.77}$$

From Lemma 3, we have

$$\begin{aligned}\eta\|f_j\|^2_{L^2(\Omega_j)} \leq & \beta_1\left(\frac{\bar{\Delta}}{\pi}\right)^2\|z_{x_1}\|^2_{L^2(\Omega_j)} + \beta_2\left(\frac{\bar{\Delta}}{\pi}\right)^2\|z_{x_2}\|^2_{L^2(\Omega_j)} \\ & + \beta_3\left(\frac{\bar{\Delta}}{\pi}\right)^4\|z_{x_1x_2}\|^2_{L^2(\Omega_j)}\end{aligned} \tag{A.78}$$

for any scalars β_1, β_2, β_3 such that (10) holds. Hence,

$$\begin{aligned}\sum_{j=1}^{N} & \left[\beta_1\left(\frac{\bar{\Delta}}{\pi}\right)^2\|z_{x_1}\|^2_{L^2(\Omega_j)} + \beta_2\left(\frac{\bar{\Delta}}{\pi}\right)^2\|z_{x_2}\|^2_{L^2(\Omega_j)}\right. \\ & \left. + \beta_3\left(\frac{\bar{\Delta}}{\pi}\right)^4\|z_{x_1x_2}\|^2_{L^2(\Omega_j)} - \eta\|f_j\|^2_{L^2(\Omega_j)}\right] \geq 0.\end{aligned} \tag{A.79}$$

Denote

$$\rho(x,t) \triangleq \frac{1}{t - t_k}\int_{t_k}^{t} z_s(x,s)ds.$$

Then, we have

$$z(x,t) = z(x,t_k) + (t - t_k)\rho(x,t).$$

By using Jensen's inequality, we have

$$\begin{aligned}& -r\int_\Omega \int_{t_k}^{t} e^{2\delta(s-t)} z_s^2(x,s)dsdx \\ & \leq -re^{-2\delta h}(t - t_k)\int_\Omega \rho^2(x,t)dx.\end{aligned} \tag{A.80}$$

Integration by parts leads to

$$\begin{aligned}& 2p_1\int_\Omega zz_t dx \\ & = 2p_1(1-\kappa)\int_\Omega z_{x_1}^2 dx - 2p_1\kappa\int_\Omega z_{x_2}^2 dx - 2p_1\int_\Omega |\Delta z|^2 dx \\ & + 2p_1\mu\sum_{j=1}^{N}\int_{\Omega_j} z\left[(t-t_k)\rho(x,t) + f_j(x,t_k)\right]dx \\ & - 2p_1\mu\int_\Omega z^2 dx,\end{aligned} \tag{A.81}$$

$$
\begin{aligned}
&2p_2 \int_\Omega \Delta z \Delta z_t dx = 2p_2 \int_\Omega \Delta^2 z \cdot z_t dx \\
&= 2p_2 \int_\Omega \Delta^2 z[-zz_{x_1} - (1-\kappa) z_{x_1x_1} + \kappa z_{x_2x_2} - \Delta^2 z] dx \\
&-2p_2\mu \sum_{j=1}^N \int_{\Omega_j} \Delta^2 z[z(x,t) - (t-t_k)\rho(x,t) - f_j(x,t_k)]dx.
\end{aligned} \tag{A.82}
$$

Then from (A.80)-(A.82), adding (A.79) into $\dot{V}_1 + 2\delta V_1$ we obtain

$$
\begin{aligned}
&\dot{V}_1(t) + 2\delta V_1(t) - \delta_1 \sup_{\theta\in[-h,0]} V_1(t+\theta) \\
&\le \dot{V}_1(t) + 2\delta V_1(t) - \delta_1 V_1(t_k) \\
&+\lambda_1 \left[\int_\Omega \nabla z^T \nabla z dx - \frac{\pi^2}{2}\int_\Omega z^2 dx\right] \\
&+\lambda_2 \left[-\int_\Omega \nabla z^T \nabla z dx - \int_\Omega z \Delta z dx\right] \\
&+ \Bigg[\beta_1 \left(\frac{\bar{\Delta}}{\pi}\right)^2 \int_\Omega z_{x_1}^2(x,t_k)dx + \beta_2 \left(\frac{\bar{\Delta}}{\pi}\right)^2 \int_\Omega z_{x_2}^2(x,t_k)dx \\
&+\beta_3 \left(\frac{\bar{\Delta}}{\pi}\right)^4 \int_\Omega z_{x_1x_2}^2(x,t_k)dx - \eta \sum_{j=1}^N \int_{\Omega_j} f_j^2(x,t_k)dx\Bigg] \\
&\le 2p_1(1-\kappa)\int_\Omega z_{x_1}^2 dx - 2p_1\kappa \int_\Omega z_{x_2}^2 dx - 2p_1 \int_\Omega |\Delta z|^2 dx \\
&+2p_1\mu \sum_{j=1}^N \int_{\Omega_j} z\left[(t-t_k)\rho + f_j(x,t_k)\right]dx - 2p_1\mu \int_\Omega z^2 dx \\
&+2p_2 \int_\Omega \Delta^2 z[-zz_{x_1} - (1-\kappa)z_{x_1x_1} + \kappa z_{x_2x_2} - \Delta^2 z]dx \\
&-2p_2\mu \int_\Omega \Delta^2 z[z - (t-t_k)\rho - f_j(x,t_k)d\xi]dx \\
&-re^{2\delta h}(t-t_k)\int_\Omega \rho^2(x,t)dx + r(t_{k+1}-t)\int_\Omega z_t^2(x,t)dx \\
&+2\delta p_1 \int_\Omega z^2 dx + 2\delta p_2 \int_\Omega |\Delta z|^2 dx - \delta_1 p_1 \int_\Omega z^2(x,t_k)dx \\
&-\delta_1 p_2 \int_\Omega |\Delta z(x,t_k)|^2 dx + (\lambda_1 - \lambda_2)\int_\Omega [z_{x_1}^2 + z_{x_2}^2]dx \\
&-\lambda_1 \frac{\pi^2}{2}\int_\Omega z^2 dx - \lambda_2 \int_\Omega z\Delta z dx + \beta_1 \left(\frac{\bar{\Delta}}{\pi}\right)^2 \int_\Omega z_{x_1}^2(x,t_k)dx \\
&+\beta_2 \left(\frac{\bar{\Delta}}{\pi}\right)^2 \int_\Omega z_{x_2}^2(x,t_k)dx + \beta_3 \left(\frac{\bar{\Delta}}{\pi}\right)^4 \int_\Omega z_{x_1x_2}^2(x,t_k)dx \\
&-\eta \sum_{j=1}^N \int_{\Omega_j} f_j^2(x,t_k)dx.
\end{aligned} \tag{A.83}
$$

Using (32), we have

$$
\begin{aligned}
&-2p_1 \int_\Omega |\Delta z|^2 dx \\
&= -2p_1 \int_\Omega [z_{x_1x_1}^2(x,t) + z_{x_2x_2}^2(x,t) + 2z_{x_1x_2}^2(x,t)]dx,
\end{aligned} \tag{A.84}
$$

$$
\begin{aligned}
&-\delta_1 p_2 \int_\Omega |\Delta z(x,t_k)|^2 dx \\
&= -\delta_1 p_2 \int_\Omega [z_{x_1x_1}^2(x,t_k) + z_{x_2x_2}^2(x,t_k) + 2z_{x_1x_2}^2(x,t_k)]dx.
\end{aligned} \tag{A.85}
$$

Set

$$
\begin{aligned}
\eta_0 = \;& \{z_{x_1}(x,t), z_{x_2}(x,t), z_{x_1x_1}(x,t), z_{x_2x_2}(x,t), \Delta^2 z(x,t), \\
& z(x,t), f_j(x,t_k)\}, \\
\eta_1 = \;& \{z_{x_1}(x,t), z_{x_2}(x,t), z_{x_1x_1}(x,t), z_{x_2x_2}(x,t), \Delta^2 z(x,t), \\
& z(x,t), f_j(x,t_k), \rho(x,t)\}, \\
\bar{\eta} = \;& \{z(x,t_k), z_{x_1x_1}(x,t_k), z_{x_2x_2}(x,t_k)\}
\end{aligned}
$$

Substituting (A.84), (A.85) into (A.83), we obtain

$$
\begin{aligned}
&\dot{V}_1(t) + 2\delta V_1(t) - \delta_1 \sup_{\theta\in[-h,0]} V_1(t+\theta) \\
&\le \sum_{j=1}^N \int_{\Omega_j} \frac{h-t+t_k}{h}\eta_0^T \Theta_0 \eta_0 + \frac{t-t_k}{h}\eta_1^T \Theta_1 \eta_1 dx \\
&+ \int_\Omega \bar{\eta}^T \bar{\Theta} \bar{\eta} dx + r(t_{k+1}-t)\int_\Omega z_t^2(x,t)dx \\
&-(4p_1 - 4\delta p_2)\int_\Omega z_{x_1x_2}^2(x,t)dx \\
&-[2\delta_1 p_2 - \beta_3(\tfrac{\bar{\Delta}}{\pi})^4]\int_\Omega z_{x_1x_2}^2(x,t_k)dx.
\end{aligned} \tag{A.86}
$$

Substitution of z_t from (69) yields

$$
\begin{aligned}
&r(t_{k+1}-t)\int_\Omega z_t^2(x,t)dx \\
&\le rh \sum_{j=1}^N \int_{\Omega_j} [-zz_{x_1} - (1-\kappa)z_{x_1x_1} + \kappa z_{x_2x_2} - \Delta^2 z \\
&-\mu z + \mu(t-t_k)\rho + \mu f_j(x,t_k)]^2 dx.
\end{aligned} \tag{A.87}
$$

Set $\bar{\psi} = \mathrm{col}\{z_{x_1}(x,t), z_{x_1x_1}(x,t), z_{x_2x_2}(x,t), \Delta^2 z(x,t), z(x,t), f_j(x,t_k), \rho(x,t)\}$ and

$$
\beta \triangleq \begin{bmatrix} -z & -(1-\kappa) & \kappa & -1 & -\mu & \mu & \mu(t-t_k) \end{bmatrix}.
$$

Then

$$
\begin{aligned}
&[-zz_{x_1} - (1-\kappa)z_{x_1x_1} + \kappa z_{x_2x_2} - \Delta^2 z - \mu z + \mu f_j(x,t_k) \\
&+\mu(t-t_k)\rho]^2 = \bar{\psi}^T \beta^T \beta \bar{\psi}.
\end{aligned} \tag{A.88}
$$

Application of Schur complement theorem to (A.88), together with (A.86) and (A.87), implies

$$
\dot{V}_1(t) + 2\delta V_1(t) - \delta_1 \sup_{\theta\in[-h,0]} V_1(t+\theta) \le 0
$$

if (70)-(72) are satisfied, and $\Lambda_1 < 0$, $\Lambda_2 < 0$ hold for all $z \in (-C, C)$. Similar to Theorem 1, LMI (73) imply $\Lambda_1 < 0$, $\Lambda_2 < 0$ for all $z \in (-C, C)$. Thus the result is established via Halanay's inequality.

Step 3: The feasibility of LMIs (70)-(73) yields that the solution of (69) admits a priori estimate $V_1(t) \le e^{-2\sigma t} \sup_{-h\le\theta\le 0} V_1(\theta)$. By Theorem 6.23.5 of [27], continuation of this solution under a priori bound to entire interval $[0, \infty)$.

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