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Constructive finite-dimensional boundary control of stochastic 1D parabolic PDEs ☆

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ABSTRACT

Recently, a constructive method for the finite-dimensional observer-based control of deterministic parabolic PDEs was suggested by employing a modal decomposition approach. In this paper, for the first time we extend this method to the stochastic 1D heat equation with nonlinear multiplicative noise. We consider the Neumann actuation and study the observer-based as well as the state-feedback controls via the modal decomposition approach. We employ either trigonometric or polynomial dynamic extension. For observer-based control we consider a noisy boundary measurement. First, we show the well-posedness of strong solutions to the closed-loop systems. Then by suggesting a direct Lyapunov method and employing Itô's formula, we provide mean-square L^2 exponential stability analysis of the full-order closed-loop system, leading to linear matrix inequality (LMI) conditions for finding the observer dimension and as large as possible noise intensity bound for the mean-square stabilizability. We prove that the LMIs are always feasible for small enough noise intensity and large enough observer dimension (for observer-based control). We further show that in the case of state-feedback and linear noise, the system is always stabilizable for noise intensities that guarantee the stabilizability of the stochastic finite-dimensional part of the closed-loop system with deterministic measurement. Numerical simulations are carried out to illustrate the efficiency of our method. For both state-feedback and observer-based controls, the trigonometric extension always allows for a larger noise than the polynomial one in the example.

1. Introduction

Stochastic PDEs are natural generalizations of PDEs and their theory has motivations coming from both mathematics and natural sciences: physics, chemistry, biology and mathematical finance (Da Prato & Zabczyk, 2014). In the application aspects, because of the inherent complexity of the underlying physical processing, many control systems in reality (such as that in the microelectronics industry, in the atmospheric motion, in communications and transportation, and so on) exhibit very complex dynamics, including substantial model uncertainty, actuator and state constraints, and high dimensionality (usually infinite). These systems are often best described by stochastic PDEs (Murray, 2003, P. 61). As stated in Lü and Zhang (2021, P. 5), control theory for stochastic PDEs is still at its very beginning stage and many tools and methods, which are effective in the deterministic case, do not work anymore in the stochastic setting. In Barbu (2018, Sec. 5.4), an infinite-dimensional internal state-feedback stabilizer was provided for stochastic parabolic PDEs with *linear* multiplicative noise, for small levels of noise and large enough gain. Inspired by Fridman and Blighovsky (2012), the control designs for stochastic PDEs with *linear* multiplicative noise by spatial decomposition have been reported (Kang, Wang, Wu, Li, & Liu, 2021; Wu & Zhang, 2020). However, spatial decomposition requires many sensors and actuators, covering the whole spatial domain.

In Duncan, Maslowski, and Pasik-Duncan (1994), adaptive boundary/point control of a linear stochastic PDE with additive noise was presented. In Liang and Wu (2022), a boundary state-feedback controller is designed for stochastic Korteweg–de Vries–Burgers equations with *linear* multiplicative noise, where the controller depends on the full information of the state. In Christofides, Armaou, Lou, and Varshney (2008) and Hu, Lou, and Christofides (2008), finite-dimensional state-feedback and output-feedback controllers for stochastic PDEs with additive

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noise under nonlocal actuation were designed by the modal decomposition approach. A singular perturbation approach that reduces the controller design to a finite-dimensional slow system was suggested, but constructive conditions for finding the dimension of the slow system that guarantees a desired closed-loop performance were not provided. In Munteanu (2018, 2019), Munteanu presented the first results on finite-dimensional boundary state-feedback stabilization for the stochastic heat equation with *nonlinear* multiplicative noise and stochastic Burgers equations with *linear* multiplicative noise, respectively, by using a fixed point argument, where the stability can be guaranteed no matter how large the level of the noise is. However, the results in Munteanu (2018, 2019) that employ modal decomposition are qualitative — for large enough number of modes the proposed controller stabilizes the system. Moreover, it is worth mentioning that the method in Munteanu (2018, 2019) requires full state knowledge and is nontrivial for only partial state knowledge (see the conclusions of Munteanu (2018, 2019)). Constructive methods for boundary or nonlocal control of systems with multiplicative noise that allows finding a bound on the number of modes (and on the observer dimension for the output-feedback case) with guaranteed performance are missing.

Finite-dimensional observers and the resulting controllers, are very attractive in applications compared to controllers that use PDE observers and need further approximation. For deterministic parabolic PDEs, recently, a constructive LMI-based method for finite-dimensional observer-based controller was introduced via modal decomposition (Katz & Fridman, 2020). A direct Lyapunov method was suggested resulting in simple LMI conditions for finding the observer dimension. In Katz and Fridman (2021) and Lhachemi and Prieur (2022), the method was extended to both unbounded operators by employing dynamic extension (Curtain & Zwart, 2012; Prieur & Trélat, 2019). Note that the above results are all focused on the linear PDEs since the nonlinearity may cause additional spillover behavior (Hagen & Mezic, 2003). In Katz and Fridman (2023), the state-feedback global stabilization of semilinear parabolic PDEs under nonlocal or Dirichlet actuation via modal decomposition approach was suggested, where the nonlinear terms are compensated by using Parseval's inequality. However, the corresponding results in Katz and Fridman (2020, 2021, 2023) and Lhachemi and Prieur (2022) cannot be extended to the stochastic case directly. The challenges for the stochastic PDEs are as follows: **(i)** The well-posedness and the regularity of solutions to the closed-loop stochastic PDE systems are essentially more challenging than in the deterministic case; **(ii)** Differently from the deterministic case, in the Lyapunov analysis, we cannot take *generator* (also called the differential operator associated with the considered stochastic equation (see Klebaner (2005, P.149) and Mao (2007, P.110))) term by term in the infinite sum since the mean-square L^2 convergence of the generators sum cannot be guaranteed. Instead, we present the Lyapunov function in the form of the one for the stochastic PDE and the other one for finite-dimensional stochastic ODEs and apply the generator to each part. Moreover, treatment of the nonlinear noise function σ_1 is challenging and is different from the treatment of nonlinearity in the deterministic case (see, e.g., Katz and Fridman (2023)) due to a quadratic term that appears in the expression for generator (see $\Sigma^{\mathrm{T}}(t)P\Sigma(t)$ in (2.49), such term does not appear in the deterministic setting); **(iii)** To prove the mean-square exponential stability, we employ corresponding Itô's formulas for stochastic ODEs and (strong solutions of) PDEs, respectively.

In this paper we aim to develop the constructive LMI-based design for stochastic parabolic PDEs. We suggest finite-dimensional observer-based and state-feedback controllers for the 1D stochastic heat equation with *nonlinear* multiplicative noise. We consider the Neumann actuation and noisy boundary measurement and study the mean-square L^2 exponential stability. We use the modal decomposition method via either trigonometric or polynomial dynamic extension. We also provide results for the linear multiplicative noise and show that for the state-feedback case, the system is always stabilizable for noise intensities that guarantee the stabilizability of the stochastic finite-dimensional part of the closed-loop system with deterministic measurement. The efficiency of the method is demonstrated by numerical simulations. For both state-feedback and observer-based controllers, the trigonometric extension always allows a larger noise than the polynomial one. The contribution of the present paper is listed as follows:

- Differently from the previous works on boundary control of stochastic PDEs that prove the well-posedness of *mild* solutions (see, e.g., Duncan et al. (1994) and Munteanu (2018, 2019)), in this paper, we apply the dynamic extension (inspired by Curtain and Zwart (2012), Karafyllis (2021) and Katz and Fridman (2021)) to get equivalent stochastic PDEs and show the well-posedness for *strong* solutions to the closed-loop systems. The latter allows us to *employ Itô's formula*.
- Differently from existing works on the finite-dimensional control of stochastic PDEs by a singular perturbation approach (Christofides et al., 2008; Hu et al., 2008) or a fixed point argument (Munteanu, 2018, 2019), we suggest for the first time a direct Lyapunov method for the mean-square L^2 exponential stabilization of stochastic parabolic PDEs with *nonlinear multiplicative noise* by finite-dimensional boundary control. Moreover, the results of Christofides et al. (2008) and Munteanu (2018, 2019) were confined to state-feedback case, whereas we present output-feedback design based on noisy boundary measurements.
- Compared with the qualitative results in Christofides et al. (2008), Hu et al. (2008) and Munteanu (2018, 2019), our method is constructive and quantitative (differently from perturbation-based approaches) with easily implementable and efficient LMI conditions for finding the number of modes of controller and observer and as large as possible noise intensity bound for the mean-square stabilizability. We prove that the derived LMIs are always feasible for small enough noise intensity and large enough number of controller and observer modes.

Preliminary results on observer-based control for deterministic boundary measurement via polynomial dynamic extension were reported in Wang, Katz, and Fridman (2022).

Notations: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of increasing sub σ-fields of $\mathcal{F}$ (see Da Prato and Zabczyk (2014, P. 71)) and let $\mathbb{E}\{\cdot\}$ be the expectation operator. For $f \in C([0,1])$, let $\|f\|_{[0,1]} = \max_{x\in[0,1]}|f(x)|$. Denote by $L^2(0,1)$ the space of square integrable functions with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\mathrm{d}x$ and induced norm $\|f\|_{L^2}^2 = \langle f, f\rangle$. Let $L^2(\Omega; L^2(0,1))$ be the set of all $\mathcal{F}_0$-measurable random variables $z \in L^2(0,1)$ with $\mathbb{E}\|z\|_{L^2}^2 < \infty$. $H^1(0,1)$ is the Sobolev space of functions $f : [0,1] \longrightarrow \mathbb{R}$ with a square integrable weak derivative. The norm defined in $H^1(0,1)$ is $\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|f'\|_{L^2}^2$. Let $\mathbb{N}$ denote the set of positive integers. The Euclidean norm is denoted by $|\cdot|$. For $P \in \mathbb{R}^{n\times n}$, $P > 0$ means that P is symmetric and positive definite. The symmetric elements of a symmetric matrix will be denoted by $*$. For $0 < P \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, we write $|x|_P^2 = x^{\mathrm{T}}Px$. For $A \in \mathbb{R}^{n\times n}$, let $\|A\|$ be the operator norm of A induced by $|\cdot|$. Let I denote the identity matrix of appropriate size.

Recall the Sturm–Liouville operator

$$\begin{aligned}&\mathcal{A}_1\phi = -\tfrac{\mathrm{d}}{\mathrm{d}x}(p(x)\tfrac{\mathrm{d}}{\mathrm{d}x}\phi(x)) + q(x)\phi(x),\\ &\mathcal{D}(\mathcal{A}_1) = \{\phi \in H^2(0,1) | \phi(0) = \phi'(1) = 0\},\end{aligned} \tag{1.1}$$

where $p \in C^2([0,1])$ and $q \in C^1([0,1])$ satisfy

$$0 < p_* \le p(x) \le p^*,\ 0 \le q(x) \le q^*,\ x \in [0,1]. \tag{1.2}$$

The Sturm–Liouville operator (1.1) has a sequence of eigenvalues $\lambda_1 < \cdots < \lambda_n < \ldots$ satisfying (see Orlov (2017))

$$\pi^2(n-1)^2 p_* \le \lambda_n \le \pi^2 n^2 p^* + q^*,\ n \ge 1, \tag{1.3}$$

with corresponding normalized eigenfunctions $\phi_n(x)$ $(n \ge 1)$ which form a complete orthonormal system in $L^2(0,1)$. Particularly, if $p(x) \equiv 1$ and $q(x) \equiv 0$, λ_n and ϕ_n are explicitly given by

$$\lambda_n = (n - \tfrac{1}{2})^2\pi^2,\ \phi_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}x),\ n \ge 1. \tag{1.4}$$

Given $N \in \mathbb{N}$ and $h \in L^2(0,1)$ satisfying $h \overset{L^2}{=} \sum_{n=1}^{\infty} h_n\phi_n$, we denote $\|h\|_N^2 = \sum_{n=N+1}^{\infty} h_n^2$. The following lemma will be used:

Lemma 1.1 (*Katz & Fridman, 2020, Lemma 2.1*)**.** *Let $h \in L^2(0,1)$ be given by $h \overset{L^2}{=} \sum_{n=1}^{\infty} h_n\phi_n$. Then $h \in H^1(0,1)$ with $h(0)=0$ iff $\sum_{n=1}^{\infty}\lambda_n h_n^2 < \infty$. Moreover, for $h \in \mathcal{D}(\mathcal{A}_1)$, we have*

$$\frac{\pi^2}{p^*\pi^2+4q^*}\sum_{n=1}^{\infty}\lambda_n h_n^2 \le \|h'\|_{L^2}^2 \le \frac{1}{p_*}\sum_{n=1}^{\infty}\lambda_n h_n^2.$$

2. Observer-based control

Consider the following stochastic 1D heat equation with nonlinear multiplicative noise under Neumann actuation:

$$\begin{aligned} &\mathrm{d}z(x,t) = [\tfrac{\partial}{\partial x}(p(x)\tfrac{\partial}{\partial x}z(x,t)) + (q_c - q(x))z(x,t)]\mathrm{d}t \\ &\qquad + \sigma_1(x,t,z(x,t))\mathrm{d}\mathcal{W}_1(t),\ t \ge 0, x \in [0,1], \\ &z(0,t) = 0,\ z_x(1,t) = u(t), \\ &z(x,0) = z_0(x), \end{aligned} \tag{2.1}$$

where $z_0 \in L^2(\Omega; L^2(0,1))$, $q_c \in \mathbb{R}$ is a constant reaction coefficient, $u(t)$ is a control input to be designed, $\mathcal{W}_1(t)$ is the 1D standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, the nonlinear noise function $\sigma_1 : [0,1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is assumed to satisfy

$$\sigma_1(x,t,0) = 0,\ |\sigma_1(x,t,z_1) - \sigma_1(x,t,z_2)| \le \bar{\sigma}_1|z_1 - z_2|, \tag{2.2}$$

for all $x \in [0,1]$, $t \in \mathbb{R}^+$, and $z_1, z_2 \in \mathbb{R}$, where $\bar{\sigma}_1 > 0$ is an upper bound on the noise intensity.

Remark 2.1. Differently from the Kalman filtering techniques developed in PDE setting (see, e.g., Falb (1967)) where the noise is independent of state (additive noise), in system (2.1) we studied the multiplicative noise which may appear due to the system parameters that undergo random perturbations of white noise process (Da Prato & Zabczyk, 2014; Mao, 2007). Specifically, one can think of system (2.1) as a stochastic version of the reaction–diffusion equations in Karafyllis (2021) and Katz and Fridman (2020), where the reaction term $(q_c - q(x))z(x,t)$ therein undergoes random perturbations and is replaced by $(q_c - q(x))z(x,t) + \sigma_1(x,t,z(x,t))\varsigma_1(t)$ (see, e.g. Haussmann (1978)). Here $\varsigma_1(t)$ is a white noise process which is *formally defined* as the derivative of the Brownian motion $\varsigma_1(t) = \frac{\mathrm{d}\mathcal{W}_1(t)}{\mathrm{d}t}$ (see Klebaner (2005, P.124)). In (2.1), we consider the white noise which is uniform in the spatial variable. Such white noise appears in many applications including filtering equations (see Da Prato and Zabczyk (2014, Sec. 13.8)) and Musiela's equation of the bond market (see Da Prato and Zabczyk (2014, Sec. 13.3)). We suggest nonlinear noise perturbation function $\sigma_1(x,t,z)$ to describe the distribution of noise with respect to space, time, and state. Similarly, we will consider the multiplicative measurement noise (see (2.3)).

In this paper we are interested in the strong solution to the closed-loop system (see Section 2.1.2) and the mean-square L^2 stability of (2.1) (see Definition 2.1). Note that the multiplicative noise always tends to destroy mean-square stability (see, e.g., Damm (2004, Remark 1.5.9), Munteanu (2018) and Wu and Zhang (2020)). Thus, we aim to study the mean-square exponential stabilization and find (as large as we can) noise intensity bound $\bar{\sigma}_1$ for the mean-square stabilizability.

We consider the following noisy boundary measurement output (see e.g., (Dragan, Morozan, & Stoica, 2006; Gershon, Shaked, & Yaesh, 2005)):

$$\mathrm{d}y(t) = z(1,t)\mathrm{d}t + \sigma_2(t, z(1,t))\mathrm{d}\mathcal{W}_2(t),\ t \ge 0, \tag{2.3}$$

where nonlinear noise function $\sigma_2 : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\sigma_2(t,0) = 0,\ |\sigma_2(t,z_1) - \sigma_2(t,z_2)| \le \bar{\sigma}_2|z_1 - z_2|, \tag{2.4}$$

for all $t \in \mathbb{R}^+$ and $z_1, z_2 \in \mathbb{R}$, and certain positive constant $\bar{\sigma}_2$, $\mathcal{W}_2(t)$ is a 1D standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Note that $\mathcal{W}_1(t)$ and $\mathcal{W}_2(t)$ are mutually independent.

The unboundedness of the control and observation operators leads to substantial technical difficulties for the well-posedness and the stability analysis of the closed-loop system. Most of the existing works are focused on the semigroup approach to the boundary control problem of stochastic PDEs, which can only guarantee the well-posedness for mild solutions (see, e.g., Duncan et al. (1994) and Munteanu (2018, 2019)). However, since the stochastic convolution is no longer a martingale, we cannot apply Itô's formula to mild solutions directly, which limits the Lyapunov stability analysis. In this section, we employ dynamic extension which is based on a change of variables to lift the control input from the boundary to the right hand side of the equivalent stochastic PDE system. This allows us to analyze the well-posedness of strong solutions to the closed-loop system and to employ Itô's formula directly. In this paper, we consider two types of dynamic extension: trigonometric (inspired by Karafyllis (2021)) and polynomial (inspired by Katz and Fridman (2021)).

2.1. Trigonometric dynamic extension

2.1.1. Controller design

Inspired by Karafyllis (2021), let $\mu > 0$ with $\mu \ne \lambda_n$ for $n \in \mathbb{N}$ be a given constant and consider a function $\psi \in C^2([0,1])$ that satisfies

$$\begin{aligned} &(p(x)\psi'(x))' - q(x)\psi(x) = -\mu\psi(x), \\ &\psi(0) = 0,\ \psi'(1) = 1. \end{aligned} \tag{2.5}$$

Since $\mu \ne \lambda_n$, it follows that the boundary-value problem (2.5) has a unique solution. In particular, if $p(x) \equiv 1$ and $q(x) \equiv 0$, we can choose $\mu = \pi^2$ and $\psi(x) = -\frac{1}{\pi}\sin(\pi x)$.

We consider the trigonometric change of variables

$$w(x,t) = z(x,t) - \psi(x)u(t) \tag{2.6}$$

to obtain the following system

$$\begin{aligned} \mathrm{d}w(x,t) &= [\tfrac{\partial}{\partial x}(p(x)\tfrac{\partial}{\partial x}w(x,t)) + (q_c - q(x))w(x,t)]\mathrm{d}t \\ &+ [(p(x)\psi'(x))' - q(x)\psi(x) + q_c\psi(x)]u(t)\mathrm{d}t \\ &- \psi(x)\mathrm{d}u(t) + \sigma_1(x,t,w(x,t) + \psi(x)u(t))\mathrm{d}\mathcal{W}_1(t) \\ &\overset{(2.5)}{=} [\tfrac{\partial}{\partial x}(p(x)\tfrac{\partial}{\partial x}w(x,t)) + (q_c - q(x))w(x,t)]\mathrm{d}t \\ &- \psi(x)[(\mu - q_c)u(t)\mathrm{d}t + \mathrm{d}u(t)] \\ &+ \sigma_1(x,t,w(x,t) + \psi(x)u(t))\mathrm{d}\mathcal{W}_1(t). \end{aligned} \tag{2.7}$$

We will henceforth treat $u(t)$ as an additional state variable, subject to the dynamics

$$\mathrm{d}u(t) = [(q_c - \mu)u(t) + v(t)]\mathrm{d}t,\ t \ge 0,\ u(0) = 0, \tag{2.8}$$

whereas $v(t) \in \mathbb{R}$ is the new control input. Note that (2.8) implies $w(\cdot, 0) = z_0(\cdot) \in L^2(\Omega, L^2(0,1))$. From (2.7) and (2.8), we obtain the equivalent system:

$$\begin{aligned} &du(t) = [(q_c - \mu)u(t) + v(t)]dt,\ t \ge 0, && (2.9a)\\ &dw(x,t) = [\tfrac{\partial}{\partial x}(p(x)\tfrac{\partial}{\partial x}w(x,t)) \\ &\quad + (q_c - q(x))w(x,t) - \psi(x)v(t)\ ]dt && (2.9b)\\ &\quad + \sigma_1(x,t,w(x,t) + \psi(x)u(t))d\mathcal{W}_1(t), \\ &w(0,t) = w_x(1,t) = 0,\ \ u(0) = 0, && (2.9c)\end{aligned}$$

with noisy boundary measurement

$$\begin{aligned} dy(t) = &[w(1,t) + \psi(1)u(t)]dt \\ &+ \sigma_2(t, w(1,t) + \psi(1)u(t))d\mathcal{W}_2(t),\ \ t \ge 0. \end{aligned} \tag{2.10}$$

In Section 2.1.2, we prove that for any initial condition $z_0 \in L^2(\Omega; L^2(0,1))$ and $z_0 \in \mathcal{D}(\mathcal{A}_1)$ almost surely, (2.9b) with boundary conditions (2.9c) possesses a unique strong solution satisfying

$$w \in L^2(\Omega; C([0,T]; L^2(0,1))) \cap L^2(\Omega \times [0,T]; H^1(0,1)) \tag{2.11}$$

for any $T > 0$. Therefore, we can present the solution to (2.9b)–(2.9c) as

$$w(x,t) \overset{L^2}{=} \sum_{n=1}^{\infty} w_n(t)\phi_n(x),\ \ w_n(t) = \langle w(\cdot,t), \phi_n\rangle, \tag{2.12}$$

with ϕ_n, $n \in \mathbb{N}$ eigenfunctions of (1.1). The convergence of series (2.12) in L^2 in mean-square follows from (2.11). Note that the Fourier expansion for solutions of stochastic PDEs has been used in the past (see e.g. Christofides et al. (2008) and Hu et al. (2008) for stochastic PDEs with additive noise and Chow (2007, P.89), Duan and Wei (2014, P.86) for stochastic PDEs with multiplicative noise).

Differentiating w_n in (2.12) and using (2.9b), we obtain

$$\begin{aligned} dw_n(t) = &\Big[\ \int_0^1 [\tfrac{\partial}{\partial x}(p(x)\tfrac{\partial}{\partial x}w(x,t)) - q(x)w(x,t)]\phi_n(x)dx \\ &+ q_c w_n(t) - b_n v(t)\ \Big]\ dt + \sigma_{1,n}(t)d\mathcal{W}_1(t),\ \ t \ge 0, \\ w_n(0) = &\langle w(\cdot,0), \phi_n\rangle, \end{aligned} \tag{2.13}$$

where

$b_n = \langle \psi, \phi_n\rangle$,
$\sigma_{1,n}(t) = \langle \sigma_1(\cdot, t, \sum_{j=1}^{\infty} w_j(t)\phi_j + \psi(\cdot)u(t)), \phi_n\rangle$.

Integrating by parts and using (1.1) and the boundary conditions (2.9c), we have

$$\begin{aligned} &\int_0^1 [\tfrac{\partial}{\partial x}(p(x)\tfrac{\partial}{\partial x}w(x,t)) - q(x)w(x,t)]\phi_n(x)dx \\ &= -\int_0^1 w(x,t)(\mathcal{A}_1\phi_n)(x)dx = -\lambda_n w_n(t), \end{aligned} \tag{2.14}$$

where the last equality is obtained from $(\mathcal{A}_1\phi_n)(x) = \lambda_n\phi_n(x)$. Then it follows from (2.13) and (2.14) that

$$\begin{aligned} dw_n(t) = &[(-\lambda_n + q_c)w_n(t) - b_n v(t)]dt \\ &+ \sigma_{1,n}(t)d\mathcal{W}_1(t),\ \ t \ge 0, \\ w_n(0) = &\langle w(\cdot,0), \phi_n\rangle. \end{aligned} \tag{2.15}$$

Let $\delta > 0$ be a desired decay rate and let $N_0 \in \mathbb{N}$ satisfy

$$-\lambda_n + q_c + \delta + \tfrac{\bar{\sigma}_1^2}{2} < 0,\ \ n > N_0, \tag{2.16}$$

where N_0 is the number of modes used for the controller design. Compared with Katz and Fridman (2020, 2021) and Lhachemi and Prieur (2022) for the deterministic PDEs, the additional term $\bar{\sigma}_1^2/2$ in (2.16) is induced by the stochastic perturbations. Let $N \in \mathbb{N}$, $N \ge N_0$, where N will be the dimension of the observer.

Remark 2.2. In (2.16), N_0 represents the number of "relatively unstable" modes that need to be stabilized. To explain this point, we present the open-loop system (2.1) (i.e., $u(t) \equiv 0$) as the following stochastic evolution equation:

$$\begin{aligned} &dz(t) = [-\mathcal{A}_1 z(t) + q_c z(t)]dt + \sigma_1(\cdot, t, z(t))d\mathcal{W}_1(t), \\ &z(0) = z_0 \in L^2(\Omega; L^2(0,1)), \end{aligned} \tag{2.17}$$

where $t \ge 0$, $\mathcal{A}_1$ is defined in (1.1). Since the nonlinear function σ_1 satisfies the global Lipschitz condition (2.2), we can conclude from Chow (2007, Theorem 6.7.4) that (2.17) has a unique strong solution $z \in L^2(\Omega; C[0,T]; L^2(0,1)) \cap L^2(\Omega \times [0,T]; H^1(0,1))$. Assume (2.16) holds for some N_0. Considering $V(z) = \|z\|^2_{L^2}$, $z \in L^2(0,1)$ and calculating the generator $\mathcal{L}$ (see Chow (2007, P. 228)) along (2.17), we have for $t \ge 0$,

$$\begin{aligned} &\mathcal{L}V(z(t)) + 2\delta V(z(t)) = \langle -\mathcal{A}_1 z(t) + q_c z(t), D_z V(z(t))\rangle_{L^2} \\ &\quad + \tfrac{1}{2}\langle D_{zz}V(z(t))\sigma_1(\cdot,t,z(t)), \sigma_1(\cdot,t,z(t))\rangle_{L^2} + 2\delta\|z(t)\|^2_{L^2} \\ &= 2\langle -\mathcal{A}_1 z(t), z(t)\rangle + (2q_c + 2\delta)\|z(t)\|^2_{L^2} + \|\sigma_1(\cdot,t,z(t))\|^2_{L^2} \\ &\overset{(2.2)}{\le} 2\langle -\mathcal{A}_1 z(t), z(t)\rangle + (2q_c + 2\delta + \bar{\sigma}_1^2)\|z(t)\|^2_{L^2}, \end{aligned} \tag{2.18}$$

where D_z, D_{zz} are the Fréchet derivatives of $V(z)$. By Parseval's equality (see Muscat (2014, Proposition 10.29)), we have

$$\begin{aligned} &\langle -\mathcal{A}_1 z(t), z(t)\rangle = \textstyle\sum_{n=1}^{\infty}\langle -\mathcal{A}_1 z(t), \phi_n\rangle\langle z(t), \phi_n\rangle \\ &= -\sum_{n=1}^{\infty}\lambda_n z_n^2(t). \end{aligned} \tag{2.19}$$

Substitution of (2.19) into (2.18) gives

$$\begin{aligned} &\mathcal{L}V(z(t)) + 2\delta V(z(t)) \\ &\le 2\sum_{n=1}^{\infty}(-\lambda_n + q_c + \delta + \frac{\bar{\sigma}_1^2}{2})z_n^2(t) \\ &\overset{(2.16)}{\le} 2\sum_{n=1}^{N_0}(-\lambda_n + q_c + \delta + \frac{\bar{\sigma}_1^2}{2})z_n^2(t),\ \ t \ge 0. \end{aligned} \tag{2.20}$$

To guarantee the mean-square L^2 exponential stability with decay rate δ (see Chow (2007, Theorem 7.4.2)), it is sufficient to control the first N_0 modes in order to guarantee that along the closed-loop system, $\mathcal{L}V(t) + 2\delta V(t) \le 0$ for all $t \ge 0$.

Following Katz and Fridman (2020) and Selivanov and Fridman (2019), we construct a N-dimensional observer of the form

$$\hat{w}(x,t) = \sum_{n=1}^{N}\hat{w}_n(t)\phi_n(x),\ \ N > N_0, \tag{2.21}$$

where $\hat{w}_n(t)$ $(1 \le n \le N)$ satisfy

$$\begin{aligned} &d\hat{w}_n(t) = [(-\lambda_n + q_c)\hat{w}_n(t) - b_n v(t)]dt \\ &\quad + l_n\{[\sum_{j=1}^{N}\phi_j(1)\hat{w}_j(t) + \psi(1)u(t)]dt - dy(t)\} \\ &\hat{w}_n(0) = 0,\ \ 1 \le n \le N, \end{aligned} \tag{2.22}$$

with $y(t)$ satisfying (2.10) and scalar observer gains $\{l_n\}_{n=1}^N$.

Introduce the notations

$$\begin{aligned} &A_0 = \mathrm{diag}\{-\lambda_n + q_c\}_{n=1}^{N_0},\ \ \tilde{A}_0 = \mathrm{diag}\{q_c - \mu, A_0\}, \\ &B_0 = [b_1, \dots, b_{N_0}]^{\mathrm{T}},\ \ \tilde{B}_0 = \mathrm{col}\{1, -B_0\}, \\ &c_n = \phi_n(1), n \in \mathbb{N},\ \ \ C_0 = [c_1, \dots, c_{N_0}]. \end{aligned} \tag{2.23}$$

From Orlov (2017) we have $c_n = O(1)$, $n \to \infty$. By Katz and Fridman (2020, Remark 3.3), we have $c_n \neq 0$, $\forall n \in \mathbb{N}$. Therefore, the pair (A_0, C_0) is observable by the Hautus lemma. Choose

$l_1,\ldots,l_{N_0}$ such that $L_0 = [l_1,\ldots,l_{N_0}]^{\mathrm{T}}$ satisfies the following Lyapunov inequality:

$$P_o(A_0+L_0C_0)+(A_0+L_0C_0)^{\mathrm{T}}P_o < -2\delta P_o, \tag{2.24}$$

where $0<P_o\in\mathbb{R}^{N_0\times N_0}$. Furthermore, we choose $l_n=0,\ n>N_0$.

By Karafyllis (2021, Lemma 2.1), the pair $(\tilde{A}_0,\tilde{B}_0)$ is controllable. Let $K_{\mathrm{T}}\in\mathbb{R}^{1\times(N_0+1)}$ satisfy

$$P_c(\tilde{A}_0+\tilde{B}_0K_{\mathrm{T}})+(\tilde{A}_0+\tilde{B}_0K_{\mathrm{T}})^{\mathrm{T}}P_c < -2\delta P_c, \tag{2.25}$$

where $0<P_c\in\mathbb{R}^{(N_0+1)\times(N_0+1)}$.

Remark 2.3. Since in many applications one cannot a priori know the noise intensity bound, here we design the observer and controller gains obtained from (2.24) and (2.25) that are independent of the noise intensity bound. To enlarge $\bar{\sigma}_1$, we can use state-feedback controller design in Section 3, where the resulting gain is related to the state noise intensity and satisfies (2.25).

We further propose a (N_0+1)-dimensional controller of the form

$$v(t)=K_{\mathrm{T}}\hat{w}^{N_0}(t),\ \ \hat{w}^{N_0}(t)=[u(t),\hat{w}_1(t),\ldots,\hat{w}_{N_0}(t)]^{\mathrm{T}}, \tag{2.26}$$

which is based on the N-dimensional observer (2.21).

2.1.2. Well-posedness of the closed-loop system

For the well-posedness we employ the following notations

$$\begin{aligned}
&\hat{w}^N(t)=[u(t),\hat{w}_1(t),\ldots,\hat{w}_N(t)]^{\mathrm{T}},\ \ \mathbb{1}_1=[1,0_{1\times N}],\\
&B_1=[b_{N_0+1},\ldots,b_N]^{\mathrm{T}},\ \tilde{B}=\mathrm{col}\{1,-B_0,-B_1\}^{\mathrm{T}},\\
&A_1=\mathrm{diag}\{-\lambda_n+q_c\}_{n=N_0+1}^{N},\ A=\mathrm{diag}\{\tilde{A}_0,A_1\},\\
&\tilde{C}=[0_{1\times1},c_1,\ldots,c_N],\ \ K_1=[K_{\mathrm{T}},0_{1\times(N-N_0)}],\\
&\tilde{L}=\mathrm{col}\{0_{1\times1},L_0,0_{(N-N_0)\times1}\}.
\end{aligned} \tag{2.27}$$

Consider $\xi(t)=\mathrm{col}\{\hat{w}^N(t),w(\cdot,t)\}$ and $\mathcal{W}(t)=[\mathcal{W}_2(t),\mathcal{W}_1(t)]^{\mathrm{T}}$. Then system (2.9) and (2.22) subject to the control input (2.26) can be presented as

$$\mathrm{d}\xi(t)=[\mathcal{A}\xi(t)+f(\xi(t))]\mathrm{d}t+g(\xi(t))\mathrm{d}\mathcal{W}(t) \tag{2.28}$$

with $\mathcal{A}=\mathrm{diag}\{\mathcal{A}_2,-\mathcal{A}_1\}$ where $\mathcal{A}_1$ is given by (1.1) and

$$\begin{aligned}
&\mathcal{A}_2=A+\tilde{B}K_1+\tilde{L}\tilde{C},\\
&f(\xi(t))=\left[\begin{smallmatrix}-\tilde{L}\int_0^1 w_x(x,t)\mathrm{d}x, & q_c w(\cdot,t)-\psi(\cdot)K_1\hat{w}^N(t)\end{smallmatrix}\right]^{\mathrm{T}},\\
&g(\xi(t))=\left[\begin{smallmatrix}g_2(\xi(t)) & 0\\ 0 & g_1(\xi(t))\end{smallmatrix}\right],\\
&g_1(\xi(t))=\sigma_1(\cdot,t,w(\cdot,t)+\psi(\cdot)\mathbb{1}_1\hat{w}^N(t)),\\
&g_2(\xi(t))=-\tilde{L}\sigma_2(t,\textstyle\int_0^1 w_x(x,t)\mathrm{d}x+\psi(1)\mathbb{1}_1\hat{w}^N(t)).
\end{aligned}$$

Let $\mathcal{H}=\mathbb{R}^{N+1}\times L^2(0,1)$ be a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}^2=|\cdot|^2+\|\cdot\|_{L^2}^2$. Take $\mathcal{V}=\mathbb{R}^{N+1}\times H^1(0,1)$ with norm $\|\cdot\|_{\mathcal{V}}^2=|\cdot|^2+\|\cdot\|_{H^1}^2$, and $\mathcal{V}'=\mathbb{R}^{N+1}\times H^{-1}(0,1)$. The duality scalar product between $\mathcal{V}'$ and $\mathcal{V}$ is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{V}',\mathcal{V}}=\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Then $\mathcal{A}:\mathcal{V}\to\mathcal{V}'$ is a closed linear operator with domain $\mathcal{D}(\mathcal{A})$ dense in $\mathcal{H}$. For any $\xi_i=\mathrm{col}\{\hat{w}_i^N,w_i\}\in\mathcal{V}$, $i=1,2$, integrating by parts and using the boundary conditions (2.9c), we have

$$\begin{aligned}
&|\langle\mathcal{A}\xi_1,\xi_2\rangle_{\mathcal{V}',\mathcal{V}}|=|\textstyle\int_0^1(-\mathcal{A}_1w_1)w_2\mathrm{d}x+(\hat{w}_1^N)^{\mathrm{T}}\mathcal{A}_2\hat{w}_2^N|\\
&=|\textstyle\int_0^1[\frac{\mathrm{d}}{\mathrm{d}x}(p(x)\frac{\mathrm{d}}{\mathrm{d}x}w_1)-q(x)w_1]w_2\mathrm{d}x\\
&\quad+(\hat{w}_1^N)^{\mathrm{T}}(A+\tilde{B}K_1+\tilde{L}\tilde{C})\hat{w}_2^N|\\
&\le p^*|\textstyle\int_0^1 w_1'w_2'\mathrm{d}x|+q^*|\int_0^1 w_1w_2\mathrm{d}x|+\mu^*|\hat{w}_1^N\|\hat{w}_2^N|\\
&\le p^*\|w_1'\|_{L^2}\|w_2'\|_{L^2}+q^*\|w_1\|_{L^2}\|w_2\|_{L^2}+\mu^*|\hat{w}_1^N\|\hat{w}_2^N|\\
&\le(p^*+q^*+\mu^*)\|\xi_1\|_{\mathcal{V}}\|\xi_2\|_{\mathcal{V}},
\end{aligned} \tag{2.29}$$

where the penultimate inequality is obtained by the Cauchy–Schwarz inequality, $\mu^*=\|A+\tilde{B}K_1+\tilde{L}\tilde{C}\|$. Similarly, for any $\xi=\mathrm{col}\{\hat{w}^N,w\}\in\mathcal{V}$, we have

$$\begin{aligned}
&\langle\mathcal{A}\xi,\xi\rangle_{\mathcal{V}',\mathcal{V}}=(\hat{w}^N)^{\mathrm{T}}(A+\tilde{B}K_1+\tilde{L}\tilde{C})\hat{w}^N\\
&\quad+\textstyle\int_0^1[\frac{\mathrm{d}}{\mathrm{d}x}(p(x)\frac{\mathrm{d}}{\mathrm{d}x}w)-q(x)w]w\mathrm{d}x\\
&\le -p_*\textstyle\int_0^1(w')^2\mathrm{d}x+\mu^*|\hat{w}^N|^2\\
&\le -p_*\|w\|_{H^1}^2+p_*\|w\|_{L^2}^2+\mu^*|\hat{w}^N|^2\\
&\le -p_*\|\xi\|_{\mathcal{V}}^2+(p_*+\mu^*)\|\xi\|_{\mathcal{H}}^2.
\end{aligned} \tag{2.30}$$

For $w_i(\cdot,t)\in\mathcal{D}(\mathcal{A}_1)$ almost surely, the application of Jensen's inequality implies

$$[\textstyle\int_0^1(\frac{\partial w_1(x,t)}{\partial x}-\frac{\partial w_2(x,t)}{\partial x})\mathrm{d}x]^2\le\|w_1(\cdot,t)-w_2(\cdot,t)\|_{H^1}^2. \tag{2.31}$$

Besides, the Wirtinger's inequality implies

$$\|w_1(\cdot,t)-w_2(\cdot,t)\|_{L^2}^2\le\tfrac{4}{\pi^2}\|w_1(\cdot,t)-w_2(\cdot,t)\|_{H^1}^2. \tag{2.32}$$

Therefore, for any $\xi_i=\mathrm{col}\{\hat{w}_i^N,w_i\}\in\mathcal{V}$, $i=1,2$, from (2.2), (2.4), (2.31) and (2.32), we can obtain

$$\begin{aligned}
&\langle f(\xi_1),\xi_1\rangle_{\mathcal{H}}+\mathrm{tr}\{g^{\mathrm{T}}(\xi_1)g(\xi_1)\}\le\kappa_1(1+\|\xi_1\|_{\mathcal{V}}^2),\\
&\|f(\xi_2)-f(\xi_1)\|_{\mathcal{H}}^2+\mathrm{tr}\{[g(\xi_2)-g(\xi_1)]^{\mathrm{T}}[g(\xi_2)-g(\xi_1)]\}\\
&\le\kappa_2\|\xi_2-\xi_1\|_{\mathcal{V}}^2,
\end{aligned} \tag{2.33}$$

for some $\kappa_1,\kappa_2>0$, where $\mathrm{tr}\{g^{\mathrm{T}}(\xi_1)g(\xi_1)\}=|g_2(\xi_1)|^2+\|g_1(\xi_1)\|_{L^2}^2$. For any $\xi(t)=\mathrm{col}\{\hat{w}^N(t),w(\cdot,t)\}\in\mathcal{V}$, by (2.2), (2.4) and (2.30), we have for some $\kappa_3>0$

$$\begin{aligned}
&\langle\mathcal{A}\xi(t),\xi(t)\rangle_{\mathcal{V}',\mathcal{V}}+\langle f(\xi(t)),\xi(t)\rangle_{\mathcal{H}}+\tfrac12\mathrm{tr}\{g^{\mathrm{T}}(\xi(t))g(\xi(t))\}\\
&=\textstyle\int_0^1[\frac{\partial}{\partial x}(p(x)\frac{\partial}{\partial x}w(x,t))-q(x)w(x,t)]w(x,t)\mathrm{d}x\\
&\quad+(\hat{w}^N(t))^{\mathrm{T}}\mathcal{A}_2\hat{w}^N(t)-\textstyle\int_0^1 w_x(x,t)\mathrm{d}x\tilde{L}^{\mathrm{T}}\hat{w}^N(t)\\
&\quad+\textstyle\int_0^1[q_cw(x,t)+a(x)\mathbb{1}_1\hat{w}^N(t)-b(x)K_1\hat{w}^N(t)]w(x,t)\mathrm{d}x\\
&\quad+\tfrac12[\|g_1(\xi(t))\|_{L^2}^2+|g_2(\xi(t))|^2]\\
&\le -p_*\|w(\cdot,t)\|_{H^1}^2-\textstyle\int_0^1 w_x(x,t)\mathrm{d}x\tilde{L}^{\mathrm{T}}\hat{w}^N(t)\\
&\quad+\tfrac{|L_0|^2\bar{\sigma}_2^2}{2}|\textstyle\int_0^1 w_x(x,t)\mathrm{d}x+\psi(1)\mathbb{1}_1\hat{w}^N(t)|^2\\
&\quad+\kappa_3(\|w(\cdot,t)\|_{L^2}^2+|\hat{w}^N(t)|^2).
\end{aligned} \tag{2.34}$$

By Young's inequality with some $\varepsilon_1,\varepsilon_2>0$, and (2.31), we obtain

$$\begin{aligned}
&-\textstyle\int_0^1 w_x(x,t)\mathrm{d}x\tilde{L}^{\mathrm{T}}\hat{w}^N(t)\\
&\le\tfrac{\varepsilon_1}{2}\|w(\cdot,t)\|_{H^1}^2+\tfrac{1}{2\varepsilon_1}|L_0|^2|\hat{w}^N(t)|^2,\\
&|\textstyle\int_0^1 w_x(x,t)\mathrm{d}x+\psi(1)\mathbb{1}_1\hat{w}^N(t)|^2\\
&\le(1+\varepsilon_2)\|w(\cdot,t)\|_{H^1}^2+(1+\tfrac{1}{\varepsilon_2})|\psi(1)\mathbb{1}_1\hat{w}^N(t)|^2.
\end{aligned} \tag{2.35}$$

Substitution of (2.35) into (2.34) gives

$$\begin{aligned}
&\langle\mathcal{A}\xi(t),\xi(t)\rangle_{\mathcal{V}',\mathcal{V}}+\langle f(\xi(t)),\xi(t)\rangle_{\mathcal{H}}+\tfrac12\mathrm{tr}\{g^{\mathrm{T}}(\xi(t))g(\xi(t))\}\\
&\le-[p_*-\tfrac{\varepsilon_1}{2}-\tfrac{|L_0|^2\bar{\sigma}_2^2}{2}(1+\varepsilon_2)]\|w(\cdot,t)\|_{H^1}^2\\
&\quad+\kappa_4(\|w(\cdot,t)\|_{L^2}^2+|\hat{w}^N(t)|^2)\\
&\le-\kappa^*\|\xi(t)\|_{\mathcal{V}}^2+\kappa_5\|\xi(t)\|_{\mathcal{H}}^2,
\end{aligned} \tag{2.36}$$

where $\kappa^*=p_*-\frac{\varepsilon_1}{2}-\frac{|L_0|^2\bar{\sigma}_2^2}{2}(1+\varepsilon_2)$ and κ_4,κ_5 are some positive constants. Choosing $\varepsilon_1,\varepsilon_2$ sufficiently small, we obtain $\kappa^*>0$ provided

$$\bar{\sigma}_2|L_0|<\sqrt{2p_*}, \tag{2.37}$$

where p_* is the lower bound of $p(x)$.

Remark 2.4. Note that the closed-loop system (2.28) contains a gradient-dependent noise with its intensity upper bounded by $\bar{\sigma}_2|L_0|$ (see g_2 component of g). It is well known that for stochastic parabolic equations, gradient-dependent noise intensity should not exceed a certain threshold set by the diffusion coefficient (see Chow (2007, P. 89)). In order to guarantee the coercivity condition (2.36) with $\kappa^* > 0$ for well-posedness, we therefore assume (2.37). Even though condition (2.37) may limit the observer gain design, violation of (2.37) (which leads to violation of the coercivity condition (2.36) with $\kappa^* > 0$) may lead to an ill-posed closed-loop system.

By arguments similar to (2.34)–(2.36), we obtain for any $\xi_i \in \mathcal{V}$, $i = 1, 2$,

$$\begin{aligned} &2\langle \mathcal{A}(\xi_1 - \xi_2), \xi_1 - \xi_2\rangle_{\mathcal{V}',\mathcal{V}} + 2\langle f(\xi_1) - f(\xi_2), \xi_1 - \xi_2\rangle_{\mathcal{H}} \\ &+\mathrm{tr}\{[g(\xi_1) - g(\xi_2)]^{\mathrm{T}}[g(\xi_1) - g(\xi_2)]\} \le \kappa_6\|\xi_1 - \xi_2\|^2_{\mathcal{H}} \end{aligned} \tag{2.38}$$

with some constant $\kappa_6 > 0$. Then by Chow (2007, Theorem 6.7.5), for any initial value $\xi_0 \in L^2(\Omega; \mathcal{H})$ and $\xi_0 \in \mathcal{D}(\mathcal{A})$ almost surely, (2.29), (2.30), (2.33), (2.36) and (2.38) guarantee that (2.28) has a unique strong solution satisfying

$$\xi \in L^2(\Omega; C([0, T]; \mathcal{H})) \cap L^2([0, T] \times \Omega; \mathcal{V})$$

for any $T > 0$, and

$$\xi(t) = \xi(0) + \int_0^t [\mathcal{A}\xi(s) + f(\xi(s))]\mathrm{d}s + \int_0^t g(\xi(s))\mathrm{d}\mathcal{W}(s),$$

almost surely, where the stochastic integral $\int_0^t g(\xi(s))\mathrm{d}\mathcal{W}(s)$ is in the sense of Itô type and a martingale. From the definition of a strong solution in Liu (2005) (see Definition 1.3.3 therein), we know that the strong solution $\xi(t) \in \mathcal{D}(\mathcal{A})$ almost surely and is adapted to $\mathcal{F}_t$, $t \ge 0$.

2.1.3. Mean-square L^2 stability analysis

First, we introduce the following mean-square L^2 stability definition for the closed-loop system (2.9) subject to control law (2.22), (2.26).

Definition 2.1. The closed-loop system (2.9) with control law (2.22), (2.26) is said to be mean-square L^2 exponentially stable with a decay rate $\delta > 0$ if there exists $M_0 > 1$ such that for any given initial value $w(\cdot, 0) \in L^2(\Omega; L^2(0, 1))$ and $w(\cdot, 0) \in \mathcal{D}(\mathcal{A}_1)$ almost surely, the corresponding strong solution $u(t)$, $w(\cdot, t)$ satisfies the following inequality for $t \ge 0$:

$$\mathbb{E}[u^2(t) + \|w(\cdot, t)\|^2_{L^2}] \le M_0 e^{-2\delta t}\mathbb{E}\|w(\cdot, 0)\|^2_{L^2}. \tag{2.39}$$

If (2.39) holds for the solutions to the closed-loop system (2.9) subject to control law (2.22), (2.26), then due to (2.6), the solution $z(\cdot, t)$ to the original system (2.1) with input $u(t)$ determined by (2.9a) satisfies

$$\mathbb{E}\|z(\cdot, t)\|^2_{L^2} \le \tilde{M}_0 e^{-2\delta t}\mathbb{E}\|z_0\|^2_{L^2}, \quad t \ge 0,$$

for some $\tilde{M}_0 \ge 1$.

Let

$$e_n(t) = w_n(t) - \hat{w}_n(t), \quad 1 \le n \le N \tag{2.40}$$

be the estimation error. The last term on the right-hand side of (2.22) can be presented as

$$\begin{aligned} &[\sum_{j=1}^{N} c_j\hat{w}_j(t) + \psi(1)u(t)]\mathrm{d}t - \mathrm{d}y(t) \\ &\overset{(2.10)}{=} [-\sum_{j=1}^{N} c_j e_j(t) - \zeta(t)]\mathrm{d}t - \sigma_2(t, \hat{\zeta}(t))\mathrm{d}\mathcal{W}_2(t), \\ &\zeta(t) = w(1, t) - \sum_{j=1}^{N} c_j w_j(t), \\ &\hat{\zeta}(t) = \zeta(t) + \sum_{j=1}^{N} c_j(e_j(t) + \hat{w}_j(t)) + \psi(1)u(t). \end{aligned} \tag{2.41}$$

Then from (2.15) and (2.22), the error system has the form

$$\begin{aligned} \mathrm{d}e_n(t) = {} & [(-\lambda_n + q_c)e_n(t) + l_n(\sum_{j=1}^{N} c_j e_j(t) + \zeta(t))]\mathrm{d}t \\ & + \sigma_{1,n}(t)\mathrm{d}\mathcal{W}_1(t) + l_n\sigma_2(t, \hat{\zeta}(t))\mathrm{d}\mathcal{W}_2(t), \quad 1 \le n \le N. \end{aligned} \tag{2.42}$$

Denote

$$\begin{aligned} & e^{N_0}(t) = [e_1(t), \dots, e_{N_0}(t)]^{\mathrm{T}}, \; C_1 = [c_{N_0+1}, \dots, c_N], \\ & e^{N-N_0}(t) = [e_{N_0+1}(t), \dots, e_N(t)]^{\mathrm{T}}, \tilde{L}_0 = \mathrm{col}\{0_{1\times 1}, L_0\}, \\ & \hat{w}^{N-N_0}(t) = [\hat{w}_{N_0+1}(t), \dots, \hat{w}_N(t)]^{\mathrm{T}}, \\ & X(t) = \mathrm{col}\{\hat{w}^{N_0}(t), e^{N_0}(t), \hat{w}^{N-N_0}(t), e^{N-N_0}(t)\}, \\ & \mathbb{L}_0 = \mathrm{col}\{-\tilde{L}_0, L_0, 0_{2(N-N_0)\times 1}\}, \; \mathcal{K}_{\mathrm{T}} = [K_{\mathrm{T}}, 0_{1\times(2N-N_0)}], \\ & F = \begin{bmatrix} \tilde{A}_0 + \tilde{B}_0 K_{\mathrm{T}} & -\tilde{L}_0 C_0 & 0 & -\tilde{L}_0 C_1 \\ 0 & A_0 + L_0 C_0 & 0 & L_0 C_1 \\ -B_1 K_{\mathrm{T}} & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \\ & \sigma^{N_0}(t) = [\sigma_{1,1}(t), \dots, \sigma_{1,N_0}(t)]^{\mathrm{T}}, \; \mathbb{1} = [1, 0_{1\times 2N}], \\ & \sigma^{N-N_0}(t) = [\sigma_{1,N_0+1}(t), \dots, \sigma_{1,N}(t)]^{\mathrm{T}}, \\ & \Sigma(t) = \mathrm{col}\{0_{(N_0+1)\times 1}, \sigma^{N_0}(t), 0_{(N-N_0)\times 1}, \sigma^{N-N_0}(t)\}, \\ & \mathcal{C}_1 = [\psi(1), C_0, C_0, C_1, C_1] \in \mathbb{R}^{1\times(2N+1)}. \end{aligned} \tag{2.43}$$

By (2.22), (2.26), (2.41), (2.42) and (2.43), we obtain the closed-loop system

$$\begin{aligned} \mathrm{d}X(t) = {} & [FX(t) + \mathbb{L}_0\zeta(t)]\mathrm{d}t + \Sigma(t)\mathrm{d}\mathcal{W}_1(t) \\ & + \mathbb{L}_0\sigma_2(t, \zeta(t) + \mathcal{C}_1 X(t))\mathrm{d}\mathcal{W}_2(t), \end{aligned} \tag{2.44a}$$

$$\begin{aligned} \mathrm{d}w_n(t) = {} & [(-\lambda_n + q_c)w_n(t) - b_n\mathcal{K}_{\mathrm{T}}X(t)]\mathrm{d}t \\ & + \sigma_{1,n}(t)\mathrm{d}\mathcal{W}_1(t), \quad n > N. \end{aligned} \tag{2.44b}$$

For mean-square L^2 exponential stability of the closed-loop system (2.44), we consider the Lyapunov function

$$V(t) = |X(t)|^2_P + \rho \sum_{n=N+1}^{\infty} w_n^2(t), \tag{2.45}$$

where $0 < P \in \mathbb{R}^{(2N+1)\times(2N+1)}$, $\rho > 0$ is a scalar. Since $u(0) = 0$ and $\hat{w}_n(t) = 0$, $1 \le n \le N$, we have

$$\begin{aligned} V(0) &\le \lambda_{\max}(P)|X(0)|^2 + \rho \sum_{n=N+1}^{\infty} w_n^2(0) \\ &\le \max\{\lambda_{\max}(P), \rho\}\|w(\cdot, 0)\|^2_{L^2}. \end{aligned} \tag{2.46}$$

Noting that $\hat{w}_n^2 + e_n^2 = (w_n - e_n)^2 + e_n^2 \ge 0.5w_n^2$, we infer that

$$\begin{aligned} V(t) &\ge \lambda_{\min}(P)[u^2(t) + \sum_{n=1}^{N}(\hat{w}_n^2(t) + e_n^2(t))] \\ &\quad + \rho \sum_{n=N+1}^{\infty} w_n^2(t) \\ &\ge \min\{\tfrac{\lambda_{\min}(P)}{2}, \rho\}[u^2(t) + \|w(\cdot, t)\|^2_{L^2}], \quad t \ge 0. \end{aligned} \tag{2.47}$$

Remark 2.5. In Katz and Fridman (2020, 2021), the boundary or point measurements were considered for the deterministic PDEs with $c_n = O(1)$, $n \to \infty$, where H^1 stability was required to compensate $\zeta(t)$ defined in (2.41). In this paper, we consider the Lyapunov function (2.45) with ρ large enough to compensate $\zeta(t)$ by using (2.60) in the Lyapunov analysis and study the L^2 exponential stability, which is justified by the regularity of solutions.

By Parseval's equality we present (2.45) as

$$\begin{aligned}&V(t)=V_1(t)-V_2(t)+V_3(w(\cdot,t)),\\&V_1(t)=|X(t)|_P^2,\ V_2(t)=\rho|\mathbb{I}X(t)|^2,\\&V_3(w(\cdot,t))=\rho\|w(\cdot,t)\|_{L^2}^2,\\&\mathbb{I}=\begin{bmatrix}0_{N_0\times 1} & I_{N_0} & I_{N_0} & 0 & 0\\ 0_{(N-N_0)\times 1} & 0 & 0 & I_{N-N_0} & I_{N-N_0}\end{bmatrix}.\end{aligned} \tag{2.48}$$

Remark 2.6. Differently from Katz and Fridman (2020, 2021) for the deterministic PDEs where the series in (2.45) was differentiated term by term, here the Lyapunov function is presented as (2.48) in order to make it suitable for application of the Itô's formula. Additionally, we use the nonlinear term in $-\mathcal{L}V_2(t)$ (see (2.49)) to compensate the nonlinear term in $\mathcal{L}V_1(t)$ (see (2.50)) by $\rho>0$ large enough.

Calculating the generators $\mathcal{L}V_1(t)$ and $\mathcal{L}V_2(t)$ along stochastic ODE (2.44a) (see Klebaner (2005, P. 149)) we have

$$\begin{aligned}&\mathcal{L}V_1(t)+2\delta V_1(t)=X^{\mathrm{T}}(t)[PF+F^{\mathrm{T}}P+2\delta P]X(t)\\&\quad+2X^{\mathrm{T}}(t)P\mathbb{L}_0\zeta(t)+\Sigma^{\mathrm{T}}(t)P\Sigma(t)\\&\quad+\sigma_2^2(t,\zeta(t)+\mathcal{C}_1X(t))\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\\&\overset{(2.4)}{\le} X^{\mathrm{T}}(t)[PF+F^{\mathrm{T}}P+2\delta P+\bar\sigma_2^2\mathcal{C}_1^{\mathrm{T}}\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\mathcal{C}_1]X(t)\\&\quad+2X^{\mathrm{T}}(t)[P\mathbb{L}_0+\bar\sigma_2^2\mathcal{C}_1^{\mathrm{T}}\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0]\zeta(t)\\&\quad+\Sigma^{\mathrm{T}}(t)P\Sigma(t)+\bar\sigma_2^2\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\zeta^2(t)\end{aligned} \tag{2.49}$$

and

$$\begin{aligned}&\mathcal{L}V_2(t)+2\delta V_2(t)=\rho X^{\mathrm{T}}(t)(\mathbb{I}^{\mathrm{T}}\mathbb{I}F+F^{\mathrm{T}}\mathbb{I}^{\mathrm{T}}\mathbb{I}+2\delta\mathbb{I}^{\mathrm{T}}\mathbb{I})X(t)\\&\quad+2\rho X^{\mathrm{T}}(t)\mathbb{I}^{\mathrm{T}}\mathbb{I}\mathbb{L}_0\zeta(t)+\rho\Sigma^{\mathrm{T}}(t)\mathbb{I}^{\mathrm{T}}\mathbb{I}\Sigma(t)\\&\quad+\sigma_2^2(t,\zeta(t)+\mathcal{C}_1X(t))\mathbb{L}_0^{\mathrm{T}}\mathbb{I}^{\mathrm{T}}\mathbb{I}\mathbb{L}_0\\&=\rho\sum_{n=1}^{N}(-2\lambda_n+2q_c+2\delta)w_n^2(t)+\rho|\Sigma(t)|^2\\&\quad-\rho\sum_{n=1}^{N}2w_n(t)b_n\mathcal{K}_{\mathrm{T}}X(t).\end{aligned} \tag{2.50}$$

Recalling $\mathcal{A}_1$ defined in (1.1), we can rewrite (2.7) subject to (2.26) as

$$\begin{aligned}dw(t)=&[-\mathcal{A}_1w(t)+q_cw(t)-\psi(\cdot)\mathcal{K}_{\mathrm{T}}X(t)]\mathrm{d}t\\&+\sigma_1(\cdot,t,w(t)+\psi(\cdot)\mathbb{1}X(t))\mathrm{d}\mathcal{W}_1(t),\end{aligned} \tag{2.51}$$

where $w(t)=w(\cdot,t)$. Note that $w(t)$ is a strong solution to (2.51) satisfying (2.11) for any $T>0$ (see Section 2.1.2). For function $V_3(w(t))$ defined in (2.48), by arguments similar to (2.18)–(2.20), we have the following expression for generator $\mathcal{L}$ of (2.51) (see Chow (2007, P. 228)):

$$\begin{aligned}&\mathcal{L}V_3(w(t))=2\rho\langle-\mathcal{A}_1w(t),w(t)\rangle\\&\quad+2\rho q_c\|w^2(t)\|_{L^2}^2-2\rho\langle\psi(\cdot)v(t),w(\cdot,t)\rangle\\&\quad+\rho\|\sigma_1(\cdot,t,w(\cdot,t)+\psi(\cdot)u(t))\|_{L^2}^2\\&\le 2\rho\sum_{n=1}^{\infty}(-\lambda_n+q_c)w_n^2(t)-2\rho\langle\psi(\cdot)v(t),w(\cdot,t)\rangle\\&\quad+2\rho\bar\sigma_1^2\|w(t)+\psi(\cdot)u(t)\|_{L^2}^2.\end{aligned} \tag{2.52}$$

By Parseval's equality (see Muscat (2014, Proposition 10.29)), we have

$$\begin{aligned}&\langle\psi(\cdot)v(t),w(\cdot,t)\rangle=\sum_{n=1}^{\infty}\langle\psi(\cdot),\phi_n\rangle\langle w(\cdot,t),\phi_n\rangle v(t)\\&\overset{(2.26)}{=}\sum_{n=1}^{\infty}w_n(t)b_n\mathcal{K}_{\mathrm{T}}X(t),\\&\|w(t)+\psi(\cdot)u(t)\|_{L^2}^2=X^{\mathrm{T}}(t)[\mathbb{B}^{\mathrm{T}}\mathbb{B}+\|b\|_N^2\mathbb{1}^{\mathrm{T}}\mathbb{1}]X(t)\\&+\sum_{n=N+1}^{\infty}2w_n(t)b_n\mathbb{1}X(t)+\sum_{n=N+1}^{\infty}w_n^2(t),\end{aligned} \tag{2.53}$$

where

$$\mathbb{B}=\begin{bmatrix}B_0 & I_{N_0} & I_{N_0} & 0 & 0\\ B_1 & 0 & 0 & I_{N-N_0} & I_{N-N_0}\end{bmatrix}.$$

By (2.52)–(2.53) we arrive at

$$\begin{aligned}&\mathcal{L}V_3(w(t))+2\delta V_3(w(t))\\&\le\sum_{n=1}^{\infty}\rho(-2\lambda_n+2q_c+2\delta)w_n^2(t)\\&-\rho\sum_{n=1}^{\infty}2w_n(t)b_n\mathcal{K}_{\mathrm{T}}X(t)+\rho\bar\sigma_1^2X^{\mathrm{T}}(t)\mathbb{B}^{\mathrm{T}}\mathbb{B}X(t)\\&+\rho\bar\sigma_1^2\sum_{n=N+1}^{\infty}[w_n^2(t)+2w_n(t)b_n\mathbb{1}X(t)+b_n^2|\mathbb{1}X(t)|^2].\end{aligned} \tag{2.54}$$

Combination of (2.48), (2.49), (2.50) and (2.54) yields

$$\begin{aligned}&\mathcal{L}V(t)+2\delta V(t)\le X^{\mathrm{T}}(t)\Xi_1X(t)+\Sigma^{\mathrm{T}}(t)(P-\rho I)\Sigma(t)\\&+2X^{\mathrm{T}}(t)[P\mathbb{L}_0+\bar\sigma_2^2\mathcal{C}_1^{\mathrm{T}}\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0]\zeta(t)+\bar\sigma_2^2\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\zeta^2(t)\\&+\sum_{n=N+1}^{\infty}2\rho(-\lambda_n+q_c+\delta+\frac{1}{2}\bar\sigma_1^2)w_n^2(t)\\&-\rho\sum_{n=1}^{\infty}2w_n(t)b_n\mathcal{K}_{\mathrm{T}}X(t)+\rho\bar\sigma_1^2\sum_{n=N+1}^{\infty}2w_n(t)b_n\mathbb{1}X(t),\end{aligned} \tag{2.55}$$

where

$$\begin{aligned}\Xi_1:=&PF+F^{\mathrm{T}}P+2\delta P+\rho\bar\sigma_1^2\mathbb{B}^{\mathrm{T}}\mathbb{B}\\&+\bar\sigma_2^2\mathcal{C}_1^{\mathrm{T}}\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\mathcal{C}_1+\rho\bar\sigma_1^2\|b\|_N^2\mathbb{1}^{\mathrm{T}}\mathbb{1}.\end{aligned} \tag{2.56}$$

Let $\alpha_1,\alpha_2>0$. Applying Young's inequality we have

$$\begin{aligned}&-\sum_{n=N+1}^{\infty}2w_n(t)b_n\mathcal{K}_{\mathrm{T}}X(t)\\&\le\sum_{n=N+1}^{\infty}\alpha_1\lambda_n^{0.75}w_n^2(t)+\sum_{n=N+1}^{\infty}\frac{b_n^2}{\alpha_1\lambda_n^{0.75}}|\mathcal{K}_{\mathrm{T}}X(t)|^2\\&\le\sum_{n=N+1}^{\infty}\alpha_1\lambda_n^{0.75}w_n^2(t)+\frac{\|b\|_N^2}{\alpha_1\lambda_{N+1}^{0.75}}|\mathcal{K}_{\mathrm{T}}X(t)|^2,\\&\sum_{n=N+1}^{\infty}2b_nw_n(t)\mathbb{1}X(t)\\&\le\sum_{n=N+1}^{\infty}\alpha_2w_n^2(t)+\frac{1}{\alpha_2}\|b\|_N^2|\mathbb{1}X(t)|^2.\end{aligned} \tag{2.57}$$

By substituting (2.57) into (2.55), we obtain

$$\begin{aligned}&\mathcal{L}V(t)+2\delta V(t)\le X^{\mathrm{T}}(t)(\Xi_1+\Xi_2)X(t)\\&+2X^{\mathrm{T}}(t)[P\mathbb{L}_0+\bar\sigma_2^2\mathcal{C}_1^{\mathrm{T}}\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0]\zeta(t)+\bar\sigma_2^2\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\zeta^2(t)\\&+\sum_{n=N+1}^{\infty}2\rho\Upsilon_nw_n^2(t)+\Sigma^{\mathrm{T}}(t)(P-\rho I)\Sigma(t),\end{aligned} \tag{2.58}$$

where

$$\begin{aligned}&\Upsilon_n:=-\lambda_n+q_c+\delta+\tfrac{1}{2}\bar\sigma_1^2+\tfrac{\alpha_1}{2}\lambda_n^{0.75}+\tfrac{\alpha_2}{2}\bar\sigma_1^2,\\&\Xi_2:=\tfrac{\rho\|b\|_N^2}{\alpha_1\lambda_{N+1}^{0.75}}\mathcal{K}_{\mathrm{T}}^{\mathrm{T}}\mathcal{K}_{\mathrm{T}}+\tfrac{\rho\bar\sigma_1^2\|b\|_N^2}{\alpha_2}\mathbb{1}^{\mathrm{T}}\mathbb{1}.\end{aligned} \tag{2.59}$$

As for $\zeta(t)$ given in (2.42), by Young's inequality and Lemma 1.1 we get

$$\begin{aligned}\zeta^2(t) &= (w(1,t)-\sum_{n=1}^{N}w_n(t)\phi_n(1))^2\\ &= (\int_0^1[w_\xi(\xi,t)-\sum_{n=1}^{N}w_n(t)\phi_n'(\xi)]\mathrm{d}\xi)^2\\ &\le \|w_x(\cdot,t)-\sum_{n=1}^{N}w_n(t)\phi_n'(\cdot)\|_{L^2}^2\\ &\le \tfrac{1}{p_*}\sum_{n=N+1}^{\infty}\lambda_n w_n^2(t).\end{aligned} \tag{2.60}$$

Then with notation

$$\theta_n = \tfrac{-2\Upsilon_n}{\lambda_n} = 2-\tfrac{\alpha_1}{\lambda_n^{0.25}}-\tfrac{2q_c+2\delta+\bar\sigma_1^2(1+\alpha_2)}{\lambda_n},\ n\ge 1,$$

from the monotonicity of λ_n and (2.60), we arrive at

$$\begin{aligned}&\sum_{n=N+1}^{\infty}2\rho\Upsilon_n w_n^2(t)\le-\rho\theta_{N+1}\sum_{n=N+1}^{\infty}\lambda_n w_n^2(t)\\ &\le-\rho\theta_{N+1}p_*\zeta^2(t)\end{aligned} \tag{2.61}$$

provided

$$\Upsilon_{N+1}=-\lambda_{N+1}+q_c+\delta+\tfrac12\bar\sigma_1^2+\tfrac{\alpha_1}{2}\lambda_{N+1}^{0.75}+\tfrac{\alpha_2}{2}\bar\sigma_1^2<0. \tag{2.62}$$

Let $\eta(t)=\mathrm{col}\{X(t),\zeta(t)\}$. From (2.55) and (2.61) we obtain

$$\begin{aligned}&\mathcal{L}V(t)+2\delta V(t)\le\eta^{\mathrm T}(t)\Xi_{\mathrm{NonL}}\eta(t)\\ &+\Sigma^{\mathrm T}(t)(P-\rho I)\Sigma(t)\le 0\end{aligned} \tag{2.63}$$

if (2.62) and

$$\begin{aligned}&\Xi_{\mathrm{NonL}}:=\begin{bmatrix}\Xi_1+\Xi_2 & P\mathbb{L}_0+\bar\sigma_2^2\mathcal{C}_1^{\mathrm T}\mathbb{L}_0^{\mathrm T}P\mathbb{L}_0\\ * & -\rho\theta_{N+1}p_*+\bar\sigma_2^2\mathbb{L}_0^{\mathrm T}P\mathbb{L}_0\end{bmatrix}<0,\\ &P<\rho I,\end{aligned} \tag{2.64}$$

hold with Ξ_1 in (2.56) and Ξ_2 in (2.59). Summarizing, we arrive at:

Theorem 2.1. *Consider (2.9) with nonlinear noise function $\sigma_1(x,t,z)$ satisfying (2.2), control law (2.26), noisy boundary measurement (2.10) with $\sigma_2(t,z)$ satisfying (2.4), (2.37), and initial value $w(\cdot,0)\in L^2(\Omega,L^2(0,1))$, $w(\cdot,0)\in\mathcal{D}(\mathcal{A}_1)$ almost surely. Let $\delta>0$ be a desired decay rate, $N_0\in\mathbb{N}$ satisfy (2.16) and $N\in\mathbb{N}$ satisfy $N\ge N_0$. Assume that L_0 and $K_{\mathrm T}$ are obtained from (2.24) and (2.25), respectively. Let $\alpha_1,\alpha_2>0$ be subject to (2.62). If there exist a matrix $0<P\in\mathbb{R}^{(2N+1)\times(2N+1)}$ and a scalar $\rho>0$ such that (2.64) hold, then the solution $u(t)$, $w(x,t)$ to (2.9) subject to the control law (2.22), (2.26) is mean-square L^2 exponentially stable and the corresponding observer $\hat w(x,t)$ given by (2.21) satisfies for $t\ge 0$*

$$\mathbb{E}\|w(\cdot,t)-\hat w(\cdot,t)\|_{L^2}^2\le M_0e^{-2\delta t}\mathbb{E}\|w(\cdot,0)\|_{L^2}^2, \tag{2.65}$$

with some constant $M_0>1$. Moreover, inequalities (2.62) and (2.64) are always feasible for small enough $\bar\sigma_1,\bar\sigma_2$ and large enough N.

Proof. First, by employing Itô's formula for $e^{2\delta t}V_i(t)$, $i=1,2$ along stochastic ODE (2.44a) (see Klebaner (2005, Theorem 4.18)), we have

$$\begin{aligned}&e^{2\delta t}V_i(t)=V_i(0)+\int_0^t e^{2\delta s}[\mathcal{L}V_i(s)+2\delta V_i(s)]\mathrm{d}s\\ &\quad+\int_0^t e^{2\delta s}2X^{\mathrm T}(s)P_i\sigma^{N_0}(s)\mathrm{d}\mathcal{W}_1(s),\quad i=1,2,\\ &P_1=P,\quad P_2=\rho\mathbb{I}^{\mathrm T}\mathbb{I}.\end{aligned} \tag{2.66}$$

Since $w(t)$ is a strong solution to (2.51) satisfying (2.11) and $X(t)$ is a solution to stochastic ODE (2.44a), we have $\mathrm{col}\{w(t),X(t)\}$, $t\in[0,T]$, $\forall T>0$ is a predictable process, and thus, an adapted process (see Da Prato and Zabczyk (2014, P. 72)). Then $q_cw-\psi(\cdot)\mathcal{K}_TX\in L^2([0,T];L^2(0,1))$ is an integrable adapted process and

$$M(t):=\int_0^t\sigma_1(\cdot,s,w(s)+\psi(\cdot)\mathbb{1}X(s))\mathrm{d}\mathcal{W}_1(s)$$

is a continuous L^2-martingale (i.e., $M(0)=0$, $\mathbb{E}|M(t)|^2<\infty$ and $\mathbb{E}(M(t)|\mathcal{F}_s)=M(s)$ for all $t\ge s\ge 0$, see Chow (2007, P. 163)) in $L^2(0,1)$. By employing Itô's formula for $e^{2\delta t}V_3(w(t))$ along (2.51) (see Chow (2007, Theorem 7.2.1)), we have

$$\begin{aligned}&e^{2\delta t}V_3(w(t))=V_3(w(0))\\ &+\int_0^t e^{2\delta s}[\mathcal{L}V_3(w(s))+2\delta V_3(w(s))]\mathrm{d}s\\ &+\int_0^t e^{2\delta s}D_wV_3(w(s))\sigma_1(\cdot,s,w(\cdot,s)+\psi(\cdot)\mathbb{1}X(t))\mathrm{d}\mathcal{W}_1(s).\end{aligned} \tag{2.67}$$

Taking expectation on both sides of (2.66) and (2.67) and using the definition $V(t)=V_1(t)-V_2(t)+V_3(w(t))$ (see (2.48)), we arrive at

$$\begin{aligned}&e^{2\delta t}\mathbb{E}V(t)=\mathbb{E}V(0)+\mathbb{E}\int_0^t e^{2\delta s}[\mathcal{L}V(s)+2\delta V(s)]\mathrm{d}s\\ &\overset{(2.63)}{\le}\mathbb{E}V(0),\ t\ge 0,\end{aligned} \tag{2.68}$$

which implies $\mathbb{E}V(t)\le e^{-2\delta t}\mathbb{E}V(0)$, $t\ge 0$. Then (2.39) follows from (2.46) and (2.47).

We show next the feasibility of (2.62) and (2.64) for large enough N and small enough $\bar\sigma_1,\bar\sigma_2$. First, for given $\alpha_1,\alpha_2>0$ and small enough $\bar\sigma_1$, (2.62) holds clearly for large enough N. Note that $|B_1K_{\mathrm T}|\le|B_1\parallel K_{\mathrm T}|\le\|b\|_{L^2}^2|K_{\mathrm T}|$, $|L_0C_1|\le|L_0\parallel C_1|\le|L_0|\cdot O(\sqrt{N})$. By arguments of Theorem 3.3 in Katz and Fridman (2020), we obtain that $P\in\mathbb{R}^{(2N+1)\times(2N+1)}$ which solves the Lyapunov equation

$$P(F+\delta I)+(F+\delta I)^{\mathrm T}P=-\tfrac1N I \tag{2.69}$$

satisfies $\|P\|=O(1)$, uniformly in N.

Next, we estimate $\|b\|_N^2$. Since $\phi_n(x)=\lambda_n^{-1}[q(x)\phi_n(x)-\{p(x)\phi_n'(x)\}']$, $n>1$, by the definition of b_n given in (2.15), we have

$$\begin{aligned}&|b_n|=\lambda_n^{-1}\int_0^1\psi(x)[q(x)\phi_n(x)-\{p(x)\phi_n'(x)\}']\mathrm{d}x\\ &=\lambda_n^{-1}[\int_0^1\psi(x)q(x)\phi_n(x)\mathrm{d}x+\int_0^1p(x)\phi_n'(x)\psi'(x)\mathrm{d}x],\end{aligned} \tag{2.70}$$

where the last equality is obtained from integration by parts and $\psi(0)=\phi_n'(1)=0$. Since $p,\psi\in C^2([0,1])$, $q\in C^1([0,1])$, and $\|\phi_n\|_{[0,1]}=O(1)$, $\|\phi_n'\|_{[0,1]}=O(\lambda_n^{0.5})$ (see Orlov (2017) and Petrovsky (1959, Sec. 23.2)), we obtain from (2.70) that there exists a positive constant M_ψ which is independent of n such that $|b_n|\le\frac{M_\psi}{\sqrt{\lambda_n}}$, $n>1$. Using (1.3) and integral convergence test, we have the following estimate

$$\|b\|_N^2\le\sum_{n=N+1}^{\infty}\frac{M_\psi^2}{\lambda_n}\le\frac{2M_\psi^2}{p_*\pi^2N},\quad N\ge 1. \tag{2.71}$$

Substituting $\alpha_1=0.5$, $\alpha_2=1$, $\rho=N^{1.2}$, $\bar\sigma_1=\bar\sigma_2=N^{-1.2}$, and (2.69) into (2.64) and applying Schur complement, we find that (2.64) hold iff

$$\begin{aligned}&-\tfrac1N I+\tfrac{1}{N^{1.2}}\mathbb{B}^{\mathrm T}\mathbb{B}+\tfrac{\mathcal{C}_1^{\mathrm T}\mathbb{L}_0^{\mathrm T}P\mathbb{L}_0\mathcal{C}_1}{N^{2.4}}\\ &+\tfrac{2N^{1.2}\|b\|_N^2}{\lambda_{N+1}^{0.75}}\mathcal{K}_{\mathrm T}^{\mathrm T}\mathcal{K}_{\mathrm T}+\tfrac{2\|b\|_N^2}{N^{1.2}}\mathbb{1}^{\mathrm T}\mathbb{1}\\ &+\tfrac{(P\mathbb{L}_0+N^{-2.4}\mathcal{C}_1^{\mathrm T}\mathbb{L}_0^{\mathrm T}P\mathbb{L}_0)(\mathbb{L}_0^{\mathrm T}P+N^{-2.4}\mathbb{L}_0^{\mathrm T}P\mathbb{L}_0\mathcal{C}_1)}{N^{1.2}\theta_{N+1}p_*-N^{-2.4}\mathbb{L}_0^{\mathrm T}P\mathbb{L}_0}<0,\\ &P<N^{1.2}I.\end{aligned} \tag{2.72}$$

Since $\|b\|_N^2$ satisfies (2.71), λ_{N+1} satisfies (1.3), $|\mathcal{C}_1|=O(\sqrt{N})$, $\|P\|=O(1)$, $|\mathbb{L}_0|=O(1)$, $N\to\infty$, (2.72) hold for large enough N.

2.1.4. Linear noise

Here we consider the case of linear noise:

$$\begin{aligned}&\sigma_1(x,t,z)=\bar{\sigma}_1 z,\ \forall (x,t,z)\in[0,1]\times\mathbb{R}^+\times\mathbb{R},\\&\sigma_2(t,z)=\bar{\sigma}_2 z,\ \forall (t,z)\in\mathbb{R}^+\times\mathbb{R},\end{aligned}\tag{2.73}$$

where $\bar{\sigma}_1,\bar{\sigma}_2$ are positive constants. In this case, the constraint $P<\rho I$ is not needed (see (2.75)). We have closed-loop system (2.44) with

$$\sigma_{1,n}(t)=\bar{\sigma}_1[w_n(t)+b_n\mathbb{1}X(t)],\ \ \Sigma(t)=\bar{\sigma}_1 GX(t),$$
$$G=\begin{bmatrix}0_{(N_0+1)\times 1}&0&0&0&0\\B_0&I_{N_0}&I_{N_0}&0&0\\0_{(N-N_0)\times 1}&0&0&0&0\\B_1&0&0&I_{N-N_0}&I_{N-N_0}\end{bmatrix}.\tag{2.74}$$

By constructing the Lyapunov function (2.48) and following arguments similar to (2.49)–(2.63) and (2.66)–(2.68), we find that if (2.62) and

$$\begin{aligned}&\Xi_{\mathrm{Lin}}:=\begin{bmatrix}\Xi_1^*+\Xi_2 & P\mathbb{L}_0+\bar{\sigma}_2^2\mathcal{C}_1^{\mathrm{T}}\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\\ * & -\rho\theta_{N+1}p_*+\bar{\sigma}_2^2\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\end{bmatrix}<0,\\&\Xi_1^*=PF+F^{\mathrm{T}}P+2\delta P+\bar{\sigma}_1^2G^{\mathrm{T}}PG\\&\qquad+\bar{\sigma}_2^2\mathcal{C}_1^{\mathrm{T}}\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\mathcal{C}_1,\ \ \Xi_2 \text{ is defined in (2.59).}\end{aligned}\tag{2.75}$$

hold, the mean-square L^2 exponential stability of the closed-loop system can be guaranteed. Moreover, (2.62) and (2.75) are always feasible for small enough $\bar{\sigma}_1,\bar{\sigma}_2$ and large enough N. Differently from the state-feedback case in Section 3.1.2, for the output-feedback case with linear noise we prove the feasibility of LMIs for small noise intensity $\bar{\sigma}_1$.

2.2. Polynomial dynamic extension

Following Katz and Fridman (2021), we employ the following change of variables

$$w(x,t)=z(x,t)-xu(t).\tag{2.76}$$

We treat $u(t)$ as an additional state variable satisfying

$$\dot{u}(t)=v(t),\ \ u(0)=0,\tag{2.77}$$

where v is the new control input. Given $v(t)$, $u(t)$ can be calculated by integrating (2.77). Note that (2.77) implies $w(\cdot,0)=z_0(\cdot)$. Then based on (2.1), (2.3), (2.76), and (2.77), we arrive at the following equivalent systems

$$\mathrm{d}u(t)=v(t)\mathrm{d}t,\ \ t\geq 0,\ \ u(0)=0,\tag{2.78a}$$

$$\begin{aligned}\mathrm{d}w(x,t)=&[\tfrac{\partial}{\partial x}(p(x)\tfrac{\partial}{\partial x}w(x,t))+a(x)u(t)\\&+(q_c-q(x))w(x,t)-b(x)v(t)]\mathrm{d}t\\&+\sigma_1(x,t,w(x,t)+b(x)u(t))\mathrm{d}\mathcal{W}_1(t),\\a(x)=&p'(x)+x(q_c-q(x)),\ b(x)=x,\end{aligned}\tag{2.78b}$$

$$w(0,t)=0,\ \ w_x(1,t)=0,\tag{2.78c}$$

with the noisy boundary measurement output

$$\begin{aligned}\mathrm{d}y(t)=&[w(1,t)+u(t)]\mathrm{d}t\\&+\sigma_2(t,w(1,t)+u(t))\mathrm{d}\mathcal{W}_2(t),\ \ t\geq 0,\end{aligned}\tag{2.79}$$

where σ_2 satisfies (2.4), (2.37). Similar to the well-posedness analysis in Section 2.1.2, we can prove also that for (2.78b) with boundary conditions (2.78c) and initial value $w(\cdot,0)\in L^2(\Omega,L^2(0,1))$ and $w(\cdot,0)\in\mathcal{D}(\mathcal{A}_1)$ almost surely, there exists a unique strong solution w satisfying (2.11). Presenting the solution to (2.78b)–(2.78c) as (2.12), we have $w_n(t)$, $n\geq 1$ satisfy

$$\begin{aligned}\mathrm{d}w_n(t)=&[(-\lambda_n+q_c)w_n(t)+a_nu(t)-b_nv(t)]\mathrm{d}t\\&+\sigma_{1,n}(t)\mathrm{d}\mathcal{W}_1(t),\ \ t\geq 0,\\w_n(0)=&\langle w(\cdot,0),\phi_n\rangle,\\\sigma_{1,n}(t)=&\langle\sigma_1(\cdot,t,\sum_{j=1}^{\infty}w_j(t)\phi_j+b(\cdot)u(t)),\phi_n\rangle,\\a_n=&\langle a,\phi_n\rangle,\ b_n=\langle b,\phi_n\rangle.\end{aligned}\tag{2.80}$$

Recall that $p\in C^2([0,1])$ and $q\in C^1([0,1])$. Hence, the following estimates on $|a_n|$ and $|b_n|$ hold (see Wang et al. (2022)):

$$|a_n|\leq\tfrac{M_1}{\sqrt{\lambda_n}},\ \ 0<|b_n|\leq\tfrac{M_2}{\sqrt{\lambda_n}},\ \ n>1,\tag{2.81}$$

where M_1 and M_2 are some positive constants which are independent of n.

Remark 2.7. In the particular case $p(x)\equiv 1$ and $q(x)\equiv 0$, a_n and b_n can be explicitly obtained by $a_n=(-1)^{n+1}\frac{\sqrt{2}q_c}{\lambda_n}$, $b_n=(-1)^{n+1}\frac{\sqrt{2}}{\lambda_n}$ with λ_n satisfying (1.4), meaning that (2.81) hold with $M_1=\sqrt{2}q_c$ and $M_2=\sqrt{2}$.

Let $\delta>0$ be a desired decay rate and $N_0\in\mathbb{N}$ be such that (2.16) holds. Let $N\in\mathbb{N}$, $N\geq N_0$, where N, N_0 are the dimensions of observer and controller, respectively.

Construct a N-dimensional observer of the form (2.21) with $\hat{w}_n(t)$ satisfying

$$\begin{aligned}\mathrm{d}\hat{w}_n(t)=&[(-\lambda_n+q_c)\hat{w}_n(t)+a_nu(t)-b_nv(t)]\mathrm{d}t\\&+l_n\{[\sum_{j=1}^{N}c_j\hat{w}_j(t)+u(t)]\mathrm{d}t-\mathrm{d}y(t)\},\ \ t\geq 0,\\\hat{w}_n(0)=&0,\ \ 1\leq n\leq N,\end{aligned}\tag{2.82}$$

where $y(t)$ is given by (2.79), $\{c_n\}_{n=1}^N$ are defined in (2.23), $\{l_n\}_{n=1}^N$ are scalar observer gains.

Recall A_0, B_0, C_0 given in (2.23), and let

$$\begin{aligned}&\mathbf{a}_0=[a_1,\ldots,a_{N_0}]^{\mathrm{T}},\ \bar{B}_0=\mathrm{col}\{1,-B_0\},\\&\bar{A}_0=\begin{bmatrix}0&0\\\mathbf{a}_0&A_0\end{bmatrix}\in\mathbb{R}^{(N_0+1)\times(N_0+1)}.\end{aligned}\tag{2.83}$$

Choose $l_1,\ldots,l_{N_0}$ such that $L_0=[l_1,\ldots,l_{N_0}]^{\mathrm{T}}$ satisfies the Lyapunov inequality (2.24). Let $l_n=0$, $n>N_0$. Since $b_n\neq 0$ (see (2.81)), the pair $(\bar{A}_0,\bar{B}_0)$ is controllable. Let $K_{\mathrm{P}}\in\mathbb{R}^{1\times(N_0+1)}$ satisfy

$$P_c(\bar{A}_0+\bar{B}_0K_{\mathrm{P}})+(\bar{A}_0+\bar{B}_0K_{\mathrm{P}})^{\mathrm{T}}P_c\leq -2\delta P_c,\tag{2.84}$$

where $0<P_c\in\mathbb{R}^{(N_0+1)\times(N_0+1)}$. We propose a (N_0+1)-dimensional controller of the form

$$v(t)=K_{\mathrm{P}}\hat{w}^{N_0}(t),\tag{2.85}$$

where $\hat{w}^{N_0}(t)$ is defined in (2.26).

Consider the error system (2.40), (2.42). With notations (2.83), we further denote

$$\begin{aligned}&\mathbf{a}_1=[a_{N_0+1},\ldots,a_N]^{\mathrm{T}},\ \ \mathcal{K}_{\mathrm{P}}=[K_{\mathrm{P}},0_{1\times(2N-N_0)}],\\&\bar{F}=\begin{bmatrix}\bar{A}_0+\bar{B}_0K_{\mathrm{P}}&-\tilde{L}_0C_0&0&-\tilde{L}_0C_1\\0&A_0+L_0C_0&0&L_0C_1\\\mathbf{a}_1\mathbb{1}_0-B_1K_{\mathrm{P}}&0&A_1&0\\0&0&0&A_1\end{bmatrix},\\&\mathcal{C}_2=[1,C_0,C_0,C_1,C_1]\in\mathbb{R}^{1\times(2N+1)}.\end{aligned}\tag{2.86}$$

By (2.42) with $\zeta(t)$ given in (2.41), (2.82), (2.85) and the notations in (2.27), (2.43), (2.86), we have the closed-loop system

$$\begin{aligned}\mathrm{d}X(t)=&[\bar{F}X(t)+\mathbb{L}_0\zeta(t)]\mathrm{d}t+\Sigma(t)\mathrm{d}\mathcal{W}_1(t)\\&+\mathbb{L}_0\sigma_2(t,\zeta(t)+\mathcal{C}_2X(t))\mathrm{d}\mathcal{W}_2(t),\ \ t\geq 0,\\\mathrm{d}w_n(t)=&[(-\lambda_n+q_c)w_n(t)+a_n\mathbb{1}X(t)\\&-b_n\mathcal{K}_{\mathrm{P}}X(t)]\mathrm{d}t+\sigma_{1,n}(t)\mathrm{d}\mathcal{W}_1(t),\ \ n>N.\end{aligned}\tag{2.87}$$

For stability analysis of the closed-loop system (2.87), we consider the Lyapunov function (2.48). Using arguments similar to (2.49)–(2.55), and applying Young's inequality

$$\begin{aligned}&\rho\sum_{n=N+1}^{\infty}2w_n(t)a_n\mathbb{1}X(t)\\&\leq\alpha_3\sum_{n=N+1}^{\infty}\rho\lambda_n^{0.75}w_n^2(t)+\frac{\rho\|a\|_N^2}{\alpha_3\lambda_{N+1}^{0.75}}|\mathbb{1}X(t)|^2,\end{aligned}\tag{2.88}$$

where $\alpha_3 > 0$, we obtain

$$\begin{aligned}&\mathcal{L}V(t)+2\delta V(t)\le X^{\mathrm{T}}(t)(\hat{\Xi}_1+\hat{\Xi}_2)X(t)\\&+2X^{\mathrm{T}}(t)[P\mathbb{L}_0+\bar{\sigma}_2^2\mathcal{C}_2^{\mathrm{T}}\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0]\zeta(t)+\bar{\sigma}_2^2\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\zeta^2(t)\\&+\sum_{n=N+1}^{\infty}2\rho\hat{\Upsilon}_n w_n^2(t)+\Sigma^{\mathrm{T}}(t)(P-\rho I)\Sigma(t).\end{aligned}\tag{2.89}$$

Here

$$\begin{aligned}&\hat{\Upsilon}_n:=-\lambda_n+\tfrac{\alpha_1+\alpha_3}{2}\lambda_n^{0.75}+q_c+\delta+\tfrac{1+\alpha_2}{2}\bar{\sigma}_1^2,\\&\hat{\Xi}_1:=P\bar{F}+\bar{F}^{\mathrm{T}}P+2\delta P+\rho\bar{\sigma}_1^2\mathbb{B}^{\mathrm{T}}\mathbb{B}+\bar{\sigma}_2^2\mathcal{C}_2^{\mathrm{T}}\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\mathcal{C}_2,\\&\hat{\Xi}_2:=\tfrac{\rho}{\lambda_{N+1}^{0.75}}\big(\tfrac{\|a\|_N^2}{\alpha_3}\mathbb{1}^{\mathrm{T}}\mathbb{1}+\tfrac{\|b\|_N^2}{\alpha_1}\mathcal{K}_{\mathrm{P}}^{\mathrm{T}}\mathcal{K}_{\mathrm{P}}\big)\\&\quad+\tfrac{(1+\alpha_2)\rho\bar{\sigma}_1^2\|b\|_N^2}{\alpha_2}\mathbb{1}^{\mathrm{T}}\mathbb{1}.\end{aligned}\tag{2.90}$$

Then with notation

$$\hat{\theta}_n=2-\tfrac{\alpha_1+\alpha_3}{\lambda_n^{0.25}}-\tfrac{2q_c+2\delta+\bar{\sigma}_1^2(1+\alpha_2)}{\lambda_n},\quad n\ge 1,$$

by using (2.60) we have

$$\sum_{n=N+1}^{\infty}2\rho\hat{\Upsilon}_n w_n^2(t)\le-\rho\hat{\theta}_{N+1}p_*\zeta^2(t)\tag{2.91}$$

provided

$$\begin{aligned}\hat{\Upsilon}_{N+1}=&-\lambda_{N+1}+q_c+\delta+\tfrac{1}{2}\bar{\sigma}_1^2\\&+\tfrac{\alpha_1+\alpha_3}{2}\lambda_{N+1}^{0.75}+\tfrac{\alpha_2}{2}\bar{\sigma}_1^2<0.\end{aligned}\tag{2.92}$$

From (2.89) and (2.91) we arrive at

$$\begin{aligned}&\mathcal{L}V(t)+2\delta V(t)\\&\le\eta^{\mathrm{T}}(t)\hat{\Xi}_{\mathrm{NonL}}\eta(t)+\Sigma^{\mathrm{T}}(t)(P-\rho I)\Sigma(t)\le 0\end{aligned}\tag{2.93}$$

if (2.92) and

$$\begin{aligned}&\hat{\Xi}_{\mathrm{NonL}}:=\begin{bmatrix}\hat{\Xi}_1+\hat{\Xi}_2 & P\mathbb{L}_0+\bar{\sigma}_2^2\mathcal{C}_2^{\mathrm{T}}\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\\ * & -\rho\hat{\theta}_{N+1}p_*+\bar{\sigma}_2^2\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\end{bmatrix}<0,\\&P<\rho I,\end{aligned}\tag{2.94}$$

hold, where $\eta(t)$ is given before (2.63) and $\hat{\Xi}_1$, $\hat{\Xi}_2$ are defined in (2.90). By arguments similar to (2.66)–(2.68), feasibility of (2.92) and (2.94) implies, by (2.93) that the solution $u(t)$, $w(x,t)$ to (2.78) subject to the control law (2.82), (2.85) is mean-square L^2 exponentially stable and the corresponding observer $\hat{w}(x,t)$ given by (2.21) satisfies (2.65).

For the feasibility of inequalities (2.92) and (2.94) for large enough N and small enough $\bar{\sigma}_1,\bar{\sigma}_2$, we need explicit upper bound estimates for $\|a\|_N^2$ and $\|b\|_N^2$. From (1.3), (2.81) and the integral convergence test, we arrive at

$$\begin{aligned}&\|a\|_N^2=\sum_{n=N+1}^{\infty}a_n^2\le\frac{2M_1^2}{p_*\pi^2 N},\\&\|b\|_N^2=\sum_{n=N+1}^{\infty}b_n^2\le\frac{2M_2^2}{p_*\pi^2 N},\quad N\ge 1.\end{aligned}\tag{2.95}$$

Then by arguments similar to the proof of Theorem 2.1, the inequalities (2.92) and (2.94) are always feasible provided N is large enough and $\bar{\sigma}_1,\bar{\sigma}_2$ are small enough. Summarizing, we have:

Theorem 2.2. *Consider (2.78) with nonlinear noise function $\sigma_1(x,t,z)$ satisfying (2.2), control law (2.85), noisy boundary measurement (2.79) with $\sigma_2(t,z)$ satisfying (2.4), (2.37), and $w(\cdot,0)\in L^2(\Omega,L^2(0,1))$, $w(\cdot,0)\in\mathcal{D}(\mathcal{A}_1)$ almost surely. Let $\delta>0$ be a desired decay rate, $N_0\in\mathbb{N}$ satisfy (2.16) and $N\in\mathbb{N}$ satisfy $N\ge N_0$. Assume that L_0 and K_{P} are obtained from (2.24) and (2.84), respectively. Let $\alpha_1,\alpha_2,\alpha_3>0$ be subject to (2.92). If there exist a matrix $0<P\in\mathbb{R}^{(2N+1)\times(2N+1)}$ and a scalar $\rho>0$ such that (2.94) hold, then the solution $u(t)$, $w(x,t)$ to (2.78) subject to the control law (2.82), (2.85) is mean-square L^2 exponentially stable and the corresponding observer $\hat{w}(x,t)$ given by (2.21) satisfies (2.65) with some constant $M_0>1$. Moreover, the inequalities (2.92) and (2.94) are always feasible for small enough $\bar{\sigma}_1$, $\bar{\sigma}_2$ and large enough N.*

Remark 2.8. For the case of linear noise where σ_1 and σ_2 are of the form (2.73), we have the closed-loop system (2.87) with $\sigma_{1,n}(t)$ and $\Sigma(t)$ given by (2.74). By constructing the Lyapunov function (2.48) and following arguments similar to (2.49)–(2.63) and (2.66)–(2.68), we find that if (2.92) and

$$\begin{aligned}&\hat{\Xi}_{\mathrm{Lin}}:=\begin{bmatrix}\hat{\Xi}_1^*+\hat{\Xi}_2 & P\mathbb{L}_0+\bar{\sigma}_2^2\mathcal{C}_2^{\mathrm{T}}\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\\ * & -\rho p_*\hat{\theta}_{N+1}+\bar{\sigma}_2^2\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\end{bmatrix}<0,\\&\hat{\Xi}_1^*=P\bar{F}+\bar{F}^{\mathrm{T}}P+2\delta P+\bar{\sigma}_1^2G^{\mathrm{T}}PG\\&\quad+\bar{\sigma}_2^2\mathcal{C}_2^{\mathrm{T}}\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0\mathcal{C}_2,\ \hat{\Xi}_2\text{ is given in (2.90)},\end{aligned}\tag{2.96}$$

hold, the mean-square L^2 exponential stability of the closed-loop system can be guaranteed. Moreover, (2.92) and (2.96) are always feasible for small enough $\bar{\sigma}_1,\bar{\sigma}_2$ and large enough N.

3. State-feedback control

In this section, we consider (2.1) subject to (2.2) and the noisy measurement of the full state. We consider the state-feedback control for two reasons: **(i).** Constructive state-feedback design has not been done yet; **(ii).** Our state-feedback LMI design is used for finding the controller gains in the output-feedback case.

We consider the state-feedback control together with the two kinds of dynamic extensions studied in Sections 2.1 and 2.2, respectively. Let $\delta>0$ be a desired decay rate and let $N_0\in\mathbb{N}$ satisfy (2.16). The state-feedback controller will be constructed by using the first N_0 modes and the additional state variable $u(t)$ (see (2.8), (2.15) for the trigonometric extension and (2.77), (2.80) for the polynomial one).

3.1. Trigonometric dynamic extension

We first consider the modal decomposition method with trigonometric dynamic extension, which is based on the change of variables (2.6) subject to (2.5) and leads to (2.7) with dynamic extension (2.8) and w_n subject to (2.15).

3.1.1. Nonlinear noise

For system (2.9), we consider the state-feedback controller of the form

$$\begin{aligned}&v(t)=\tilde{K}_{\mathrm{T}}y(t),\ \ y(t)=\bar{X}(t)+\bar{\sigma}_2\bar{X}(t)\varsigma_2(t),\\&\bar{X}(t)=\mathrm{col}\{u(t),w_1(t),\ldots,w_{N_0}(t)\},\\&w_n(t)=\langle w(\cdot,t),\phi_n\rangle,\end{aligned}\tag{3.1}$$

where $\tilde{K}_{\mathrm{T}}\in\mathbb{R}^{1\times(N_0+1)}$ is the controller gain which will be obtained from LMIs below, $y(t)$ is the noisy measurement, $\bar{\sigma}_2\bar{X}(t)\varsigma_2(t)$ is the multiplicative random perturbation to $\bar{X}(t)$ with $\bar{\sigma}_2>0$ representing an upper bound on the noise intensity and $\varsigma_2(t)=\frac{\mathrm{d}\mathcal{W}_2(t)}{\mathrm{d}t}$ being a white noise process.

For well-posedness of the closed-loop system (2.9) subject to the control input (3.1), we consider the state $\xi(t)=\mathrm{col}\{u(t),w(\cdot,t)\}$ and $\mathcal{W}(t)=\mathrm{col}\{\mathcal{W}_1(t),\mathcal{W}_2(t)\}$ to obtain the following stochastic evolution equation

$$\mathrm{d}\xi(t)=[\mathcal{A}\xi(t)+f(\xi(t))]\mathrm{d}t+g(\xi(t))\mathrm{d}\mathcal{W}(t),\tag{3.2}$$

with $\mathcal{A}=\mathrm{diag}\{\mathcal{A}_2,-\mathcal{A}_1\}$ where $\mathcal{A}_1$ is given by (1.1), $\mathcal{A}_2=q_c-\mu$, and

$$\begin{aligned}&f(\xi(t))=\begin{bmatrix}0\\ q_c w(\cdot,t)\end{bmatrix}+\begin{bmatrix}1\\ -\psi(\cdot)\end{bmatrix}\tilde{K}_{\mathrm{T}}\begin{bmatrix}u(t)\\ \mathrm{col}\{\langle w(\cdot,t),\phi_n\rangle\}_{n=1}^{N_0}\end{bmatrix},\\&g(\xi(t))=[g_1(\xi(t)),\quad g_2(\xi(t))],\\&g_1(\xi(t))=\begin{bmatrix}0\\ \sigma(\cdot,t,w(\cdot,t)+\psi(\cdot)u(t))\end{bmatrix},\\&g_2(\xi(t))=\bar{\sigma}_2\begin{bmatrix}1\\ -\psi(\cdot)\end{bmatrix}\tilde{K}_{\mathrm{T}}\begin{bmatrix}u(t)\\ \mathrm{col}\{\langle w(\cdot,t),\phi_n\rangle\}_{n=1}^{N_0}\end{bmatrix}.\end{aligned}$$

Define spaces $\mathcal{H}$, $\mathcal{V}$ and $\mathcal{V}'$ as in Section 2.1.2 with $N+1$ therein replaced by 1. Then $\mathcal{A}:\mathcal{V}\to\mathcal{V}'$ is a closed linear operator with domain $\mathcal{D}(\mathcal{A})$ dense in $\mathcal{H}$. For $\xi_1,\xi_2\in\mathcal{V}$, integrating by parts and using the boundary conditions $w(0,t)=w_x(1,t)=0$, we can check that there exist constants $\alpha>0$, $\beta>0$ and γ such that

$$|\langle\mathcal{A}\xi_1,\xi_2\rangle_{\mathcal{V}',\mathcal{V}}|\le\alpha\|\xi_1\|_{\mathcal{V}}\|\xi_2\|_{\mathcal{V}},\\ \langle\mathcal{A}\xi_1,\xi_1\rangle_{\mathcal{V}',\mathcal{V}}\le-\beta\|\xi_1\|^2_{\mathcal{V}}+\gamma\|\xi_1\|^2_{\mathcal{H}}.$$

For any $\xi_1,\xi_2\in\mathcal{H}$, from (2.2) we can check that there exist positive constants κ_1, κ_2 such that (2.33) is satisfied. Then by Chow (2007, Theorem 6.7.4), for any initial value $\xi_0\in L^2(\Omega;\mathcal{H})$ and $\xi_0\in\mathcal{D}(\mathcal{A})$ almost surely, (3.2) has a unique strong solution $\xi\in L^2(\Omega;C([0,T];\mathcal{H}))\cap L^2([0,T]\times\Omega;\mathcal{V})$ such that $\xi(t)\in\mathcal{D}(\mathcal{A})$, $0\le t\le T$, almost surely and is adapted to $\mathcal{F}_t$, $t\ge0$. Thus, we can present the solution as (2.13) with w_n satisfying (2.15).

With notations $\tilde{A}_0$, $\tilde{B}_0$ defined in (2.23) and $\Sigma^{N_0}=[0,\sigma_{1,1}(t),\dots,\sigma_{1,N_0}(t)]^{\mathrm{T}}$, from (2.8), (2.15), and (3.1) we have the following closed-loop system:

$$\begin{aligned}&\mathrm{d}\bar{X}(t)=[\tilde{A}_0+\tilde{B}_0\tilde{K}_{\mathrm{T}}]\bar{X}(t)\mathrm{d}t+\Sigma^{N_0}(t)\mathrm{d}\mathcal{W}_1(t)\\&\qquad+\bar{\sigma}_2\tilde{B}_0\tilde{K}_{\mathrm{T}}\bar{X}(t)\mathrm{d}\mathcal{W}_2(t),\ t\ge0,\\&\mathrm{d}w_n(t)=[(-\lambda_n+q_c)w_n(t)-b_n\tilde{K}_{\mathrm{T}}\bar{X}(t)]\mathrm{d}t\\&\qquad+\sigma_{1,n}(t)\mathrm{d}\mathcal{W}_1(t)-\bar{\sigma}_2b_n\tilde{K}_{\mathrm{T}}\bar{X}(t)\mathrm{d}\mathcal{W}_2(t),\ n>N_0.\end{aligned}\tag{3.3}$$

For the mean-square L^2 exponential stability of the closed-loop system (3.3), we consider the Lyapunov function

$$\begin{aligned}V(t)&=|\bar{X}(t)|^2_P+\rho\sum_{n=N_0+1}^{\infty}w_n^2(t)\\&=|\bar{X}(t)|^2_P-\rho|\mathbb{I}_0\bar{X}(t)|^2+\rho\|w(\cdot,t)\|^2_{L^2},\\&\mathbb{I}_0=[0_{N_0\times1},I_{N_0}].\end{aligned}\tag{3.4}$$

Using arguments similar to (2.49)–(2.55), we have

$$\begin{aligned}&\mathcal{L}V(t)+2\delta V(t)\le\bar{X}^{\mathrm{T}}(t)\Theta_{\mathrm{NonL}}\bar{X}(t)\\&+\Sigma^{\mathrm{T}}(t)(P-\rho I)\Sigma(t)+\sum_{n=N_0+1}^{\infty}2\rho\Upsilon_nw_n^2(t)\le0\end{aligned}\tag{3.5}$$

provided

$$\begin{aligned}\Upsilon_{N_0+1}&=-\lambda_{N_0+1}+q_c+\delta+\tfrac{1}{2}\bar{\sigma}_1^2\\&\quad+\tfrac{\alpha_1}{2}\lambda^{0.75}_{N_0+1}+\tfrac{\alpha_2}{2}\bar{\sigma}_1^2<0\end{aligned}\tag{3.6}$$

and

$$\Theta_{\mathrm{NonL}}:=\Theta_1+\Theta_2<0,\quad P<\rho I\tag{3.7}$$

hold, where

$$\begin{aligned}\Theta_1&=P(\tilde{A}_0+\tilde{B}_0\tilde{K}_{\mathrm{T}})+(\tilde{A}_0+\tilde{B}_0\tilde{K}_{\mathrm{T}})^{\mathrm{T}}P\\&\quad+2\delta P+\rho\bar{\sigma}_1^2\mathbb{B}_0^{\mathrm{T}}\mathbb{B}_0,\\\Theta_2&=\tfrac{\rho\|b\|^2_{N_0}}{\alpha_1\lambda^{0.75}_{N_0+1}}\tilde{K}_{\mathrm{T}}^{\mathrm{T}}\tilde{K}_{\mathrm{T}}+\rho\bar{\sigma}_1^2(1+\tfrac{1}{\alpha_2})\|b\|^2_{N_0}\mathbb{1}_0^{\mathrm{T}}\mathbb{1}_0\\&\quad+\bar{\sigma}_2^2\tilde{K}_{\mathrm{T}}^{\mathrm{T}}\tilde{B}_0^{\mathrm{T}}P\tilde{B}_0\tilde{K}_{\mathrm{T}}+\rho\bar{\sigma}_2^2\|b\|^2_{N_0}\tilde{K}_{\mathrm{T}}^{\mathrm{T}}\tilde{K}_{\mathrm{T}},\\\mathbb{B}_0&=[B_0,I_{N_0}],\quad\mathbb{1}_0=[1,0_{1\times N_0}].\end{aligned}\tag{3.8}$$

Then by arguments similar to (2.66)–(2.68), feasibility of (3.6) and (3.7) implies, by (3.5) the mean-square L^2 exponential stability of the closed-loop system (3.3).

To obtain equivalent LMIs for the design of the gain $\tilde{K}_{\mathrm{T}}$, we multiply Θ_{NonL} from the left and right by P^{-1}. Then, introducing the notations

$$Q=P^{-1},\quad Y=P^{-1}\tilde{K}_{\mathrm{T}}^{\mathrm{T}}=Q\tilde{K}_{\mathrm{T}}^{\mathrm{T}},\quad\bar{\rho}=\rho^{-1}\tag{3.9}$$

and applying Schur complement, we find that (3.7) hold iff

$$\bar{\rho}I<Q\tag{3.10}$$

and

$$\begin{aligned}&\left[\begin{array}{c|ccccc}\chi_{\mathrm{T}_1}&Y&\bar{\sigma}_1Q\mathbb{B}_0^{\mathrm{T}}&\bar{\sigma}_1Q\mathbb{1}_0^{\mathrm{T}}&\bar{\sigma}_2Y\tilde{B}_0^{\mathrm{T}}&\bar{\sigma}_2Y\\\hline *&&&-\chi_{\mathrm{T}_2}&&\end{array}\right]<0,\\&\chi_{\mathrm{T}_1}=\tilde{A}_0Q+Q\tilde{A}_0^{\mathrm{T}}+\tilde{B}_0Y^{\mathrm{T}}+Y\tilde{B}_0^{\mathrm{T}}+2\delta Q,\\&\chi_{\mathrm{T}_2}=\mathrm{diag}\left\{\tfrac{\bar{\rho}\alpha_1\lambda^{0.75}_{N_0+1}}{\|b\|^2_{N_0}},\bar{\rho}I,\tfrac{\alpha_2\bar{\rho}}{(1+\alpha_2)\|b\|^2_{N_0}},Q,\tfrac{\bar{\rho}}{\|b\|^2_{N_0}}\right\},\end{aligned}\tag{3.11}$$

hold. If (3.10) and (3.11) are feasible, the controller gain is obtained by $\tilde{K}_{\mathrm{T}}=Y^{\mathrm{T}}Q^{-1}$.

We show next that inequalities (3.6) and (3.7) are always feasible for small enough $\bar{\sigma}_1,\bar{\sigma}_2>0$ and large enough N_0. Fix $\bar{N}_0$ such that (2.16) holds with N_0 replaced by $\bar{N}_0$. Then fix $\alpha_1,\alpha_2>0$ and let $N_0\ge\bar{N}_0$ such that (3.6) holds. We can rewrite $\tilde{A}_0$ and $\tilde{B}_0$ as

$$\tilde{A}_0=\mathrm{diag}\{\tilde{A}_{0u},\tilde{A}_{0s}\},\quad\tilde{B}_0=\mathrm{col}\{\tilde{B}_{0u},\tilde{B}_{0s}\},$$

such that $\tilde{A}_{0s}\in\mathbb{R}^{(N_0+1-\bar{N}_0)\times(N_0+1-\bar{N}_0)}$ is Hurwitz. Let $\tilde{K}_{\mathrm{T}}$ be of the form

$$\tilde{K}_{\mathrm{T}}=[\hat{K}_{\mathrm{T}},0_{1\times(N_0+1-\bar{N}_0)}]\in\mathbb{R}^{1\times(N_0+1)}.\tag{3.12}$$

We have $\|\tilde{K}_{\mathrm{T}}^{\mathrm{T}}\tilde{K}_{\mathrm{T}}\|=O(1)$, $N_0\to\infty$. Then

$$\tilde{A}_0+\tilde{B}_0\tilde{K}_{\mathrm{T}}=\begin{bmatrix}\tilde{A}_{0u}+\tilde{B}_{0u}\hat{K}_{\mathrm{T}}&0\\\tilde{B}_{0s}\hat{K}_{\mathrm{T}}&\tilde{A}_{0s}\end{bmatrix}.$$

Since the pair $(\tilde{A}_0,\tilde{B}_0)$ is controllable and $\{b_n\}_{n=1}^{\infty}\in\ell^2(\mathbb{N})$, we can obtain from Katz and Fridman (2020, Theorem 3.1) that the solution $0<P\in\mathbb{R}^{(N_0+1)\times(N_0+1)}$ to

$$P(\tilde{A}_0+\tilde{B}_0\tilde{K}_{\mathrm{T}}+\delta I)+(\tilde{A}_0+\tilde{B}_0\tilde{K}_{\mathrm{T}}+\delta I)^{\mathrm{T}}P=-I$$

satisfies $\|P\|=O(1)$ uniformly in N_0. Choose $\rho=N_0^{\gamma}$ and $\bar{\sigma}_1=\bar{\sigma}_2=N_0^{-\gamma}$, $0<\gamma<1$. Substituting P, ρ and $\bar{\sigma}_1$ back into (3.7) we arrive at

$$\begin{aligned}&-I+N_0^{-\gamma}\mathbb{B}_0^{\mathrm{T}}\mathbb{B}_0+\frac{N_0^{\gamma}\|b\|^2_{N_0}\tilde{K}_{\mathrm{T}}^{\mathrm{T}}\tilde{K}_{\mathrm{T}}}{\lambda^{0.75}_{N_0+1}\alpha_1}+N_0^{-\gamma}\|b\|^2_{N_0}\tilde{K}_{\mathrm{T}}^{\mathrm{T}}\tilde{K}_{\mathrm{T}}\\&+N_0^{-\gamma}(1+\tfrac{1}{\alpha_2})\|b\|^2_{N_0}\mathbb{1}_0^{\mathrm{T}}\mathbb{1}_0+N^{-2\gamma}\tilde{K}_{\mathrm{T}}^{\mathrm{T}}\tilde{B}_0^{\mathrm{T}}P\tilde{B}_0\tilde{K}_{\mathrm{T}}<0,\\&P<N_0^{\gamma}I,\end{aligned}$$

where λ_{N_0+1} satisfies (1.3) and $\|b\|^2_{N_0}$ satisfies (2.71). Taking $N_0\to\infty$ we get the feasibility. Summarizing, we have:

Proposition 3.1. *Consider (2.9) with nonlinear noise function $\sigma_1(x,t,z)$ satisfying (2.2), state-feedback controller (3.1), and $w(\cdot,0)\in L^2(\Omega,L^2(0,1))$, $w(\cdot,0)\in\mathcal{D}(\mathcal{A}_1)$ almost surely. Let $\delta>0$ be a desired decay rate and $N_0\in\mathbb{N}$ satisfy (2.16). Let $\alpha_1,\alpha_2>0$ subject to (3.6) and there exist matrices $0<Q\in\mathbb{R}^{(N_0+1)\times(N_0+1)}$, $Y\in\mathbb{R}^{(N_0+1)\times1}$, and a scalar $\bar{\rho}>0$ such that LMIs (3.10) and (3.11) hold. Then the solution $u(t)$, $w(x,t)$ to (2.9) subject to nonlinear noise function $\sigma_1(x,t,z)$ satisfying (2.2) and the control law (3.1) with controller gain $\tilde{K}_{\mathrm{T}}=Y^{\mathrm{T}}Q^{-1}$ is mean-square L^2 exponentially stable. Moreover, (3.6) and (3.7) are always feasible for small enough $\bar{\sigma}_1$, $\bar{\sigma}_2$ and $1/N_0$.*

3.1.2. Linear noise

For the case of linear noise with σ_1 in the form (2.73), we have the closed-loop system (3.3) with

$$\begin{aligned}&\Sigma^{N_0}(t)=\bar{\sigma}_1G_0\bar{X}(t),\quad G_0=\begin{bmatrix}0_{1\times1}&0_{1\times N_0}\\B_0&I_{N_0}\end{bmatrix},\\&\sigma_{1,n}(t)=\bar{\sigma}_1[w_n(t)+b_n\mathbb{1}_0\bar{X}(t)],\end{aligned}\tag{3.13}$$

where $\mathbb{1}_0$ is defined in (3.8). Consider the Lyapunov function (3.4). Similar to the estimate (3.5), we have $\mathcal{L}V(t)+2\delta V(t)\le0$ provided (3.6) and

$$\begin{aligned}&P(\tilde{A}_0+\tilde{B}_0\tilde{K}_{\mathrm{T}})+(\tilde{A}_0+\tilde{B}_0\tilde{K}_{\mathrm{T}})^{\mathrm{T}}P+2\delta P\\&+\bar{\sigma}_1^2G_0^{\mathrm{T}}PG_0+\Theta_2<0,\end{aligned}\tag{3.14}$$

hold, where Θ_2 is defined in (3.8). Since we do not need the condition $P \leq \rho I$, we take ρ as the tuning parameter and α_1, α_2 as the variables. By introducing notations

$$Q = P^{-1},\ Y = P^{-1}\tilde{K}_{\mathrm{T}}^{\mathrm{T}} = Q\tilde{K}_{\mathrm{T}}^{\mathrm{T}}, \tag{3.15}$$

and applying Schur complement, we find that (3.14) holds iff

$$\begin{aligned}
&\left[\begin{array}{c|cccccc} \chi_{\mathrm{T}_1}^* & Y & \bar{\sigma}_1 Q G_0^{\mathrm{T}} & \bar{\sigma}_1 Q \mathbb{1}_0^{\mathrm{T}} & \bar{\sigma}_1 Q \mathbb{1}_0^{\mathrm{T}} & \bar{\sigma}_2 Y \tilde{B}_0^{\mathrm{T}} & \bar{\sigma}_2 Y \\ \hline * & \multicolumn{6}{c}{-\chi_{\mathrm{T}_2}^*} \end{array}\right] < 0,\\
&\chi_{\mathrm{T}_1}^* = \tilde{A}_0 Q + Q\tilde{A}_0^{\mathrm{T}} + \tilde{B}_0 Y^{\mathrm{T}} + Y\tilde{B}_0^{\mathrm{T}} + 2\delta Q,\\
&\chi_{\mathrm{T}_2}^* = \mathrm{diag}\left\{\frac{\alpha_1\lambda_{N_0+1}^{0.75}}{\rho\|b\|_{N_0}^2}, Q, \frac{1}{\rho\|b\|_{N_0}^2}, \frac{\alpha_2}{\rho\|b\|_{N_0}^2}, Q, \frac{1}{\rho\|b\|_{N_0}^2}\right\}.
\end{aligned} \tag{3.16}$$

If (3.6) and (3.16) are feasible, the control gain is obtained by $\tilde{K}_{\mathrm{T}} = Y^{\mathrm{T}}Q^{-1}$.

The triple $(\tilde{A}+\delta I, \bar{\sigma}_1 G_0, \tilde{B}_0)$ is called stabilizable if there exists $\tilde{K}_{\mathrm{T}} \in \mathbb{R}^{1\times(N_0+1)}$ and a $(N_0+1)\times(N_0+1)$ matrix $P > 0$ that satisfy the generalized Lyapunov equation (see Damm (2004, Definition 1.7.1))

$$\begin{aligned}
&P(\tilde{A}_0 + \tilde{B}_0\tilde{K}_{\mathrm{T}} + \delta I) + (\tilde{A}_0 + \tilde{B}_0\tilde{K}_{\mathrm{T}} + \delta I)^{\mathrm{T}}P\\
&+\bar{\sigma}_1^2 G_0^{\mathrm{T}} P G_0 = -I.
\end{aligned} \tag{3.17}$$

Note that the controllability of $(\tilde{A}_0, \tilde{B}_0)$ does not imply stabilizability of $(\tilde{A}_0 + \delta I, \bar{\sigma}_1 G_0, \tilde{B}_0)$ for any $\bar{\sigma}_1$ (see Damm (2004, P. 24)). Little is known about the conditions that guarantee the existence of $P > 0$ that satisfies (3.17) (Zhang & Chen, 2012). However, if the triple $(\tilde{A}_0 + \delta I, \bar{\sigma}_1 G_0, \tilde{B}_0)$ is stabilizable for a certain noise intensity $\bar{\sigma}_1$, then we claim that inequalities (3.6) and (3.14) are feasible for small enough measurement noise $\bar{\sigma}_2$. Fix α_1 and α_2 such that (3.6) holds. Substituting (3.17) into (3.14), we find that (3.14) holds iff

$$\begin{aligned}
&-I + \frac{\rho\|b\|_{N_0}^2 \tilde{K}_{\mathrm{T}}^{\mathrm{T}}\tilde{K}_{\mathrm{T}}}{\alpha_1\lambda_{N_0+1}^{0.75}} + \rho(1+\tfrac{1}{\alpha_2})\bar{\sigma}_1^2\|b\|_{N_0}^2 \mathbb{1}_0^{\mathrm{T}}\mathbb{1}_0\\
&+\bar{\sigma}_2^2\tilde{K}_{\mathrm{T}}^{\mathrm{T}}[\tilde{B}_0^{\mathrm{T}}P\tilde{B}_0 + \rho\|b\|_{N_0}^2 I]\tilde{K}_{\mathrm{T}} < 0.
\end{aligned} \tag{3.18}$$

The latter clearly holds for small enough ρ and $\bar{\sigma}_2$. In addition, increasing the dimension of the controller (3.1) does not deteriorate the performance of the resulting closed-loop system. Indeed, let $\tilde{K}_{\mathrm{T}}$ be obtained from the LMIs, Considering (3.1) with $\tilde{K}_{\mathrm{T}}$ and N_0 replaced by $[\tilde{K}_{\mathrm{T}}, 0]$ and N_0+1, we have the controller $v(t)$ unchanged, which implies that the resulting closed-loop system for $t \geq 0$ is still presented as (3.3). The same Lyapunov function (3.4) leads to LMIs (3.6) and (3.16). Summarizing, we arrive at:

Proposition 3.2. *Consider (2.9) with linear noise perturbation (2.73), state-feedback controller (3.1), and $w(\cdot,0) \in L^2(\Omega, L^2(0,1))$, $w(\cdot,0) \in \mathcal{D}(\mathcal{A}_1)$ almost surely. Let $\delta > 0$ be a desired decay rate and $N_0 \in \mathbb{N}$ satisfy (2.16). Let $\rho > 0$ be given and there exist matrices $0 < Q \in \mathbb{R}^{(N_0+1)\times(N_0+1)}$, $Y \in \mathbb{R}^{(N_0+1)\times 1}$, and scalars $\alpha_1, \alpha_2 > 0$ such that LMIs (3.6) and (3.16) hold. Then the solution $u(t)$, $w(x,t)$ to (2.9) with linear noise (2.73) subject to the control law (3.1) with controller gain $\tilde{K}_{\mathrm{T}} = Y^{\mathrm{T}}Q^{-1}$ is mean-square L^2 exponentially stable. Moreover, for given $\bar{\sigma}_1$ such that the triple $(\tilde{A}_0 + \delta I, \bar{\sigma}_1 G_0, \tilde{B}_0)$ is stabilizable, the LMIs (3.6) and (3.16) are always feasible for small enough $\bar{\sigma}_2$ and ρ. In addition, if (3.6) and (3.16) hold, the increasing dimension of the controller does not deteriorate the performance of the resulting closed-loop system.*

3.2. Polynomial dynamic extension

We proceed with the state-feedback control for system (2.78) using polynomial dynamic extension defined by change of variables (2.76) with dynamic extension (2.77), and leading to the ODEs for w_n given by (2.80).

3.2.1. Nonlinear noise

For system (2.77), (2.80), we consider the state-feedback controller of the form (3.1) with $\tilde{K}_{\mathrm{T}}$ replaced by $\tilde{K}_{\mathrm{P}}$. With the notations $\bar{A}_0$, $\bar{B}_0$ given in (2.83) and $\Sigma^{N_0} = [0, \sigma_1(t), \ldots, \sigma_{N_0}(t)]^{\mathrm{T}}$, we have the following closed-loop system:

$$\begin{aligned}
\mathrm{d}\bar{X}(t) &= [\bar{A}_0 + \bar{B}_0\tilde{K}_{\mathrm{P}}]\bar{X}(t)\mathrm{d}t + \Sigma^{N_0}(t)\mathrm{d}\mathcal{W}_1(t)\\
&\quad + \bar{\sigma}_2\bar{B}_0\tilde{K}_P\bar{X}(t)\mathrm{d}\mathcal{W}_2(t),\ t \geq 0,\\
\mathrm{d}w_n(t) &= [(-\lambda_n + q_c)w_n(t) + a_n\mathbb{1}_0\bar{X}(t)\\
&\quad - b_n\tilde{K}_P\bar{X}(t)\ ]\mathrm{d}t + \sigma_{1,n}(t)\mathrm{d}\mathcal{W}_1(t)\\
&\quad - \bar{\sigma}_2 b_n\tilde{K}_P\bar{X}(t)\mathrm{d}\mathcal{W}_2(t),\ n > N_0,
\end{aligned} \tag{3.19}$$

For the mean-square L^2 exponential stability of (3.19), consider the Lyapunov function (3.4). By arguments similar to (2.49)–(2.55), we have

$$\begin{aligned}
&\mathcal{L}V(t) + 2\delta V(t) \leq \bar{X}^{\mathrm{T}}(t)\hat{\Theta}_{\mathrm{NonL}}\bar{X}(t)\\
&+\Sigma^{\mathrm{T}}(t)(P-\rho I)\Sigma(t) + \sum_{n=N_0+1}^{\infty} 2\rho\hat{\Upsilon}_n w_n^2(t) \leq 0
\end{aligned} \tag{3.20}$$

provided

$$\begin{aligned}
\hat{\Upsilon}_{N_0+1} &= -\lambda_{N_0+1} + q_c + \delta + \tfrac{1}{2}\bar{\sigma}_1^2\\
&\quad + \tfrac{\alpha_1+\alpha_3}{2}\lambda_{N_0+1}^{0.75} + \tfrac{\alpha_2}{2}\bar{\sigma}_1^2 < 0
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\hat{\Theta}_{\mathrm{NonL}} := \hat{\Theta}_1 + \hat{\Theta}_2 < 0,\quad P < \rho I,\\
&\hat{\Theta}_1 = P(\bar{A}_0 + \bar{B}_0\tilde{K}_{\mathrm{P}}) + (\bar{A}_0 + \bar{B}_0\tilde{K}_{\mathrm{P}})^{\mathrm{T}}P\\
&\quad + 2\delta P + \rho\bar{\sigma}_1^2\mathbb{B}_0^{\mathrm{T}}\mathbb{B}_0\\
&\hat{\Theta}_2 = \rho\bar{\sigma}_1^2(1+\tfrac{1}{\alpha_2})\|b\|_{N_0}^2\mathbb{1}_0^{\mathrm{T}}\mathbb{1}_0\\
&\quad + \frac{\rho}{\lambda_{N_0+1}^{0.75}}\left(\frac{\|a\|_{N_0}^2}{\alpha_3}\mathbb{1}_0^{\mathrm{T}}\mathbb{1}_0 + \frac{\|b\|_{N_0}^2}{\alpha_1}\tilde{K}_{\mathrm{P}}^{\mathrm{T}}\tilde{K}_{\mathrm{P}}\right)\\
&\quad + \bar{\sigma}_2^2\tilde{K}_{\mathrm{P}}^{\mathrm{T}}\bar{B}_0^{\mathrm{T}}P\bar{B}_0\tilde{K}_{\mathrm{P}} + \rho\bar{\sigma}_2^2\|b\|_{N_0}^2\tilde{K}_{\mathrm{P}}^{\mathrm{T}}\tilde{K}_{\mathrm{P}},
\end{aligned} \tag{3.22}$$

where $\mathbb{B}_0$ and $\mathbb{1}_0$ are defined in (3.8). Feasibility of (3.20) guarantees the mean-square L^2 exponential stability of the solution $u(t)$, $w(x,t)$ to (2.78) subject to the state-feedback controller (3.1) with $\tilde{K}_{\mathrm{T}}$ replaced by $\tilde{K}_{\mathrm{P}}$. By introducing the notations (3.9) with $\tilde{K}_{\mathrm{T}}$ replaced by $\tilde{K}_{\mathrm{P}}$ and applying Schur complement, we find that (3.22) hold iff

$$\begin{aligned}
&\bar{\rho}I < Q,\\
&\left[\begin{array}{c|cccccc} \chi_{\mathrm{P}_1} & Q\mathbb{1}_0^{\mathrm{T}} & Y & \bar{\sigma}_1 Q\mathbb{B}_0^{\mathrm{T}} & \bar{\sigma}_1 Q\mathbb{1}_0^{\mathrm{T}} & \bar{\sigma}_2 Y\bar{B}_0^{\mathrm{T}} & \bar{\sigma}_2 Y \\ \hline * & \multicolumn{6}{c}{-\chi_{\mathrm{P}_2}} \end{array}\right] < 0,\\
&\chi_{\mathrm{P}_1} = \bar{A}_0 Q + Q\bar{A}_0^{\mathrm{T}} + \bar{B}_0 Y^{\mathrm{T}} + Y\bar{B}_0^{\mathrm{T}} + 2\delta Q,\\
&\chi_{\mathrm{P}_2} = \mathrm{diag}\left\{\frac{\bar{\rho}\alpha_3\lambda_{N_0+1}^{0.75}}{\|a\|_{N_0}^2}, \frac{\bar{\rho}\alpha_1\lambda_{N_0+1}^{0.75}}{\|b\|_{N_0}^2}, \bar{\rho}I, \frac{\alpha_2\bar{\rho}}{(1+\alpha_2)\|b\|_{N_0}^2}, Q, \frac{\bar{\rho}}{\|b\|_{N_0}^2}\right\}.
\end{aligned} \tag{3.23}$$

Moreover, the inequalities (3.21) and (3.22) are always feasible for small enough $\bar{\sigma}_1, \bar{\sigma}_2 > 0$ and $1/N_0$.

3.2.2. Linear noise

For the case of linear noise perturbation with σ_1 in the form (2.73), we have the closed-loop system (3.19) with $\Sigma^{N_0}(t)$ and $\sigma_{1,n}(t)$ given in (3.13). By arguments similar to (3.20), we have that if (3.21) and

$$\begin{aligned}
&P(\bar{A}_0 + \bar{B}_0\tilde{K}_{\mathrm{P}}) + (\bar{A}_0 + \bar{B}_0\tilde{K}_{\mathrm{P}})^{\mathrm{T}}P + 2\delta P\\
&+\bar{\sigma}_1^2 G_0^{\mathrm{T}}PG_0 + \hat{\Theta}_2 < 0,
\end{aligned} \tag{3.24}$$

hold, where $\hat{\Theta}_2$ is defined in (3.22), the mean-square L^2 exponential stability of the closed-loop system can be guaranteed. By introducing notations (3.15) with $\tilde{K}_{\mathrm{T}}$ replaced by $\tilde{K}_{\mathrm{P}}$ and applying

Table 1
Nonlinear noise: $\bar{\sigma}_{\max}^{1}$ for state-feedback control with $\bar{\sigma}_2 \in \{0.1, 0.2\}$ and $N_0 \in \{2, 4, 6, 8, 10, 12\}$: P-DE vs. T-DE.

N_0	2	4	6	8	10	12
$\bar{\sigma}_2 = 0.1$: T-DE	2.793	3.439	3.579	3.623	3.640	3.647
$\bar{\sigma}_2 = 0.1$: P-DE	2.438	3.065	3.231	3.287	3.309	3.318
$\bar{\sigma}_2 = 0.2$: T-DE	2.146	2.452	2.483	2.490	2.492	2.493
$\bar{\sigma}_2 = 0.2$: P-DE	1.714	1.949	1.976	1.982	1.984	1.985

Table 2
Linear noise: $\bar{\sigma}_{\max}^{1}$ for state-feedback control with $N_0 \in \{1, 2, 3, 4, 5, 6\}$.

δ	N_0					
	1	2	3	4	5	6
0.1	5.67	10.54	15.15	19.68	24.18	28.66
1	5.51	10.45	15.09	19.63	24.14	28.63
10	3.52	9.55	14.48	19.17	23.77	28.31

Schur complement, we find that (3.24) holds iff

$$\left[\begin{array}{c|ccccccc} \chi_{P_1}^{*} & Y & \bar{\sigma}_1 Q G_0^{T} & \bar{\sigma}_1 Q \mathbb{1}_0^{T} & \bar{\sigma}_1 Q \mathbb{1}_0^{T} & Q \mathbb{1}_0^{T} & \bar{\sigma}_2 Y \bar{B}_0^{T} & \bar{\sigma}_2 Y \\ \hline * & \multicolumn{7}{c}{-\chi_{P_2}^{*}} \end{array}\right] < 0,$$
$$\chi_{P_1}^{*} = \bar{A}_0 Q + Q \bar{A}_0^{T} + \bar{B}_0 Y^{T} + Y \bar{B}_0^{T} + 2\delta Q, \tag{3.25}$$
$$\chi_{P_2}^{*} = \mathrm{diag}\left\{ \frac{\alpha_1 \lambda_{N_0+1}^{0.75}}{\rho \|b\|_{N_0}^{2}}, Q, \frac{1}{\rho \|b\|_{N_0}^{2}}, \frac{\alpha_2}{\rho \|b\|_{N_0}^{2}}, \frac{\alpha_3 \lambda_{N_0+1}^{0.75}}{\rho \|a\|_{N_0}^{2}}, Q, \frac{1}{\rho \|b\|_{N_0}^{2}} \right\}.$$

If (3.21) and (3.25) are feasible, the control gain is obtained by $\tilde{K}_P = Y^{T} Q^{-1}$. For given $\bar{\sigma}_1 > 0$ such that the triple $(\bar{A}_0 + \delta I, \bar{\sigma}_1 G_0, \bar{B}_0)$ is stabilizable, the feasibility of (3.21) and (3.24) for small enough $\bar{\sigma}_2$ and ρ follows directly from the analysis above Proposition 3.2.

4. Numerical example

In this section, to illustrate the effectiveness of the proposed design method, we consider a 1D rod of length 1 whose one end is maintained at 0° and another end is controlled by the heat flow. Assume that there is an exothermic reaction taking place inside the rod. Then the temperature (denoted by $z(x,t)$) in the rod is modeled as (2.1) with $p(x) \equiv 1$, $q(x) \equiv 0$ (see, e.g., Haussmann (1978) and Wu and Zhang (2020)), where q_c depends on the rate of reaction and the stochastic term $\sigma_1(t, x, z(x,t))\mathrm{d}\mathcal{W}_1(t)$ is due to the random parameter variation of the reaction term $q_c z(x,t)$. We consider $q_c = 6$, which results in an unstable open-loop system in the sense of mean-square stability for any noise intensity.

We start with the boundary state-feedback control studied in Section 3. First, we measure the temperature at the controlled end with the measurement noise intensity bound $\bar{\sigma}_2 = 0.1$ and 0.2, respectively. Take $\alpha_1 = \alpha_3 = 1$, $\alpha_2 = 5$ and $\delta = 0.001$. The LMIs (3.10), (3.11) (via trigonometric dynamic extension (T-DE)) and (3.23) (via polynomial dynamic extension (P-DE)) were verified, respectively, for different values of N_0 to obtain $\bar{\sigma}_{\max}^{1}$ (the maximal value of $\bar{\sigma}_1$) which preserves the feasibility. The results are given in Table 1. From Table 1, we can see that the method via T-DE always allows larger $\bar{\sigma}_{\max}^{1}$ than the method via P-DE.

For linear state-dependent noise with deterministic measurement (i.e., $\bar{\sigma}_2 = 0$), we choose $\rho = 0.1$ and decay rate $\delta \in \{0.1, 1, 10\}$. The LMIs (3.6), (3.16) (via T-DE) and (3.21), (3.25) (via P-DE) were verified, respectively, for different values of N_0 to obtain $\bar{\sigma}_{\max}^{1}$ which preserves the feasibility. The obtained values for T-DE and P-DE are the same and given in Table 2. Compared with Liang and Wu (2022), the merits of our method are that (i) we can manage with any decay rate; (ii) our controller depends on the first N_0 "relatively unstable" modes; (iii) our method is robust with respect to delays.

For simulations of closed-loop system (2.9) subject to state-feedback control (3.1) and closed-loop system (2.78) subject to state-feedback control (3.1) with $\tilde{K}_T$ replaced by $\tilde{K}_P$, choose initial condition $w(x,0) = x - 0.5x^2$, $\sigma_1(x,t,z) = \bar{\sigma}_1 \sin z$ and $\bar{\sigma}_2 = 0.1$. Clearly, (2.2) is satisfied. Take $N_0 = 2$. From Table 1 we have $\bar{\sigma}_{\max}^{1} = 2.793$ for T-DE and $\bar{\sigma}_{\max}^{1} = 2.438$ for P-DE, respectively. The controller gains $\tilde{K}_T$ (obtained from (3.11)) and $\tilde{K}_P$ (obtained from (3.23)) are given by

$$\begin{aligned} \tilde{K}_T &= [40.7622, -413.1891, 40.0737], \\ \tilde{K}_P &= [-271.9261, -405.8638, 38.3947]. \end{aligned} \tag{4.1}$$

By using the FTCS (Forward Time Centered Space) finite-difference scheme and Euler–Maruyama method (see Higham (2001)) with time step 0.001 and space step 0.05, the evolutions of $\mathbb{E}[u^2(t) + \|w(\cdot,t)\|_{L^2}^2]$ and a surface plot of the solution $\mathbb{E}w(x,t)$ are given in Fig. 1 for the T-DE and in Fig. 2 for the P-DE (here and in the following simulations, $\mathbb{E}$ means taking average over 500 sample trajectories). The simulation results confirm our theoretical results. In simulations, stability of the closed-loop system with the same given gains is preserved up to $\bar{\sigma}_{\max}^{1} = 40$ (for T-DE) and $\bar{\sigma}_{\max}^{1} = 37$ (for P-DE), respectively, which may illustrate some conservatism of our method.

We next consider the boundary observer-based control. Consider $\delta = 10$, which results in $N_0 \geq 1$ by (2.16). Take $N_0 = 2$. The observer gain L_0 and controller gains are found from (2.24) and given by

$$\begin{aligned} L_0 &= [-11.3738, -5.2525]^{T}, \\ K_T &= [81.370, -641.700, 5.522], \\ K_P &= [-249.394, -383.730, 20.592]. \end{aligned} \tag{4.2}$$

Table 3
$\bar{\sigma}_{\max}^{1}$ for observer-based control with $N_0 = 2$ and $N \in \{4, 6, 8, 10, 12\}$: T-DE vs. P-DE.

N	Controller gains (4.2)		Controller gains (4.1)	
	T-DE	P-DE	T-DE	P-DE
4	0.734	0.586	0.921	0.825
6	0.823	0.655	0.966	0.881
8	0.839	0.675	0.979	0.904
10	0.846	0.686	0.986	0.918
12	0.850	0.693	0.990	0.927

We choose the controller gains obtained in the state-feedback case (4.1), i.e., $K_T = \tilde{K}_T$, $K_P = \tilde{K}_P$. Take $\alpha_1 = \alpha_3 = 0.7$, $\alpha_2 = 6$. For the deterministic measurement (i.e., $\bar{\sigma}_2 = 0$) the LMIs (2.64) and (2.94) were verified, respectively, with $\delta = 10^{-3}$ and gains (4.1), (4.2) for different values of N to obtain $\bar{\sigma}_{\max}^{1}$ which preserves the feasibility. The results are given in Table 3. For the noisy measurement with $\bar{\sigma}_2 = 0.1$, with the observer gain (4.2), we find that (2.37) holds. Then the LMIs (2.64) and (2.94) were verified, respectively, for different values of N to obtain $\bar{\sigma}_{\max}^{1}$ which preserves the feasibility. The results are given in Table 3. From Table 3, we can see that the method via T-DE always allows larger $\bar{\sigma}_{\max}^{1}$ than the method via P-DE and the state-feedback controller designs allow larger noise than the controller design (2.25) that used in Katz and Fridman (2020, 2021).

For simulations of the closed-loop system with $N_0 = 2$, $N = 4$ and $\sigma_1(t,x,z) = \bar{\sigma}_1 \sin z$, $\sigma_2(t,z) = \bar{\sigma}_2 z$, we have that (2.2) and (2.4) are satisfied. Taking $\bar{\sigma}_2 = 0.1$, from Table 3 we have the upper bounds of $\bar{\sigma}_1$ are 0.921 for the T-DE and 0.825 for the P-DE, respectively. We fix the initial condition $w(x,0) = x - 0.5x^2$. The evolutions of $\mathbb{E}[u^2(t) + \|w(\cdot,t)\|_{L^2}^2]$ and a surface plot of the solution $\mathbb{E}w(x,t)$ are given in Fig. 3 for the T-DE and in Fig. 4 for the P-DE. The numerical simulations validate the theoretical results. In simulations, stability of the closed-loop system is preserved up to $\bar{\sigma}_{\max}^{1} = 28$ for the T-DE and $\bar{\sigma}_{\max}^{1} = 27$ for the P-DE, respectively, that may illustrate the conservatism of our LMI-based conditions.

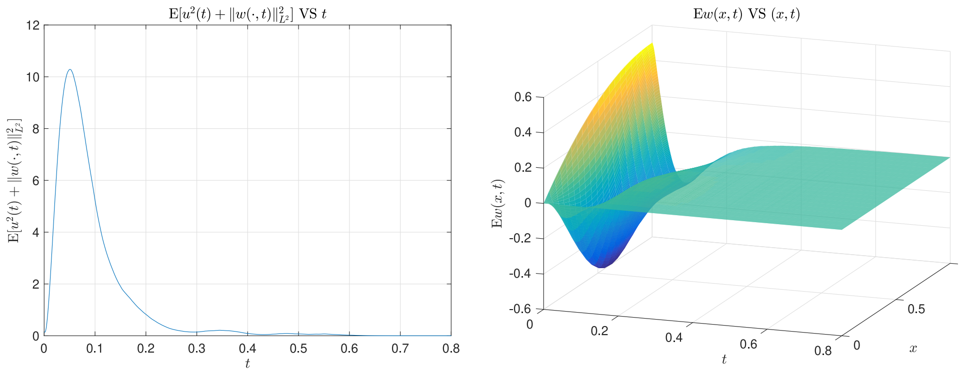


Fig. 1. State-feedback control via T-DE: $\mathbb{E}[u^2(t)+\|w(\cdot,t)\|^2_{L^2}]$ vs. t and $\mathbb{E}w(x,t)$ vs. (x,t).

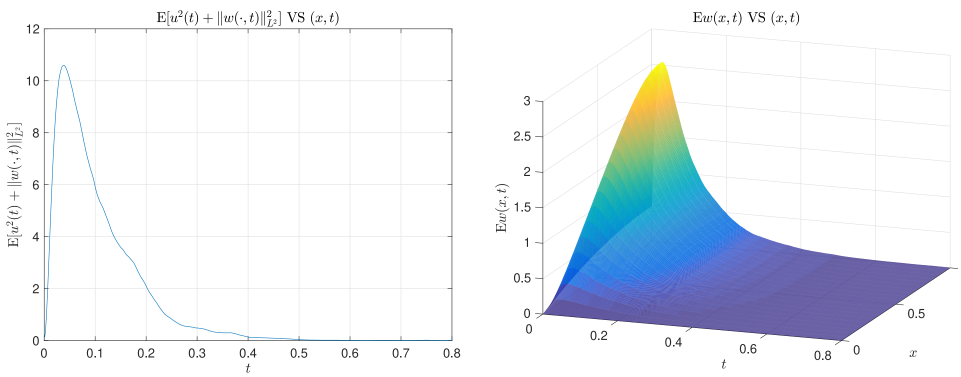


Fig. 2. State-feedback control via P-DE: $\mathbb{E}[u^2(t)+\|w(\cdot,t)\|^2_{L^2}]$ vs. t and $\mathbb{E}w(x,t)$ vs. (x,t).

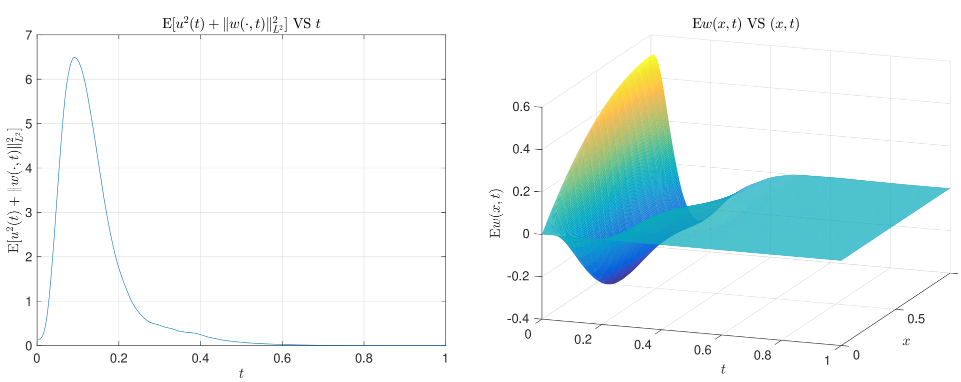


Fig. 3. Observer-based control via T-DE: $\mathbb{E}[u^2(t)+\|w(\cdot,t)\|^2_{L^2}]$ vs. t and $\mathbb{E}w(x,t)$ vs. (x,t).

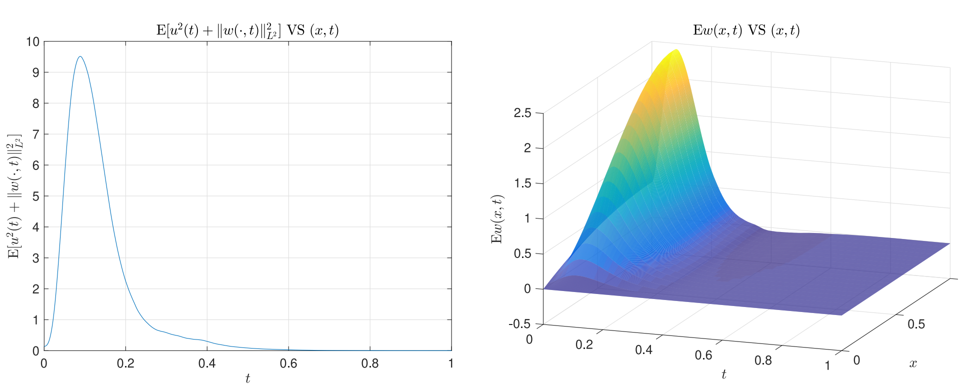


Fig. 4. Observer-based control P-DE: $\mathbb{E}[u^2(t)+\|w(\cdot,t)\|^2_{L^2}]$ vs. t and $\mathbb{E}w(x,t)$ vs. (x,t).

5. Conclusions

This paper presented the first LMI-based method for finite-dimensional observer-based and state-feedback boundary control for stochastic parabolic PDEs via the modal decomposition method. Our Lyapunov stability analysis results in constructive LMI conditions for finding the dimension of observers. The LMIs are accompanied by rigorous feasibility guarantees. The presented method can be extended in the future to various control problems for stochastic PDEs.

References

Barbu, V. (2018). *Controllability and stabilization of parabolic equations*. Springer.

Chow, P.-L. (2007). *Stochastic partial differential equations*. Chapman and Hall/CRC.

Christofides, P. D., Armaou, A., Lou, Y., & Varshney, A. (2008). *Control and optimization of multiscale process systems*. Springer Science & Business Media.

Curtain, R. F., & Zwart, H. (2012). *An introduction to infinite-dimensional linear systems theory, vol. 21*. Springer Science & Business Media.

Da Prato, G., & Zabczyk, J. (2014). *Stochastic equations in infinite dimensions*. Cambridge University Press.

Damm, T. (2004). *Rational matrix equations in stochastic control, vol. 297*. Springer Science & Business Media.

Dragan, V., Morozan, T., & Stoica, A.-M. (2006). *Mathematical methods in robust control of linear stochastic systems, vol. 50*. Springer.

Duan, J., & Wei, W. (2014). *Effective dynamics of stochastic partial differential equations*. Elsevier.

Duncan, T. E., Maslowski, B., & Pasik-Duncan, B. (1994). Adaptive boundary and point control of linear stochastic distributed parameter systems. *SIAM Journal on Control and Optimization*, *32*(3), 648–672.

Falb, P. L. (1967). Infinite-dimensional filtering: The Kalman-Bucy filter in Hilbert space. *Information and Control*, *11*(1-2), 102–137.

Fridman, E., & Blighovsky, A. (2012). Robust sampled-data control of a class of semilinear parabolic systems. *Automatica*, *48*(5), 826–836.

Gershon, E., Shaked, U., & Yaesh, I. (2005). *H-infinity control and estimation of state-multiplicative linear systems, vol. 318*. Springer Science & Business Media.

Hagen, G., & Mezic, I. (2003). Spillover stabilization in finite-dimensional control and observer design for dissipative evolution equations. *SIAM Journal on Control and Optimization*, *42*(2), 746–768.

Haussmann, U. G. (1978). Asymptotic stability of the linear Itô equation in infinite dimensions. *Journal of Mathematical Analysis and Applications*, *65*(1), 219–235.

Higham, D. J. (2001). An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Review*, *43*(3), 525–546.

Hu, G., Lou, Y., & Christofides, P. D. (2008). Dynamic output feedback covariance control of stochastic dissipative partial differential equations. *Chemical Engineering Science*, *63*(18), 4531–4542.

Kang, W., Wang, X.-N., Wu, K.-N., Li, Q., & Liu, Z. (2021). Observer-based $H\infty$ control of a stochastic Korteweg-de Vries-Burgers equation. *International Journal of Robust and Nonlinear Control*, *31*(12), 5943–5961.

Karafyllis, I. (2021). Lyapunov-based boundary feedback design for parabolic PDEs. *International Journal of Control*, *94*(5), 1247–1260.

Katz, R., & Fridman, E. (2020). Constructive method for finite-dimensional observer-based control of 1-D parabolic PDEs. *Automatica*, *122*, Article 109285.

Katz, R., & Fridman, E. (2021). Finite-dimensional control of the heat equation: Dirichlet actuation and point measurement. *European Journal of Control*, *62*, 158–164.

Katz, R., & Fridman, E. (2023). Global stabilization of a 1D semilinear heat equation via modal decomposition and direct Lyapunov approach. *Automatica*.

Klebaner, F. C. (2005). *Introduction to stochastic calculus with applications*. World Scientific Publishing Company.

Lhachemi, H., & Prieur, C. (2022). Finite-dimensional observer-based boundary stabilization of reaction–diffusion equations with either a Dirichlet or Neumann boundary measurement. *Automatica*, *135*, Article 109955.

Liang, S., & Wu, K.-N. (2022). Boundary control of stochastic Korteweg-de Vries-Burgers equations. *Nonlinear Dynamics*, *108*(4), 4093–4102.

Liu, K. (2005). *Stability of infinite dimensional stochastic differential equations with applications*. CRC Press.

Lü, Q., & Zhang, X. (2021). *Mathematical control theory for stochastic partial differential equations*. Springer.

Mao, X. (2007). *Stochastic differential equations and applications*. Elsevier.

Munteanu, I. (2018). Boundary stabilization of the stochastic heat equation by proportional feedbacks. *Automatica*, *87*, 152–158.

Munteanu, I. (2019). Exponential stabilization of the stochastic Burgers equation by boundary proportional feedback. *Discrete & Continuous Dynamical Systems*, *39*(4), 2173.

Murray, R. M. (2003). *Control in an information rich world: Report of the panel on future directions in control, dynamics, and systems*. SIAM.

Muscat, J. (2014). *Functional analysis: An introduction to metric spaces, hilbert spaces, and banach algebras*. Springer.

Orlov, Y. (2017). On general properties of eigenvalues and eigenfunctions of a Sturm–Liouville operator: comments on ‘‘ISS with respect to boundary disturbances for 1-D parabolic PDEs’’. *IEEE Transactions on Automatic Control*, *62*(11), 5970–5973.

Petrovsky, I. G. (1959). *Lectures on partial differential equations*. Courier Corporation.

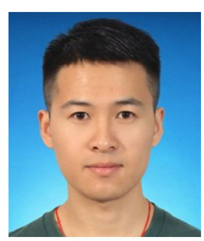
Prieur, C., & Trélat, E. (2019). Feedback stabilization of a 1-D linear reaction-diffusion equation with delay boundary control. *IEEE Transactions on Automatic Control*, *64*(4), 1415–1425.

Selivanov, A., & Fridman, E. (2019). Boundary observers for a reaction-diffusion system under time-delayed and sampled-data measurements. *IEEE Transactions on Automatic Control*, *64*(8), 3385–3390.

Wang, P., Katz, R., & Fridman, E. (2022). Constructive finite-dimensional observer-based boundary control of stochastic parabolic PDEs. In *2022 American control conference*, IEEE.

Wu, H.-N., & Zhang, X.-M. (2020). Exponential stabilization for 1-D linear Itô-type state-dependent stochastic parabolic PDE systems via static output feedback. *Automatica*, *121*, Article 109173.

Zhang, W., & Chen, B.-S. (2012). $\mathcal{H}$-Representation and applications to generalized Lyapunov equations and linear stochastic systems. *IEEE Transactions on Automatic Control*, *57*(12), 3009–3022.



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