



Extremum seeking via a time-delay approach to averaging[☆]

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ABSTRACT

In this paper, we present a constructive approach to extremum seeking (ES) by using a time-delay approach to averaging. We consider gradient-based ES of static maps in the case of one and two variables, and we study two ES methods: the classical one and a more recent bounded ES method. By transforming the ES dynamics into a time-delay system where the delay is the period of averaging, we derive the practical stability conditions for the resulting time-delay system. The time-delay system stability guarantees the stability of the original ES plant. Under assumption of some known bounds on the extremum point, the extremum value and the Hessian, the time-delay approach provides a quantitative calculation on the lower bound of the frequency and on the upper bound of the resulting ultimate bound. We also give a bound on the neighborhood of the extremum point starting from which the solution is ultimately bounded. When the extremum value is unknown, we provide, for the first time, the asymptotic ultimate bound in terms of the frequency in the case of bounded ES. Moreover, our explicit bound on the seeking error of ES control systems allows to select appropriate tuning parameters (such as dither frequency, magnitude, and control gain). Two numerical examples illustrate the efficiency of our method. Particularly, our quantitative bounds are more efficient for the classical ES than for the bounded one. However, the latter bounds correspond to a more general case with unknown extremum value.

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1. Introduction

Extremum Seeking (ES) is a powerful real-time optimization method without requesting a knowledge of system model. Although the first ES-based power transfer mechanism might date back to 1922 (Tan, Moase, Manzie, Netic, & Mareels, 2010), it often requires the advanced tools to understand the behavior of ES plants from the viewpoint of system theory. The publication (Krstic & Wang, 2000) in 2000 gave the first stability analysis of extremum seeking in a rigorous way, which shed new light on ES by making use of averaging theory.

Since the turn of the 21st century, ES has greeted its golden age, both in terms of the development of theory (Ariyur & Krstic, 2003; Guay & Dochain, 2014; Liu & Krstic, 2012; Scheinker & Krstic, 2017) and its application in industry (Cochran, Kanso,

Kelly, Xiong & Krstic, 2009; Cochran, Siranosian, Ghods & Krstic, 2009; Ye & Hu, 2016). A proof that expands the feasibility of extremum seeking from local to global stability was given in Tan, Netic, and Mareels (2006). The article (Durr, Stankovic, Ebenbauer, & Johansson, 2013) introduced an interpretation of extremum seeking by using Lie bracket approximation. A Newton-based ES algorithm, which overcomes the shortcoming that the convergence rate depends upon the unknown Hessian, was proposed by Ghaffari, Krstic, and Netic (2012). In Adetola and Guay (2007), Guay and Dochain (2015), Guay and Zhang (2003), the authors solved adaptive ES control problems for nonlinear systems with uncertain parameters. The Refs. Choi, Krstic, Ariyur, and Lee (2002), Khong, Netic, Tan, and Manzie (2013), Poveda and A. (2017) applied ES to discrete-time, sampled-data and hybrid systems, respectively. In Feiling, Koga, Krstic, and Oliveira (2018), Oliveira, Feiling, Koga, and Krstic (2020), Oliveira, Krstic, and Tsubakino (2016), the authors addressed ES control for PDE dynamics and time-delay systems (Zhu & Fridman, 2020; Zhu & Krstic, 2020). The publications (Frihauf, Krstic, & Basar, 2011; Guay, Vandermeulen, Dougherty, & McLellan, 2018; Ye & Hu, 2017; Ye, Hu, & Xu, 2020) considered distributed ES for coupled or non-coupled networks and games.

A majority of existing literature employ the averaging theory to prove the stability of ES systems. It is well-known that ES

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control systems are practically stable provided that the dither frequency is large enough whereas the dither magnitude is sufficiently small. The existing methods for asymptotic averaging guarantee the stability of the system for small enough values of the parameter provided the averaged system is stable (Ariyur & Krstic, 2003; Khalil, 2002). However, these methods do not suggest quantitative upper bounds on the parameter that preserves the stability. Recently a new constructive approach to periodic averaging was presented in Fridman and Zhang (2020) with efficient bounds on the small parameter that preserves the stability of the original system provided the averaged system is stable. In Fridman and Zhang (2020) backward averaging of the original system is suggested, and the resulting system is presented as a time-delay system. Then the corresponding Lyapunov functionals lead to linear matrix inequalities (LMI) conditions for an upper bound on the small parameter that preserves the stability.

In this paper, motivated by Fridman and Zhang (2020), we propose a constructive time-delay approach for stability analysis of gradient-based ES algorithms in the case of static maps. We convert the ES dynamics into a model with time-delay. The delay length is equal to the minimal period of the dither signals. The stability of the resulting time-delay system guarantees the stability of the original ES plant. We construct a Lyapunov functional to find sufficient practical stability conditions in the form of LMIs. Through the solution of LMIs, we find lower bounds on the dither frequency that guarantee the practical stability. Different from the conventional averaging method (Ariyur & Krstic, 2003; Krstic & Wang, 2000; Scheinker & Krstic, 2014, 2017) which provides a qualitative analysis, the time-delay based method gives a quantitative calculation of the ultimate bound of seeking error. The expression of the error bound in an accurate manner is intuitional in the performance evaluation of ES algorithms and provides useful details for the selection of user-assignable parameters (see Remark 1).

A conference version will be presented in Zhu and Fridman (2021), where the gradient-based ES and bounded ES in the scalar case are considered.

The paper's rest organization is as follows: In Sections 2 and 3, we apply the time-delay approach to the gradient-based classical ES and bounded ES (as introduced in Scheinker and Krstic (2014, 2017)) in the case of static maps, respectively. Each section consists of two subsections: single-variable static map and two-variable static map. Section 4 provides examples with simulation results, and Section 5 summarizes some conclusions.

Before ending Section 1, we introduce the following lemma on Jensen's inequality (see (1)) and its extended version (see (2)), which will be used in later sections:

Lemma 1 (Fridman, 2014a, 2014b; Solomon & Fridman, 2013). For any $n \times n$ matrix $R > 0$, scalars $\alpha \leq \beta$, functions $\kappa(\tau) : [\alpha, \beta] \rightarrow \mathbb{R}$ and $\Phi(\tau) : [\alpha, \beta] \rightarrow \mathbb{R}^n$, such that the integrations concerned are well defined, the following hold:

$$\int_{\alpha}^{\beta} \Phi^T(\tau) d\tau R \int_{\alpha}^{\beta} \Phi(\tau) d\tau \leq (\beta - \alpha) \int_{\alpha}^{\beta} \Phi^T(\tau) R \Phi(\tau) d\tau, \quad (1)$$

and

$$\begin{aligned} & \int_{\alpha}^{\beta} \kappa(\tau) \Phi^T(\tau) d\tau R \int_{\alpha}^{\beta} \kappa(\tau) \Phi(\tau) d\tau \\ & \leq \int_{\alpha}^{\beta} |\kappa(\tau)| d\tau \int_{\alpha}^{\beta} |\kappa(\tau)| \Phi^T(\tau) R \Phi(\tau) d\tau. \end{aligned} \quad (2)$$

Proof. see Fridman (2014a, Page 87) and Solomon and Fridman (2013, Page 3469).

2. A time-delay approach to classical ES

For conceptional clearness, we first apply a time-delay approach to gradient-based classical ES.

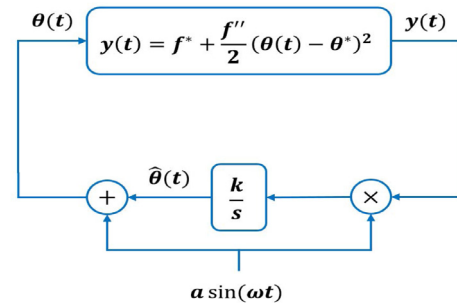


Fig. 1. Classical ES for a single-variable static map.

2.1. Scalar systems

Consider single-variable static maps of the quadratic form as follows (Ariyur & Krstic, 2003; Scheinker & Krstic, 2017):

$$y(t) = f(\theta(t)) = f^* + \frac{f''}{2} (\theta(t) - \theta^*)^2, \quad (3)$$

where $y(t) \in \mathbb{R}$ is the measurable output, $\theta(t) \in \mathbb{R}$ is the scalar input, f^* and θ^* are constants, f'' is the gradient which is a non-zero constant. It is seen that the quadratic map (3) has a maximum or minimum value $y(t) = f^*$ at $\theta(t) = \theta^*$ such that

$$\begin{cases} \frac{\partial f}{\partial \theta} |_{\theta=\theta^*} = 0, \\ \frac{\partial^2 f}{\partial \theta^2} |_{\theta=\theta^*} = f'' < 0 \quad \text{or} \quad > 0. \end{cases}$$

Usually, the cost function (3) is unknown, but the sign of Hessian f'' is known. In the present paper, in order to derive efficient LMI conditions, we assume the extremum point θ^* to be sought is uncertain from a known interval $\theta^* \in [\underline{\theta}^*, \bar{\theta}^*]$ with $\bar{\theta}^* - \underline{\theta}^* = \sigma_0$, whereas the extremum value f^* and the Hessian f'' are known.

As clarified in Ariyur and Krstic (2003), smooth function $f(\theta(t))$ in many cases can be approximated locally by the quadratic map (3). We define the real-time estimate $\hat{\theta}(t)$ of θ^* with the estimation error

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*. \quad (4)$$

The purpose of ES is to render the error towards zero. As illustrated in Fig. 1, the gradient-based classical ES algorithm is selected as follows (Ariyur & Krstic, 2003):

$$\begin{aligned} \theta(t) &= \hat{\theta}(t) + a \sin(\omega t), \\ \dot{\hat{\theta}}(t) &= k \cdot a \sin(\omega t) \cdot y(t), \\ &= ka \sin(\omega t) \left[f^* + \frac{f''}{2} \left(\hat{\theta}(t) + a \sin(\omega t) - \theta^* \right)^2 \right], \end{aligned} \quad (5)$$

where $\hat{\theta}(0) \in [\underline{\theta}^*, \bar{\theta}^*]$, a and ω are the amplitude and frequency of the dither signal, respectively, k is the adaptation gain satisfying $\text{sgn}(k) = -\text{sgn}(f'')$.

From (4)–(5), the estimation error is governed by

$$\begin{aligned} \dot{\tilde{\theta}}(t) &= ka \sin(\omega t) \left[f^* + \frac{f''}{2} \left(\tilde{\theta}(t) + a \sin(\omega t) \right)^2 \right], \\ &= \frac{ka^2 f''}{2} [1 - \cos(2\omega t)] \tilde{\theta}(t) + \frac{kaf''}{2} \sin(\omega t) \tilde{\theta}^2(t) \\ &\quad + kaf^* \sin(\omega t) + \frac{ka^3 f''}{2} \sin^3(\omega t). \end{aligned} \quad (6)$$

To analyze the ES control system (6), the existing literature on ES start (and finish) with the derivation of the averaged system (by Lie brackets, etc.) (Ariyur & Krstic, 2003; Durr et al., 2013; Krstic & Wang, 2000) and then apply the theorem on averaging (Khalil, 2002). To be specific, defining $\omega = \frac{2\pi}{\epsilon}$, the backward

averaged system of (6) (which is consistent with Assumption A1 in Fridman and Zhang (2020)) is derived as follows:

$$\begin{aligned} \dot{\tilde{\theta}}_{av}(t) &= \frac{ka^2f''}{2} \frac{1}{\varepsilon} \int_{- \varepsilon}^0 [1 - \cos(\frac{4\pi}{\varepsilon}\tau)] d\tau \tilde{\theta}_{av}(t) \\ &+ \frac{kaf''}{2} \frac{1}{\varepsilon} \int_{- \varepsilon}^0 \sin(\frac{2\pi}{\varepsilon}\tau) d\tau \tilde{\theta}_{av}^2(t) \\ &+ kaf'' \frac{1}{\varepsilon} \int_{- \varepsilon}^0 \sin(\frac{2\pi}{\varepsilon}\tau) d\tau + \frac{ka^3f''}{2} \frac{1}{\varepsilon} \int_{- \varepsilon}^0 \sin^3(\frac{2\pi}{\varepsilon}\tau) d\tau, \\ &= \frac{ka^2f''}{2} \tilde{\theta}_{av}(t), \end{aligned} \tag{7}$$

where we utilize the averaging

$$\begin{aligned} \int_{- \varepsilon}^0 \cos(\frac{4\pi}{\varepsilon}\tau) d\tau &= 0, \quad \int_{- \varepsilon}^0 \sin(\frac{2\pi}{\varepsilon}\tau) d\tau = 0, \\ \int_{- \varepsilon}^0 \sin^3(\frac{2\pi}{\varepsilon}\tau) d\tau &= 0. \end{aligned} \tag{8}$$

As clarified in Khalil (2002, Chapter 10.4), the essential problem in the averaging method is to determine in what sense the behavior of the averaged system (7) approximates the behavior of the original system (6), which may not be intuitively clear.

Inspired by Fridman and Zhang (2020), we apply the time-delay method to averaging of (6). We integrate (6) in $t \geq \varepsilon$ from $t - \varepsilon$ to t . Taking into account (8), we have

$$\begin{aligned} \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \dot{\tilde{\theta}}(\tau) d\tau &= \frac{ka^2f''}{2\varepsilon} \int_{t-\varepsilon}^t \tilde{\theta}(\tau) d\tau \\ &- \frac{ka^2f''}{2\varepsilon} \int_{t-\varepsilon}^t \cos(\frac{4\pi}{\varepsilon}\tau) \tilde{\theta}(\tau) d\tau \\ &+ \frac{kaf''}{2\varepsilon} \int_{t-\varepsilon}^t \sin(\frac{2\pi}{\varepsilon}\tau) \tilde{\theta}^2(\tau) d\tau. \end{aligned} \tag{9}$$

In the remainder of this paper when we write $x \pm y$ we mean $x + y - y$, not the set $\{x + y, x - y\}$. To handle the first term on the right-hand side of (9), we have

$$\begin{aligned} \frac{ka^2f''}{2\varepsilon} \int_{t-\varepsilon}^t \tilde{\theta}(\tau) d\tau &= \frac{ka^2f''}{2\varepsilon} \int_{t-\varepsilon}^t [\tilde{\theta}(\tau) \pm \tilde{\theta}(t)] d\tau, \\ &= \frac{ka^2f''}{2} \tilde{\theta}(t) - \frac{ka^2f''}{2\varepsilon} \int_{t-\varepsilon}^t [\tilde{\theta}(t) - \tilde{\theta}(\tau)] d\tau, \\ &= \frac{ka^2f''}{2} \tilde{\theta}(t) - \frac{ka^2f''}{2\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t \dot{\tilde{\theta}}(s) ds d\tau. \end{aligned} \tag{10}$$

where we employed the equality $\tilde{\theta}(t) - \tilde{\theta}(\tau) = \int_{\tau}^t \dot{\tilde{\theta}}(s) ds$. If $\dot{\tilde{\theta}}(s)$ is bounded, then the double-integral term $-\frac{1}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t \dot{\tilde{\theta}}(s) ds d\tau$ is of the order of $O(\varepsilon)$. Thus, as long as ε is chosen small enough, this double-integral term is close to zero. To address the second term on the right-hand side of (9), we employ the fact that $\int_{t-\varepsilon}^t \cos(\frac{4\pi}{\varepsilon}\tau) d\tau \cdot \tilde{\theta}(t) = 0$ that leads to

$$\begin{aligned} &-\frac{ka^2f''}{2\varepsilon} \int_{t-\varepsilon}^t \cos(\frac{4\pi}{\varepsilon}\tau) \tilde{\theta}(\tau) d\tau \\ &= \frac{ka^2f''}{2\varepsilon} \int_{t-\varepsilon}^t \cos(\frac{4\pi}{\varepsilon}\tau) [\tilde{\theta}(t) - \tilde{\theta}(\tau)] d\tau, \\ &= \frac{ka^2f''}{2\varepsilon} \int_{t-\varepsilon}^t \cos(\frac{4\pi}{\varepsilon}\tau) \int_{\tau}^t \dot{\tilde{\theta}}(s) ds d\tau. \end{aligned} \tag{11}$$

For the third term on the right-hand side of (9) we have

$$\begin{aligned} &\frac{kaf''}{2\varepsilon} \int_{t-\varepsilon}^t \sin(\frac{2\pi}{\varepsilon}\tau) \tilde{\theta}^2(\tau) d\tau \\ &= -\frac{kaf''}{2\varepsilon} \int_{t-\varepsilon}^t \sin(\frac{2\pi}{\varepsilon}\tau) [\tilde{\theta}^2(t) - \tilde{\theta}^2(\tau)] d\tau, \\ &= -\frac{kaf''}{\varepsilon} \int_{t-\varepsilon}^t \sin(\frac{2\pi}{\varepsilon}\tau) \int_{\tau}^t \tilde{\theta}(s) \dot{\tilde{\theta}}(s) ds d\tau, \end{aligned} \tag{12}$$

since $\int_{t-\varepsilon}^t \sin(\frac{2\pi}{\varepsilon}\tau) d\tau \cdot \tilde{\theta}^2(t) = 0$. We have utilized the equality $\tilde{\theta}^2(t) - \tilde{\theta}^2(\tau) = 2 \int_{\tau}^t \tilde{\theta}(s) \dot{\tilde{\theta}}(s) ds$. Note that the above techniques used in (10)–(12) will be frequently employed throughout the article.

Denoting

$$\begin{aligned} G(t) &= \frac{1}{\varepsilon} \int_{t-\varepsilon}^t (\tau - t + \varepsilon) \dot{\tilde{\theta}}(\tau) d\tau, \\ Y_1(t) &= \frac{ka^2}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t [1 - \cos(\frac{4\pi}{\varepsilon}\tau)] \dot{\tilde{\theta}}(s) ds d\tau, \\ Y_2(t) &= \frac{ka}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t \sin(\frac{2\pi}{\varepsilon}\tau) \tilde{\theta}(s) \dot{\tilde{\theta}}(s) ds d\tau, \end{aligned} \tag{13}$$

and employing the relation (Fridman & Zhang, 2020)

$$\frac{1}{\varepsilon} \int_{t-\varepsilon}^t \dot{\tilde{\theta}}(\tau) d\tau = \frac{d}{dt} [\tilde{\theta}(t) - G(t)], \tag{14}$$

we present the closed-loop system as

$$\begin{aligned} \frac{d}{dt} [\tilde{\theta}(t) - G(t)] &= \frac{ka^2f''}{2} \tilde{\theta}(t) - \frac{f''}{2} Y_1(t) - f'' Y_2(t), \\ t &\geq \varepsilon. \end{aligned} \tag{15}$$

where $\dot{\tilde{\theta}}(t)$ is defined by the right-hand side of (6).

If we substitute to (15) the right-hand side of (6), we have a differential equation with delays. That is to say, the system (6) has been transformed into the time-delay system (15) for $t \geq \varepsilon$, which is a perturbation of the stable averaged system (7). Note that, if $\tilde{\theta}(t)$ and $\dot{\tilde{\theta}}(t)$ are bounded, then the integral terms $G(t)$, $Y_1(t)$, $Y_2(t)$ are of the order of $O(\varepsilon)$. The plant (15) is a kind of neutral type system that depends on the past values of $\tilde{\theta}(s)$, $\dot{\tilde{\theta}}(s)$, $s \in [t - \varepsilon, t]$. The solution $\tilde{\theta}(t)$ of the system (6) is also a solution of the time-delay system (15). Hence the stability of the time-delay system guarantees the stability of the original delay-free ES system.

Theorem 1. Assume that the Hessian f'' and the extremum value f^* are known, the uncertain extremum point θ^* belongs to the known interval $[\underline{\theta}^*, \bar{\theta}^*]$. Consider the closed-loop system consisting of the scalar plant (3) and classical ES controller (5), with the initial condition $|\tilde{\theta}(0)| \leq \sigma_0$. Given tuning parameters k, a and $q, \delta, \varepsilon^* > 0$ as well as $\sigma > \sigma_0$, let scalars $P > 1$ and $R, \gamma_1, \gamma_2 > 0$ satisfy the LMIs:

$$\begin{aligned} \Phi_1 &= \begin{bmatrix} P-1 & -P \\ * & P+e^{-2\delta\varepsilon^*}R \end{bmatrix} > 0, \\ \Phi_2 &= \begin{bmatrix} (ka^2f''+2\delta)P & -\frac{(ka^2f''+4\delta)P}{2} & -\frac{f''}{2}P & -f''P \\ * & -\frac{4}{\varepsilon^*}e^{-2\delta\varepsilon^*}R+2\delta P & \frac{f''}{2}P & f''P \\ * & * & -\frac{\gamma_1}{\varepsilon^*} & 0 \\ * & * & * & -\frac{\gamma_2}{\varepsilon^*} \end{bmatrix} < 0, \end{aligned} \tag{16}$$

$$\begin{aligned} \Phi_3 &= \left(1 + \frac{1}{q}\right)^2 P \sigma_0^2 \\ &+ \varepsilon^{*2} \left[\left(1 + \frac{1}{q}\right) (1 + q)P + \frac{1+q}{4}P + \frac{1}{3}R \right] \Delta^2 \\ &+ \frac{\varepsilon^*}{2\delta} (\gamma_1 k^2 a^4 + \frac{\gamma_2}{4} k^2 a^2 \sigma^2 + R) \Delta^2 < \sigma^2, \\ \Phi_4 &= \sigma_0 + \varepsilon^* \Delta < \sigma, \end{aligned}$$

where

$$\Delta = |ka| \bar{\Delta}, \quad \bar{\Delta} = |f^*| + \frac{|f''|}{2} (\sigma + |a|)^2. \tag{17}$$

Then $\forall \varepsilon \in (0, \varepsilon^*]$ the solution of the closed-loop system (6) satisfies

$$\begin{aligned} \tilde{\theta}^2(t) &< (|\tilde{\theta}(0)| + \varepsilon \Delta)^2 < \sigma^2, \quad t \in [0, \varepsilon], \\ \tilde{\theta}^2(t) &< \left(1 + \frac{1}{q}\right)^2 P e^{-2\delta(t-\varepsilon)} \tilde{\theta}^2(0) + \varepsilon^2 e^{-2\delta(t-\varepsilon)} \\ &\times \left[\left(1 + \frac{1}{q}\right) (1 + q)P + \frac{1+q}{4}P + \frac{1}{3}R \right] \Delta^2 \\ &+ (1 - e^{-2\delta(t-\varepsilon)}) \\ &\times \frac{\varepsilon}{2\delta} (\gamma_1 k^2 a^4 + \frac{\gamma_2}{4} k^2 a^2 \sigma^2 + R) \Delta^2 < \sigma^2, \quad t \geq \varepsilon. \end{aligned} \tag{18}$$

Moreover, for all $\varepsilon \in (0, \varepsilon^*]$ and all initial conditions $|\tilde{\theta}(0)| \leq \sigma_0$ the ball

$$\Theta = \left\{ \tilde{\theta} \in \mathbb{R} : \tilde{\theta}^2 < \frac{\varepsilon}{2\delta} (\gamma_1 k^2 a^4 + \frac{\gamma_2}{4} k^2 a^2 \sigma^2 + R) \Delta^2 \right\} \tag{19}$$

is exponentially attractive with a decay rate δ . ■

Proof. The proof will follow the argument of the more general case of the vector system in Theorem 2 (See Appendix A). ■

The LMIs (16) are always feasible for small enough ε^* . When ε^* is sufficiently small, the feasibility of Φ_3 and Φ_4 are self-evident. Next we check the feasibility for Φ_1 and Φ_2 . Firstly, applying the Schur complement to $\Phi_1 > 0$, we have

$$P - 1 - \frac{p^2}{p + e^{-2\delta\varepsilon^*}R} \approx P - 1 - \frac{p^2}{p+R} > 0, \quad \varepsilon^* \rightarrow 0. \quad (20)$$

As long as the decision variables satisfy $R > \frac{p}{p-1}$, the above inequality is feasible. Secondly, applying the Schur complement to $\Phi_2 > 0$, we get

$$\begin{aligned} & \begin{bmatrix} (ka^2f''+2\delta)P & -\frac{(ka^2f''+4\delta)P}{2} \\ * & -\frac{4}{\varepsilon^*}e^{-2\delta\varepsilon^*}R+2\delta P \end{bmatrix} \\ & + \frac{\varepsilon^*}{\gamma_1} \begin{bmatrix} -\frac{f''}{2}P \\ \frac{f''}{2}P \end{bmatrix} \begin{bmatrix} -\frac{f''}{2}P & \frac{f''}{2}P \end{bmatrix} + \frac{\varepsilon^*}{\gamma_2} \begin{bmatrix} -f''P \\ f''P \end{bmatrix} \begin{bmatrix} -f''P & f''P \end{bmatrix} \\ & \approx \begin{bmatrix} (ka^2f''+2\delta)P & -\frac{(ka^2f''+4\delta)P}{2} \\ * & -\frac{4}{\varepsilon^*}e^{-2\delta\varepsilon^*}R+2\delta P \end{bmatrix} < 0, \quad \varepsilon^* \rightarrow 0. \end{aligned} \quad (21)$$

We further apply the Schur complement to the above LMI such that

$$\begin{aligned} & (ka^2f'' + 2\delta)P + \frac{\varepsilon^*}{4e^{-2\delta\varepsilon^*}R - 2\varepsilon^*\delta P} \frac{(ka^2f''+4\delta)^2 P^2}{4} \\ & \approx (ka^2f'' + 2\delta)P < 0, \quad \varepsilon^* \rightarrow 0. \end{aligned} \quad (22)$$

For any $ka^2f'' < -2\delta$, the inequality is always feasible. The similar argument for the LMI feasibility is applicable in later sections.

Assume now that the Hessian f'' is not known but its sign is known together with its bounds

$$0 < f''_m \leq |f''| \leq f''_M, \quad (23)$$

Note that in LMIs (16) Φ_2 is affine in f'' . So, for feasibility of $\Phi_2 < 0$, it is sufficient to verify two LMIs in the vertices:

$$\Phi_2|_{f''=\text{sgn}(f'')f''_m} < 0, \quad \Phi_2|_{f''=\text{sgn}(f'')f''_M} < 0. \quad (24)$$

Assume further that the extremum value f^* is unknown, but it is subject to

$$|f^*| \leq f^*_M, \quad (25)$$

where f^*_M is known. Then from (17) we can choose

$$\bar{\Delta} = f^*_M + \frac{f''_M}{2}(\sigma + |a|)^2 \quad (26)$$

Thus, we have the following corollary.

Corollary 1. *Under the assumption that the sign of Hessian is known, and the Hessian f'' and the extremum value f^* are subject to the bounds (23) and (25) respectively, let the LMIs in (16) hold where Φ_2 is replaced by two LMIs (24) and Φ_3, Φ_4 are renewed with the bound (26). Then $\forall \varepsilon \in (0, \varepsilon^*]$ the solution of the closed-loop system (6) satisfies (18) and the attractive ball is defined by (19) with $\bar{\Delta}$ defined by (26). ■*

Remark 1. From (19), it is observed that the ultimate bound on the estimation error is of the order of $O(\sqrt{\varepsilon})$ provided that a, k are of the order of $O(1)$ leading to δ of the order of $O(1)$. This is larger than $O(\varepsilon)$ achieved in Ariyur and Krstic (2003). However, the time-delay approach gives a precise bound on the estimation error (see Eq. (18)). The ultimate bound can be reduced by decreasing k and a whereas the decay rate is improved by increasing k and a , so that there exists a trade-off between the ultimate bound and the convergence rate. A lower bound on ω , which indicates how large the frequency of the perturbation signal may be selected, is found through the LMIs (16). Our quantitative analysis via time-delay method provides more details for tuning parameters than the existing results that provide qualitative averaging-based analysis.

Theoretically, given any initial state $\tilde{\theta}(0)$, we can always find σ for small enough ε to let (18) hold. Therefore, the result is semiglobal (from examples it is seen that even if theoretically we can start with any $\tilde{\theta}(0)$, to have a practical (not too high) bound on ω we have to choose a small enough $\tilde{\theta}(0)$). ■

Remark 2. The time-delay approach proposed in the paper is applicable to the classical gradient ES of time-varying static map, provided that the change rate of the cost function is not fast. As a counterpart of (3), the time-varying static map is considered as follows (Ariyur & Krstic, 2003):

$$y(t) = f(\theta(t)) = f^*(t) + \frac{f''}{2}(\theta(t) - \theta^*(t))^2, \quad (27)$$

where $f^*(t)$ and $\theta^*(t)$ are the time-varying maxima or minima. The gradient ES is identical with (5) and the estimation error is defined as

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*(t). \quad (28)$$

The dynamics of the estimation error is

$$\begin{aligned} \dot{\tilde{\theta}}(t) &= \frac{ka^2f''}{2} [1 - \cos(2\omega t)] \tilde{\theta}(t) + \frac{kaf''}{2} \sin(\omega t) \tilde{\theta}^2(t) \\ &+ kaf^*(t) \sin(\omega t) + \frac{ka^3f''}{2} \sin^3(\omega t) - \dot{\theta}^*(t). \end{aligned} \quad (29)$$

Comparing (29) with (6), it is seen that the differences are in two terms: $kaf^*(t) \sin(\omega t)$ and $\dot{\theta}^*(t)$. Setting $\omega = \frac{2\pi}{\varepsilon}$ and in parallel with (9)–(12), we apply the time-delay approach to (29), and arrive at the closed-loop system

$$\begin{aligned} \frac{d}{dt} [\tilde{\theta}(t) - G(t)] &= \frac{ka^2f''}{2} \tilde{\theta}(t) - \frac{f''}{2} Y_1(t) - f'' Y_2(t) \\ &- \frac{ka}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \hat{f}^*(s) ds d\tau - \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \dot{\theta}^*(\tau) d\tau, \\ &t \geq \varepsilon. \end{aligned} \quad (30)$$

where $G(t), Y_1(t), Y_2(t)$ have been defined in (13) and are of order of $O(\varepsilon)$. Assume that $\hat{f}^*(s)$ is bounded. Then, in (30), the double integral term on $\hat{f}^*(s)$ is of order of $O(\varepsilon)$. Thus, as long as $\dot{\theta}^*(\tau) = O(\varepsilon)$ is small enough, which means the optimal point $\theta^*(\tau)$ is slowly time-varying, the time-delay system (30) is still practically stable. ■

2.2. Vector systems

In this section we apply time-delay approach to gradient-based classical ES for vector systems. To avoid notational complexity, we address the case of two variables. The method can be extended to any $n > 2$ variables by using the same arguments, but derivations are much longer. Consider multi-variable static maps given by,

$$y(t) = Q(\theta(t)) = Q^* + \frac{1}{2}(\theta(t) - \theta^*)^T H(\theta(t) - \theta^*), \quad (31)$$

where $y(t) \in \mathbb{R}$ is the measurable output, $\theta(t) = [\theta_1(t), \theta_2(t)]^T \in \mathbb{R}^2$ is the vector input, Q^* is a constant, $\theta^* = [\theta_1^*, \theta_2^*]^T$ is a constant vector, $H = \begin{bmatrix} h_{11} & h_{12} \\ * & h_{22} \end{bmatrix}$ is the Hessian matrix which is either positive definite or negative definite. It is observed that the quadratic map (31) has a maximum (when $H < 0$) or minimum (when $H > 0$) value $y(t) = Q^*$ if $\theta(t) = \theta^*$. Usually, Q^*, H, θ^* are unknown, whereas the sign of the Hessian H is available. In the present paper, we assume the extremum point θ^* to be sought is uncertain from a known interval where each element satisfies $\theta_i^* \in [\underline{\theta}_i^*, \bar{\theta}_i^*], i = 1, 2$ with $\sum_{i=1}^2 (\bar{\theta}_i^* - \underline{\theta}_i^*)^2 = \sigma_0^2$, the extremum value Q^* and the Hessian H are known that allows to derive efficient LMI conditions.

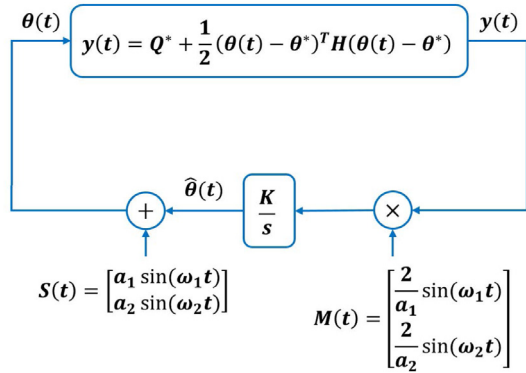


Fig. 2. Classical ES for a multi-variable static map.

The gradient-based classical ES update law shown in Fig. 2 is given by

$$\begin{aligned} \theta(t) &= \hat{\theta}(t) + S(t), \\ \dot{\hat{\theta}}(t) &= KM(t)y(t) = KM(t) \left[Q^* + \frac{1}{2}(\hat{\theta}(t) + S(t) - \theta^*)^T \right. \\ &\quad \left. \times H(\hat{\theta}(t) + S(t) - \theta^*) \right], \end{aligned} \quad (32)$$

where $\hat{\theta}_i(0) \in [\bar{\theta}_i^*, \bar{\theta}_i^*]$, $i = 1, 2$, and

$$\begin{aligned} S(t) &= [a_1 \sin(\omega_1 t), a_2 \sin(\omega_2 t)]^T, \\ M(t) &= \left[\frac{2}{a_1} \sin(\omega_1 t), \frac{2}{a_2} \sin(\omega_2 t) \right]^T, \end{aligned} \quad (33)$$

are the dither signals in which $\omega_1 \neq \omega_2$ are non-zero and $\frac{\omega_1}{\omega_2}$ is rational, $K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$ is the adaptation gain matrix whose sign is opposite to that of H . The estimation error defined in (4) is governed by

$$\begin{aligned} \dot{\hat{\theta}}(t) &= KM(t) \left[Q^* + \frac{1}{2}(\tilde{\theta}(t) + S(t))^T H(\tilde{\theta}(t) + S(t)) \right], \\ &= KM(t) \left[Q^* + \frac{1}{2}\tilde{\theta}^T(t)H\tilde{\theta}(t) + S^T(t)H\tilde{\theta}(t) \right. \\ &\quad \left. + \frac{1}{2}S^T(t)HS(t) \right], \end{aligned} \quad (34)$$

of which each element is expanded as

$$\begin{aligned} \dot{\tilde{\theta}}_i(t) &= k_i(h_{ii}\tilde{\theta}_i(t) + h_{ij}\tilde{\theta}_j(t)) \\ &\quad - k_i \cos(2\omega_i t)(h_{ii}\tilde{\theta}_i(t) + h_{ij}\tilde{\theta}_j(t)) \\ &\quad + \frac{2k_i a_j}{a_i} \sin(\omega_i t) \sin(\omega_j t)(h_{ij}\tilde{\theta}_i(t) + h_{ij}\tilde{\theta}_j(t)) \\ &\quad + \frac{2k_i}{a_i} \sin(\omega_i t)Q^* + \frac{k_i}{a_i} \sin(\omega_i t)\tilde{\theta}^T(t)H\tilde{\theta}(t) \\ &\quad + \frac{k_i}{a_i} \sin(\omega_i t)S^T(t)HS(t), \quad i, j = 1, 2, \quad i \neq j. \end{aligned} \quad (35)$$

First of all, defining $\omega_1 = \frac{2\pi l_1}{\varepsilon}$, $\omega_2 = \frac{2\pi l_2}{\varepsilon}$, $l_1, l_2 \in \mathbb{N}$, $l_1 \neq l_2$, we apply the conventional averaging method to (35) to get its

backward averaged system below,

$$\begin{aligned} \dot{\tilde{\theta}}_i^{av}(t) &= k_i(h_{ii}\tilde{\theta}_i^{av}(t) + h_{ij}\tilde{\theta}_j^{av}(t)) \\ &\quad - k_i \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \cos\left(\frac{4\pi l_i}{\varepsilon} \tau\right) d\tau (h_{ii}\tilde{\theta}_i^{av}(t) + h_{ij}\tilde{\theta}_j^{av}(t)) \\ &\quad + \frac{2k_i a_j}{a_i} \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) \sin\left(\frac{2\pi l_j}{\varepsilon} \tau\right) d\tau \\ &\quad \times (h_{ij}\tilde{\theta}_i^{av}(t) + h_{ij}\tilde{\theta}_j^{av}(t)) + \frac{2k_i}{a_i} \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) d\tau Q^* \\ &\quad + \frac{k_i}{a_i} \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) d\tau \tilde{\theta}_{av}^T(t)H\tilde{\theta}_{av}(t) \\ &\quad + \frac{k_i}{a_i} \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) S^T(\tau)HS(\tau) d\tau, \\ &= k_i(h_{ii}\tilde{\theta}_i^{av}(t) + h_{ij}\tilde{\theta}_j^{av}(t)), \quad i, j = 1, 2, \quad i \neq j, \end{aligned} \quad (36)$$

where we utilize the averaging

$$\begin{aligned} \int_{-\varepsilon}^0 \cos\left(\frac{4\pi l_i}{\varepsilon} \tau\right) d\tau &= 0, \quad \int_{-\varepsilon}^0 \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) d\tau = 0, \\ \int_{-\varepsilon}^0 \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) \sin\left(\frac{2\pi l_j}{\varepsilon} \tau\right) d\tau &= 0, \\ \int_{-\varepsilon}^0 \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) S^T(\tau)HS(\tau) d\tau &= \int_{-\varepsilon}^0 \left[h_{ii}a_i^2 \sin^3\left(\frac{2\pi l_i}{\varepsilon} \tau\right) + 2h_{ij}a_i a_j \sin^2\left(\frac{2\pi l_i}{\varepsilon} \tau\right) \right. \\ &\quad \left. \times \sin\left(\frac{2\pi l_j}{\varepsilon} \tau\right) + h_{ij}a_j^2 \sin\left(\frac{2\pi l_j}{\varepsilon} \tau\right) \sin^2\left(\frac{2\pi l_i}{\varepsilon} \tau\right) \right] = 0. \end{aligned} \quad (37)$$

Grouping (36) into the vector, we get the averaged vector system

$$\dot{\tilde{\theta}}_{av}(t) = KH\tilde{\theta}_{av}(t). \quad (38)$$

Next, we apply the time-delay method to averaging of (35). Integrating (35) in $t \geq \varepsilon$ from $t - \varepsilon$ to t , and taking into account (37), we get

$$\begin{aligned} \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \dot{\tilde{\theta}}_i(\tau) d\tau &= \frac{k_i}{\varepsilon} \int_{t-\varepsilon}^t (h_{ii}\tilde{\theta}_i(\tau) + h_{ij}\tilde{\theta}_j(\tau)) d\tau \\ &\quad - \frac{k_i}{\varepsilon} \int_{t-\varepsilon}^t \cos\left(\frac{4\pi l_i}{\varepsilon} \tau\right) (h_{ii}\tilde{\theta}_i(\tau) + h_{ij}\tilde{\theta}_j(\tau)) d\tau \\ &\quad + \frac{2k_i a_j}{a_i \varepsilon} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) \sin\left(\frac{2\pi l_j}{\varepsilon} \tau\right) \\ &\quad \times (h_{ij}\tilde{\theta}_i(\tau) + h_{ij}\tilde{\theta}_j(\tau)) d\tau \\ &\quad + \frac{k_i}{a_i \varepsilon} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) \tilde{\theta}^T(\tau)H\tilde{\theta}(\tau) d\tau, \end{aligned} \quad (39)$$

Employing the similar technique to (10), the first term on the right-hand side of (39) is handled as

$$\begin{aligned} \frac{k_i}{\varepsilon} \int_{t-\varepsilon}^t (h_{ii}\tilde{\theta}_i(\tau) + h_{ij}\tilde{\theta}_j(\tau)) d\tau &= \frac{k_i}{\varepsilon} \int_{t-\varepsilon}^t [h_{ii}\tilde{\theta}_i(\tau) + h_{ij}\tilde{\theta}_j(\tau) \pm h_{ii}\tilde{\theta}_i(t) \pm h_{ij}\tilde{\theta}_j(t)] \\ &\quad \times d\tau, \\ &= k_i(h_{ii}\tilde{\theta}_i(t) + h_{ij}\tilde{\theta}_j(t)) - \frac{k_i}{\varepsilon} \int_{t-\varepsilon}^t \\ &\quad \times [h_{ii}\tilde{\theta}_i(t) + h_{ij}\tilde{\theta}_j(t) - h_{ii}\tilde{\theta}_i(\tau) - h_{ij}\tilde{\theta}_j(\tau)] d\tau, \\ &= k_i(h_{ii}\tilde{\theta}_i(t) + h_{ij}\tilde{\theta}_j(t)) \\ &\quad - \frac{k_i}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t (h_{ii}\dot{\tilde{\theta}}_i(s) + h_{ij}\dot{\tilde{\theta}}_j(s)) ds d\tau. \end{aligned} \quad (40)$$

Referring to (11), the second term on the right-hand side of (39) is given by

$$\begin{aligned} -\frac{k_i}{\varepsilon} \int_{t-\varepsilon}^t \cos\left(\frac{4\pi l_i}{\varepsilon} \tau\right) (h_{ii}\tilde{\theta}_i(\tau) + h_{ij}\tilde{\theta}_j(\tau)) d\tau &= \frac{k_i}{\varepsilon} \int_{t-\varepsilon}^t \cos\left(\frac{4\pi l_i}{\varepsilon} \tau\right) \\ &\quad \times [h_{ii}\tilde{\theta}_i(t) + h_{ij}\tilde{\theta}_j(t) - h_{ii}\tilde{\theta}_i(\tau) - h_{ij}\tilde{\theta}_j(\tau)] d\tau, \\ &= \frac{k_i}{\varepsilon} \int_{t-\varepsilon}^t \cos\left(\frac{4\pi l_i}{\varepsilon} \tau\right) \int_{\tau}^t (h_{ii}\dot{\tilde{\theta}}_i(s) + h_{ij}\dot{\tilde{\theta}}_j(s)) ds d\tau. \end{aligned} \quad (41)$$

Similar to (11), the third term on the right-hand side of (39) is addressed as

$$\begin{aligned} & \frac{2k_i a_j}{a_i \varepsilon} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) \sin\left(\frac{2\pi l_j}{\varepsilon} \tau\right) \\ & \quad \times (h_{ij} \tilde{\theta}_i(\tau) + h_{ij} \tilde{\theta}_j(\tau)) d\tau \\ &= -\frac{2k_i a_j}{a_i \varepsilon} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) \sin\left(\frac{2\pi l_j}{\varepsilon} \tau\right) \\ & \quad \times [h_{ij} \tilde{\theta}_i(t) + h_{ij} \tilde{\theta}_j(t) - h_{ij} \tilde{\theta}_i(\tau) - h_{ij} \tilde{\theta}_j(\tau)] d\tau, \\ &= -\frac{2k_i a_j}{a_i \varepsilon} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) \sin\left(\frac{2\pi l_j}{\varepsilon} \tau\right) \\ & \quad \times \int_{\tau}^t (h_{ij} \dot{\tilde{\theta}}_i(s) + h_{ij} \dot{\tilde{\theta}}_j(s)) ds d\tau. \end{aligned} \tag{42}$$

In parallel with (12), the fourth term on the right-hand side of (39) is given by

$$\begin{aligned} & \frac{k_i}{a_i \varepsilon} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) \tilde{\theta}^T(\tau) H \tilde{\theta}(\tau) d\tau \\ &= -\frac{k_i}{a_i \varepsilon} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) [\tilde{\theta}^T(t) H \tilde{\theta}(t) - \tilde{\theta}^T(\tau) H \tilde{\theta}(\tau)] d\tau, \\ &= -\frac{2k_i}{a_i \varepsilon} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi l_i}{\varepsilon} \tau\right) \int_{\tau}^t \tilde{\theta}^T(s) H \dot{\tilde{\theta}}(s) ds d\tau. \end{aligned} \tag{43}$$

Substituting (40)–(43) into (39), employing the same relation with (14), we get the closed-loop vector system

$$\frac{d}{dt} [\tilde{\theta}(t) - G(t)] = KH\tilde{\theta}(t) - Y_1(t) - Y_2(t), \tag{44}$$

where $\dot{\tilde{\theta}}(t)$ is defined by the right-hand side of (34) and

$$\begin{aligned} G(t) &= \frac{1}{\varepsilon} \int_{t-\varepsilon}^t (\tau - t + \varepsilon) \dot{\tilde{\theta}}(\tau) d\tau, \\ Y_1(t) &= \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t N_1(\tau) H \dot{\tilde{\theta}}(s) ds d\tau, \\ Y_2(t) &= \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t N_2(\tau) \tilde{\theta}^T(s) H \dot{\tilde{\theta}}(s) ds d\tau, \\ N_1(\tau) &= \begin{bmatrix} k_1 [1 - \cos(\frac{4\pi l_1}{\varepsilon} \tau)] & \frac{2k_1 a_2}{a_1} \sin(\frac{2\pi l_1}{\varepsilon} \tau) \sin(\frac{2\pi l_2}{\varepsilon} \tau) \\ \frac{2k_2 a_1}{a_2} \sin(\frac{2\pi l_1}{\varepsilon} \tau) \sin(\frac{2\pi l_2}{\varepsilon} \tau) & k_2 [1 - \cos(\frac{4\pi l_2}{\varepsilon} \tau)] \end{bmatrix}, \\ N_2(\tau) &= \begin{bmatrix} \frac{2k_1}{a_1} \sin(\frac{2\pi l_1}{\varepsilon} \tau) \\ \frac{2k_2}{a_2} \sin(\frac{2\pi l_2}{\varepsilon} \tau) \end{bmatrix}. \end{aligned} \tag{45}$$

It is evident that the system (44) is a perturbation of the stable averaged system (38).

Theorem 2. Assume that the Hessian H and the extremum value Q^* are known, each element of the uncertain extremum point θ^* belongs to the known interval $\theta_i^* \in [\underline{\theta}_i^*, \bar{\theta}_i^*]$, $i = 1, 2$. Consider the closed-loop system consisting of the vector plant (31) and classical ES controller (32), with the initial condition $|\tilde{\theta}(0)| \leq \sigma_0$. Given tuning parameters k_1, k_2, a_1, a_2 and $q, \delta, \varepsilon^* > 0$ as well as σ_1, σ_2 satisfying $\sigma_1^2 + \sigma_2^2 > \sigma_0^2$, let matrices $P > I, R > 0$ and scalars

$\lambda_P, \lambda_R, \gamma_1, \gamma_2 > 0$ satisfy the LMIs:

$$\begin{aligned} \Phi_1 &= \begin{bmatrix} P-I & -P \\ * & P+e^{-2\delta\varepsilon^*} R \end{bmatrix} > 0, \\ \Phi_2 &= \begin{bmatrix} H^T K^T P + PKH + 2\delta P & -(HK + 2\delta I)P & -P & -P \\ * & -\frac{4}{\varepsilon^*} e^{-2\delta\varepsilon^*} R + 2\delta P & P & P \\ * & * & -\frac{\gamma_1}{\varepsilon^*} I & 0 \\ * & * & * & -\frac{\gamma_2}{\varepsilon^*} I \end{bmatrix} < 0, \\ \Phi_3 &= \left(1 + \frac{1}{q}\right)^2 \lambda_P \sigma_0^2 + \varepsilon^{*2} \left[\left(1 + \frac{1}{q}\right) (1 + q) \right. \\ & \quad \times \lambda_P + \frac{1+q}{4} \lambda_P + \frac{1}{3} \lambda_R \left. \right] \Delta^2 \\ & \quad + \frac{\varepsilon^*}{2\delta} \left[\lambda_R \left(\frac{4k_1^2}{a_1^2} + \frac{4k_2^2}{a_2^2} \right) + \gamma_1 \Delta_1^2 + \gamma_2 \Delta_2^2 \right] \bar{\Delta}^2 < \sigma^2, \\ P - \lambda_P I &< 0, \quad R - \lambda_R I < 0, \\ \Phi_4 &= \sigma_0 + \varepsilon^* \Delta < \sigma, \end{aligned} \tag{46}$$

where

$$\begin{aligned} \sigma &= \sqrt{\sigma_1^2 + \sigma_2^2}, \quad \Delta = \sqrt{\left(\frac{4k_1^2}{a_1^2} + \frac{4k_2^2}{a_2^2}\right) \bar{\Delta}}, \\ \bar{\Delta} &= |Q^*| + \frac{\lambda(H)}{2} \left(\sigma + \sqrt{a_1^2 + a_2^2} \right)^2, \\ \lambda(H) &= \max\{|\lambda_{\min}(H)|, |\lambda_{\max}(H)|\}, \\ \Delta_1 &= \sqrt{\left(4k_1^2 + \frac{4k_2^2 a_1^2}{a_2^2}\right) \left(\left| \frac{h_{11} k_1}{a_1} \right| + \left| \frac{h_{12} k_2}{a_2} \right| \right)} \\ & \quad + \sqrt{\left(\frac{4k_1^2 a_2^2}{a_1^2} + 4k_2^2\right) \left(\left| \frac{h_{12} k_1}{a_1} \right| + \left| \frac{h_{22} k_2}{a_2} \right| \right)}, \\ \Delta_2 &= \sqrt{\left(\frac{4k_1^2}{a_1^2} + \frac{4k_2^2}{a_2^2}\right)} \\ & \quad \times \left[\frac{(\sigma_1 |h_{11}| + \sigma_2 |h_{12}|) k_1}{|a_1|} + \frac{(\sigma_1 |h_{12}| + \sigma_2 |h_{22}|) k_2}{|a_2|} \right]. \end{aligned} \tag{47}$$

Then $\forall \varepsilon \in (0, \varepsilon^*]$ the solution of the closed-loop system (34) satisfies

$$\begin{aligned} |\tilde{\theta}(t)|^2 &< (|\tilde{\theta}(0)| + \varepsilon \Delta)^2 < \sigma^2, \quad t \in [0, \varepsilon], \\ |\tilde{\theta}(t)|^2 &< \left(1 + \frac{1}{q}\right)^2 \lambda_P e^{-2\delta(t-\varepsilon)} |\tilde{\theta}(0)|^2 \\ & \quad + \varepsilon^2 e^{-2\delta(t-\varepsilon)} \left[\left(1 + \frac{1}{q}\right) (1 + q) \lambda_P \right. \\ & \quad \left. + \frac{1+q}{4} \lambda_P + \frac{1}{3} \lambda_R \right] \Delta^2 \\ & \quad + (1 - e^{-2\delta(t-\varepsilon)}) \frac{\varepsilon}{2\delta} \left[\lambda_R \left(\frac{4k_1^2}{a_1^2} + \frac{4k_2^2}{a_2^2} \right) \right. \\ & \quad \left. + \gamma_1 \Delta_1^2 + \gamma_2 \Delta_2^2 \right] \bar{\Delta}^2 < \sigma^2, \quad t \geq \varepsilon. \end{aligned} \tag{48}$$

Moreover, for all $\varepsilon \in (0, \varepsilon^*]$ and all initial conditions $|\tilde{\theta}(0)| \leq \sigma_0$ the ball

$$\begin{aligned} \Theta &= \left\{ \tilde{\theta} \in \mathbb{R}^2 : |\tilde{\theta}|^2 < \frac{\varepsilon}{2\delta} \left[\lambda_R \left(\frac{4k_1^2}{a_1^2} + \frac{4k_2^2}{a_2^2} \right) \right. \right. \\ & \quad \left. \left. + \gamma_1 \Delta_1^2 + \gamma_2 \Delta_2^2 \right] \bar{\Delta}^2 \right\} \end{aligned} \tag{49}$$

is exponentially attractive with a decay rate δ . The LMIs (46) are always feasible for small enough ε^* . ■

Proof. See Appendix A. ■

Assume now that the extremum value Q^* is unknown, but it is subject to

$$|Q^*| \leq Q_M^*, \tag{50}$$

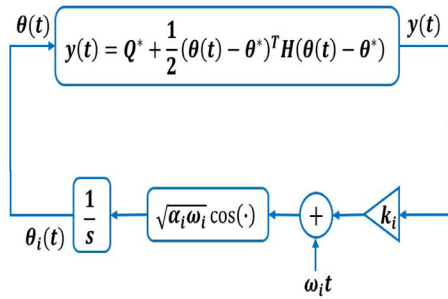


Fig. 3. Bounded ES for a static map.

where Q_M^* is known. Then from (A.10) we have

$$\bar{\Delta} = Q_M^* + \frac{\lambda(H)}{2} \left(\sigma + \sqrt{a_1^2 + a_2^2} \right)^2, \quad (51)$$

Thus Φ_3, Φ_4 in (46) are renewed with the bounds (51). Moreover, $\forall \varepsilon \in (0, \varepsilon^*]$ the solution of the closed-loop system (34) satisfies (48) and the attractive ball is defined by (49) with Δ defined by (51). If the Hessian H is not known either, the LMLs in the vertices will be more complicated and we do not go into details due to the page limit.

3. A time-delay approach to bounded ES

In this section we apply time-delay approach to gradient-based bounded ES proposed in Scheinker and Krstic (2012, 2014, 2017), Scheinker and Scheinker (2016, 2018).

3.1. Scalar systems

We start with scalar systems and then extend the method to vector systems in next section. As shown in Fig. 3, we consider the gradient-based bounded ES as follows:

$$\dot{\theta}(t) = \sqrt{\alpha\omega} \cos(\omega t + ky(t)), \quad (52)$$

where ω is the frequency of the dither signal whose magnitude is proportional to α , k is the controller gain (after averaging), and $y(t)$ is a measurable output function defined by (3) such that

$$y(t) = f^* + \frac{f''}{2} (\theta(t) - \theta^*)^2 = f^* + \frac{f''}{2} \tilde{\theta}^2(t), \quad (53)$$

with the estimation error given by

$$\tilde{\theta}(t) = \theta(t) - \theta^*. \quad (54)$$

The sign of the adaptation gain k is selected to be identical with that of the Hessian f'' . Taking the time derivative of (54) along (52)–(53), we have

$$\begin{aligned} \dot{\tilde{\theta}}(t) &= \sqrt{\alpha\omega} \cos(\omega t + ky(t)), \\ &= \sqrt{\alpha\omega} \cos(\omega t) \cos(ky(t)) - \sqrt{\alpha\omega} \sin(\omega t) \sin(ky(t)). \end{aligned} \quad (55)$$

The averaged system of (55) is derived as

$$\dot{\tilde{\theta}}_{av}(t) = -\frac{kf''\alpha}{2} \tilde{\theta}_{av}(t) \quad (56)$$

The detailed derivation to obtain (56) is given by Scheinker and Krstic (2017, Chapter 2.3) and Scheinker and Scheinker (2016).

Defining $\omega = \frac{2\pi}{\varepsilon}$, we apply the time-delay method to averaging of (55). Integrating (55) in $t \geq \varepsilon$ from $t - \varepsilon$ to t , we get

$$\begin{aligned} \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \dot{\tilde{\theta}}(\tau) d\tau &= \frac{1}{\varepsilon} \sqrt{\frac{2\pi\alpha}{\varepsilon}} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \cos(ky(\tau)) d\tau \\ &\quad - \frac{1}{\varepsilon} \sqrt{\frac{2\pi\alpha}{\varepsilon}} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \sin(ky(\tau)) d\tau. \end{aligned} \quad (57)$$

Firstly, we deal with the first term on the right-hand side of (57) below,

$$\begin{aligned} &\frac{1}{\varepsilon} \sqrt{\frac{2\pi\alpha}{\varepsilon}} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \cos(ky(\tau)) d\tau \\ &= -\frac{1}{\varepsilon} \sqrt{\frac{2\pi\alpha}{\varepsilon}} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \left[\cos(ky(t)) \right. \\ &\quad \left. - \cos(ky(\tau)) \right] d\tau, \\ &= \frac{kf''}{\varepsilon} \sqrt{\frac{2\pi\alpha}{\varepsilon}} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin(ky(s)) \tilde{\theta}(s) \dot{\tilde{\theta}}(s) ds d\tau, \\ &= \frac{kf''}{\varepsilon} \sqrt{\frac{2\pi\alpha}{\varepsilon}} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin(ky(s)) \tilde{\theta}(s) \\ &\quad \times \left[\sqrt{\frac{2\pi\alpha}{\varepsilon}} \cos\left(\frac{2\pi}{\varepsilon}s\right) \cos(ky(s)) \right. \\ &\quad \left. - \sqrt{\frac{2\pi\alpha}{\varepsilon}} \sin\left(\frac{2\pi}{\varepsilon}s\right) \sin(ky(s)) \right] ds d\tau, \\ &= \frac{kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) \sin(2ky(s)) \\ &\quad \times \tilde{\theta}(s) ds d\tau - \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) \\ &\quad \times \sin^2(ky(s)) \tilde{\theta}(s) ds d\tau. \end{aligned} \quad (58)$$

The first term on the right-hand side of (58) is calculated as follows:

$$\begin{aligned} &\frac{kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) \sin(2ky(s)) \tilde{\theta}(s) ds d\tau \\ &= \frac{kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) ds d\tau \cdot \sin(2ky(t)) \\ &\quad \times \tilde{\theta}(t) - \frac{kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) \\ &\quad \times \left[\sin(2ky(t)) \tilde{\theta}(t) - \sin(2ky(s)) \tilde{\theta}(s) \right] ds d\tau, \\ &= \frac{kf''\alpha}{2\varepsilon} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \left[\sin\left(\frac{2\pi}{\varepsilon}t\right) - \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \right] d\tau \\ &\quad \times \sin(2ky(t)) \tilde{\theta}(t) - \frac{kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) \\ &\quad \times \int_s^t \left[\sin(2ky(\xi)) + \cos(2ky(\xi)) 2kf''\tilde{\theta}^2(\xi) \right] \\ &\quad \times \dot{\tilde{\theta}}(\xi) d\xi ds d\tau, \end{aligned} \quad (59)$$

where $\int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \left[\sin\left(\frac{2\pi}{\varepsilon}t\right) - \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \right] d\tau = 0$ via averaging. The second term on the right-hand side of (58) is calculated as follows:

$$\begin{aligned} &-\frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) \sin^2(ky(s)) \tilde{\theta}(s) ds d\tau \\ &= -\frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) ds d\tau \cdot \sin^2(ky(t)) \\ &\quad \times \tilde{\theta}(t) + \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) \\ &\quad \times \left[\sin^2(ky(t)) \tilde{\theta}(t) - \sin^2(ky(s)) \tilde{\theta}(s) \right] ds d\tau, \\ &= \frac{kf''\alpha}{\varepsilon} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \left[\cos\left(\frac{2\pi}{\varepsilon}t\right) - \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \right] d\tau \\ &\quad \times \sin^2(ky(t)) \tilde{\theta}(t) + \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \\ &\quad \times \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) \int_s^t \left[\sin^2(ky(\xi)) \right. \\ &\quad \left. + \sin(2ky(\xi)) kf''\tilde{\theta}^2(\xi) \right] \dot{\tilde{\theta}}(\xi) d\xi ds d\tau, \\ &= -\frac{kf''\alpha}{\varepsilon} \int_{t-\varepsilon}^t \left[\frac{1}{2} + \frac{1}{2} \cos\left(\frac{4\pi}{\varepsilon}\tau\right) \right] d\tau \cdot \sin^2(ky(t)) \tilde{\theta}(t) \\ &\quad + \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) \int_s^t \left[\sin^2(ky(\xi)) \right. \\ &\quad \left. + \sin(2ky(\xi)) kf''\tilde{\theta}^2(\xi) \right] \dot{\tilde{\theta}}(\xi) d\xi ds d\tau, \\ &= -\frac{kf''\alpha}{2} \sin^2(ky(t)) \tilde{\theta}(t) + \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \\ &\quad \times \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) \int_s^t \left[\sin^2(ky(\xi)) \right. \\ &\quad \left. + \sin(2ky(\xi)) kf''\tilde{\theta}^2(\xi) \right] \dot{\tilde{\theta}}(\xi) d\xi ds d\tau. \end{aligned} \quad (60)$$

Secondly, we address the second term on the right-hand side of (57) below,

$$\begin{aligned}
 & -\frac{1}{\varepsilon} \sqrt{\frac{2\pi\alpha}{\varepsilon}} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \sin(ky(\tau)) d\tau \\
 &= \frac{1}{\varepsilon} \sqrt{\frac{2\pi\alpha}{\varepsilon}} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \left[\sin(ky(t)) \right. \\
 &\quad \left. - \sin(ky(\tau)) \right] d\tau, \\
 &= \frac{kf''}{\varepsilon} \sqrt{\frac{2\pi\alpha}{\varepsilon}} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos(ky(s)) \tilde{\theta}(s) \dot{\tilde{\theta}}(s) ds d\tau, \\
 &= \frac{kf''}{\varepsilon} \sqrt{\frac{2\pi\alpha}{\varepsilon}} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos(ky(s)) \tilde{\theta}(s) \\
 &\quad \times \left[\sqrt{\frac{2\pi\alpha}{\varepsilon}} \cos\left(\frac{2\pi}{\varepsilon}s\right) \cos(ky(s)) \right. \\
 &\quad \left. - \sqrt{\frac{2\pi\alpha}{\varepsilon}} \sin\left(\frac{2\pi}{\varepsilon}s\right) \sin(ky(s)) \right] ds d\tau, \tag{61} \\
 &= -\frac{kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) \sin(2ky(s)) \\
 &\quad \times \tilde{\theta}(s) ds d\tau + \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) \\
 &\quad \times \cos^2(ky(s)) \tilde{\theta}(s) ds d\tau.
 \end{aligned}$$

The first term on the right-hand side of (61) is calculated as follows:

$$\begin{aligned}
 & -\frac{kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) \sin(2ky(s)) \tilde{\theta}(s) ds d\tau \\
 &= -\frac{kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) ds d\tau \cdot \sin(2ky(t)) \\
 &\quad \times \tilde{\theta}(t) + \frac{kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) \\
 &\quad \times \left[\sin(2ky(t)) \tilde{\theta}(t) - \sin(2ky(s)) \tilde{\theta}(s) \right] ds d\tau, \tag{62} \\
 &= \frac{kf''\alpha}{2\varepsilon} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \left[\cos\left(\frac{2\pi}{\varepsilon}t\right) - \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \right] d\tau \\
 &\quad \times \sin(2ky(t)) \tilde{\theta}(t) + \frac{kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) \\
 &\quad \times \int_s^t \left[\sin(2ky(\xi)) + \cos(2ky(\xi)) 2kf''\tilde{\theta}^2(\xi) \right] \\
 &\quad \times \dot{\tilde{\theta}}(\xi) d\xi ds d\tau.
 \end{aligned}$$

where $\int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \left[\cos\left(\frac{2\pi}{\varepsilon}t\right) - \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \right] d\tau = 0$ via averaging. The second term on the right-hand side of (61) is calculated as follows:

$$\begin{aligned}
 & \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) \cos^2(ky(s)) \tilde{\theta}(s) ds d\tau \\
 &= \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) ds d\tau \cdot \cos^2(ky(t)) \\
 &\quad \times \tilde{\theta}(t) - \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) \\
 &\quad \times \left[\cos^2(ky(t)) \tilde{\theta}(t) - \cos^2(ky(s)) \tilde{\theta}(s) \right] ds d\tau, \\
 &= \frac{kf''\alpha}{\varepsilon} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \left[\sin\left(\frac{2\pi}{\varepsilon}t\right) - \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \right] d\tau \\
 &\quad \times \cos^2(ky(t)) \tilde{\theta}(t) - \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \\
 &\quad \times \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) \int_s^t \left[\cos^2(ky(\xi)) \right. \\
 &\quad \left. - \sin(2ky(\xi)) kf''\tilde{\theta}^2(\xi) \right] \dot{\tilde{\theta}}(\xi) d\xi ds d\tau, \tag{63} \\
 &= -\frac{kf''\alpha}{\varepsilon} \int_{t-\varepsilon}^t \left[\frac{1}{2} - \frac{1}{2} \cos\left(\frac{4\pi}{\varepsilon}\tau\right) \right] d\tau \cdot \cos^2(ky(t)) \tilde{\theta}(t) \\
 &\quad - \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) \int_s^t \left[\cos^2(ky(\xi)) \right. \\
 &\quad \left. - \sin(2ky(\xi)) kf''\tilde{\theta}^2(\xi) \right] \dot{\tilde{\theta}}(\xi) d\xi ds d\tau, \\
 &= -\frac{kf''\alpha}{2} \cos^2(ky(t)) \tilde{\theta}(t) - \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \\
 &\quad \times \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) \int_s^t \left[\cos^2(ky(\xi)) \right. \\
 &\quad \left. - \sin(2ky(\xi)) kf''\tilde{\theta}^2(\xi) \right] \dot{\tilde{\theta}}(\xi) d\xi ds d\tau.
 \end{aligned}$$

Substituting (59)–(60) into (58) and (62)–(63) into (61), and further substituting (58) and (61) into (57), employing the identical relation with (14), we get the closed-loop system

$$\begin{aligned}
 \frac{d}{dt} [\tilde{\theta}(t) - G(t)] &= -\frac{kf''\alpha}{2} \tilde{\theta}(t) \\
 &\quad - \frac{kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) \int_s^t \left[\sin(2ky(\xi)) \right. \\
 &\quad \left. + \cos(2ky(\xi)) 2kf''\tilde{\theta}^2(\xi) \right] \dot{\tilde{\theta}}(\xi) d\xi ds d\tau \\
 &\quad + \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \cos\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) \int_s^t \left[\sin^2(ky(\xi)) \right. \\
 &\quad \left. + \sin(2ky(\xi)) kf''\tilde{\theta}^2(\xi) \right] \dot{\tilde{\theta}}(\xi) d\xi ds d\tau \tag{64} \\
 &\quad + \frac{kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \sin\left(\frac{2\pi}{\varepsilon}s\right) \int_s^t \left[\sin(2ky(\xi)) \right. \\
 &\quad \left. + \cos(2ky(\xi)) 2kf''\tilde{\theta}^2(\xi) \right] \dot{\tilde{\theta}}(\xi) d\xi ds d\tau \\
 &\quad - \frac{2kf''\pi\alpha}{\varepsilon^2} \int_{t-\varepsilon}^t \sin\left(\frac{2\pi}{\varepsilon}\tau\right) \int_{\tau}^t \cos\left(\frac{2\pi}{\varepsilon}s\right) \int_s^t \left[\cos^2(ky(\xi)) \right. \\
 &\quad \left. - \sin(2ky(\xi)) kf''\tilde{\theta}^2(\xi) \right] \dot{\tilde{\theta}}(\xi) d\xi ds d\tau \\
 &= -\frac{kf''\alpha}{2} \tilde{\theta}(t) - f''\pi Y_1(t) - 2f''^2\pi Y_2(t).
 \end{aligned}$$

where $\dot{\tilde{\theta}}(t)$ is defined by the right-hand side of (55) and

$$\begin{aligned}
 Y_1(t) &= \frac{\alpha k}{\varepsilon^2} \int_{t-\varepsilon}^t \int_{\tau}^t \int_s^t \left[\sin\left(\frac{2\pi}{\varepsilon}(\tau+s) + 2ky(\xi)\right) \right. \\
 &\quad \left. + \sin\left(\frac{2\pi}{\varepsilon}(\tau-s)\right) \right] \dot{\tilde{\theta}}(\xi) d\xi ds d\tau, \tag{65} \\
 Y_2(t) &= \frac{\alpha k^2}{\varepsilon^2} \int_{t-\varepsilon}^t \int_{\tau}^t \int_s^t \cos\left(\frac{2\pi}{\varepsilon}(\tau+s) + 2ky(\xi)\right) \\
 &\quad \times \tilde{\theta}^2(\xi) \dot{\tilde{\theta}}(\xi) d\xi ds d\tau.
 \end{aligned}$$

The system (64) is a perturbation of the stable averaged system (56).

Theorem 3. Assume that the Hessian f'' is known, whereas the extremum value f^* is unknown, the extremum point θ^* is uncertain but belongs to the known interval $\theta^* \in [\underline{\theta}^*, \bar{\theta}^*]$. Consider the closed-loop system consisting of the scalar plant (53) and bounded ES controller (52), with the initial condition $|\tilde{\theta}(0)| \leq \sigma_0$. Given tuning parameters $q, \delta, \varepsilon^*, k, \alpha > 0$ and $\sigma > \sigma_0$, let scalars $P > 1$ and $R, \gamma_1, \gamma_2 > 0$ satisfy the LMIs:

$$\begin{aligned}
 \Phi_1 &= \begin{bmatrix} P-1 & -P \\ * & P+e^{-2\delta\varepsilon^*}R \end{bmatrix} > 0, \\
 \Phi_2 &= \begin{bmatrix} -(\alpha kf''-2\delta)P & (\frac{\alpha kf''}{2}-2\delta)P & -f''\pi P & -2f''^2\pi P \\ * & -\frac{4}{\varepsilon^*}e^{-2\delta\varepsilon^*}R+2\delta P & f''\pi P & 2f''^2\pi P \\ * & * & -\frac{\gamma_1}{\sqrt{\varepsilon^*}} & 0 \\ * & * & * & -\frac{\gamma_2}{\sqrt{\varepsilon^*}} \end{bmatrix} \\
 &< 0, \tag{66} \\
 \Phi_3 &= \left(1 + \frac{1}{q}\right)^2 P\sigma_0^2 + 2\pi\alpha\varepsilon^* \left[\left(1 + \frac{1}{q}\right) (1 + q) \right. \\
 &\quad \times P + \frac{1+q}{4}P + \frac{1}{3}R \left. \right] + \left[\frac{\pi\alpha}{\delta}R + \frac{\pi\alpha^3k^2\sqrt{\varepsilon^*}}{\delta} \right. \\
 &\quad \left. \times \left(\frac{\gamma_1}{9} + \frac{\gamma_2}{36}k^2\sigma^4 \right) \right] < \sigma^2, \\
 \Phi_4 &= \sigma_0 + \sqrt{2\pi\alpha\varepsilon^*} < \sigma.
 \end{aligned}$$

Then $\forall \varepsilon \in (0, \varepsilon^*]$ the solution of the closed-loop system (55) satisfies

$$\begin{aligned} \tilde{\theta}^2(t) &\leq \left(|\tilde{\theta}(0)| + \sqrt{2\pi\alpha\varepsilon} \right)^2 < \sigma^2, \quad t \in [0, \varepsilon], \\ \tilde{\theta}^2(t) &< \left(1 + \frac{1}{q} \right)^2 Pe^{-2\delta(t-\varepsilon)}\tilde{\theta}^2(0) + 2\pi\alpha\varepsilon e^{-2\delta(t-\varepsilon)} \\ &\times \left[\left(1 + \frac{1}{q} \right) (1+q)P + \frac{1+q}{4}P + \frac{1}{3}R \right] \\ &+ (1 - e^{-2\delta(t-\varepsilon)}) \left[\frac{\pi\alpha}{\delta}R + \frac{\pi\alpha^3k^2\sqrt{\varepsilon}}{\delta} \left(\frac{\gamma_1}{9} + \frac{\gamma_2}{36}k^2\sigma^4 \right) \right] \\ &< \sigma^2, \quad t \geq \varepsilon. \end{aligned} \tag{67}$$

Moreover, for all $\varepsilon \in (0, \varepsilon^*]$ and all initial conditions $|\tilde{\theta}(0)| \leq \sigma_0$ the ball

$$\Theta = \left\{ \tilde{\theta} \in \mathbb{R} : \tilde{\theta}^2 < \frac{\pi\alpha}{\delta}R + \frac{\pi\alpha^3k^2\sqrt{\varepsilon}}{\delta} \left(\frac{\gamma_1}{9} + \frac{\gamma_2}{36}k^2\sigma^4 \right) \right\} \tag{68}$$

is exponentially attractive with a decay rate δ . ■

Proof. The proof will follow the argument of the more general case of the vector system in Theorem 4 (See Appendix B). ■

Referring to Corollary 1 for the classical ES, let f'' is subject to the bound (23). Applying Schur complement to the last column and row of Φ_2 , we see that in this column we can replace f'' by f''_M , while all the other elements of Φ_2 are affine in f'' . We arrive at

Corollary 2. Under the assumption that the sign of Hessian is known, and the Hessian f'' is subject to the bound (23), let the LMIs in (66) hold with $\Phi_2 < 0$ replaced by two LMIs (24). Then $\forall \varepsilon \in (0, \varepsilon^*]$ the solution of the closed-loop system (55) satisfies (67) and the attractive ball is defined by (68). ■

3.2. Vector systems

In this section we apply time-delay approach to gradient-based bounded ES for vector systems. For notational simplicity, we handle the two dimensional system. It is possible to conceptually extend the result to arbitrary n -dimension system, but computations are much longer. As revealed in Fig. 3, we consider the two dimensional dynamics whose element is

$$\dot{\theta}_i(t) = \sqrt{\alpha_i\omega_i} \cos(\omega_i t + k_i y(t)), \quad i = 1, 2 \tag{69}$$

where $y(t)$ is a measurable output function to be optimized which is of the form of (31) such that

$$\begin{aligned} y(t) &= Q^* + \frac{1}{2}(\theta(t) - \theta^*)^T H(\theta(t) - \theta^*), \\ &= Q^* + \frac{1}{2}\tilde{\theta}^T(t)H\tilde{\theta}(t), \end{aligned} \tag{70}$$

with

$$\tilde{\theta}(t) = \theta(t) - \theta^*. \tag{71}$$

Combining (69)–(70) with (71), we get

$$\begin{aligned} \dot{\tilde{\theta}}_i(t) &= \sqrt{\alpha_i\omega_i} \cos(\omega_i t + k_i y(t)), \\ &= \sqrt{\alpha_i\omega_i} \cos(\omega_i t) \cos(k_i y(t)) \\ &\quad - \sqrt{\alpha_i\omega_i} \sin(\omega_i t) \sin(k_i y(t)), \quad i = 1, 2 \end{aligned} \tag{72}$$

From Scheinker and Krstic (2014, 2017), grouping (72) into the vector, the averaged vector system is given by

$$\dot{\tilde{\theta}}_{av}(t) = -\frac{k\alpha}{2}H\tilde{\theta}_{av}(t). \tag{73}$$

Setting $k_1 = k_2 = k$ (the sign of k is the same with the sign of H), $\alpha_1 = \alpha_2 = \alpha$, $\omega_i = \frac{2\pi l_i}{\varepsilon}$, $i = 1, 2$, $l_1 \neq l_2$, $l_1, l_2 \in \mathbb{N}$, we apply the time-delay method to averaging of (72). After a lengthy

calculation similar to the scalar case (57)–(64), we arrive at the closed-loop system

$$\frac{d}{dt} [\tilde{\theta}(t) - G(t)] = -\frac{\alpha k}{2}H\tilde{\theta}(t) - \pi Y_1(t) - 2\pi Y_2(t), \tag{74}$$

where $\dot{\tilde{\theta}}(t)$ is defined by the right-hand side of (72) and

$$\begin{aligned} Y_1(t) &= \frac{\alpha k}{\varepsilon^2} \int_{t-\varepsilon}^t \int_{\tau}^t \int_s^t M(\tau, s, \xi) H \dot{\tilde{\theta}}(\xi) d\xi ds d\tau, \\ Y_2(t) &= \frac{\alpha k^2}{\varepsilon^2} \int_{t-\varepsilon}^t \int_{\tau}^t \int_s^t N(\tau, s, \xi) H \tilde{\theta}(\xi) \tilde{\theta}^T(\xi) H \dot{\tilde{\theta}}(\xi) \\ &\quad \times d\xi ds d\tau. \end{aligned} \tag{75}$$

in which

$$\begin{aligned} M(\tau, s, \xi) &= \begin{bmatrix} M_{11}(\tau, s, \xi) & M_{12}(\tau, s, \xi) \\ M_{21}(\tau, s, \xi) & M_{22}(\tau, s, \xi) \end{bmatrix}, \\ M_{ii}(\tau, s, \xi) &= l_i \left[\sin\left(\frac{2\pi}{\varepsilon} l_i(\tau + s) + 2ky(\xi)\right) \right. \\ &\quad \left. + \sin\left(\frac{2\pi}{\varepsilon} l_i(\tau - s)\right) \right], \\ M_{ij}(\tau, s, \xi) &= \sqrt{l_i l_j} \left[\sin\left(\frac{2\pi}{\varepsilon} (l_i \tau + l_j s) + 2ky(\xi)\right) \right. \\ &\quad \left. + \sin\left(\frac{2\pi}{\varepsilon} (l_i \tau - l_j s)\right) \right], \\ N(\tau, s, \xi) &= \begin{bmatrix} N_{11}(\tau, s, \xi) & N_{12}(\tau, s, \xi) \\ N_{21}(\tau, s, \xi) & N_{22}(\tau, s, \xi) \end{bmatrix}, \\ N_{ii}(\tau, s, \xi) &= l_i \cos\left(\frac{2\pi}{\varepsilon} l_i(\tau + s) + 2ky(\xi)\right), \\ N_{ij}(\tau, s, \xi) &= \sqrt{l_i l_j} \cos\left(\frac{2\pi}{\varepsilon} (l_i \tau + l_j s) + 2ky(\xi)\right), \\ &\quad i, j = 1, 2, \quad i \neq j. \end{aligned} \tag{76}$$

It is seen that the system (74) is a perturbation of the stable averaged system (73).

Theorem 4. Assume that the Hessian H is known, whereas the extremum value Q^* is unknown, the extremum point is uncertain but each of its elements belongs to the known interval $\theta_i^* \in [\underline{\theta}_i^*, \bar{\theta}_i^*]$, $i = 1, 2$. Consider the closed-loop system consisting of the vector plant (70) and bounded ES controller (69), with the initial condition $|\tilde{\theta}(0)| \leq \sigma_0$. Given tuning parameters $q, \delta, \varepsilon^*, k, l_1, l_2, \alpha > 0$ and σ_1, σ_2 satisfying $\sigma_1^2 + \sigma_2^2 > \sigma_0^2$, let matrices $P > I, R > 0$ and scalars $\lambda_P, \lambda_R, \gamma_1, \gamma_2 > 0$ satisfy the LMIs:

$$\begin{aligned} \Phi_1 &= \begin{bmatrix} P-I & -P \\ * & P+e^{-2\delta\varepsilon^*}R \end{bmatrix} > 0, \\ \Phi_2 &= \begin{bmatrix} -P(\alpha kH-2\delta I) & \left(\frac{\alpha k}{2}H-2\delta I\right)P & -\pi P & -2\pi P \\ * & -\frac{4}{\varepsilon^*}e^{-2\delta\varepsilon^*}R+2\delta P & \pi P & 2\pi P \\ * & * & -\frac{\gamma_1}{\sqrt{\varepsilon^*}}I & 0 \\ * & * & * & -\frac{\gamma_2}{\sqrt{\varepsilon^*}}I \end{bmatrix} < 0, \\ \Phi_3 &= \left(1 + \frac{1}{q} \right)^2 \lambda_P \sigma_0^2 + 2\pi\alpha\varepsilon^*(l_1 + l_2) \left[\left(1 + \frac{1}{q} \right) \right. \\ &\quad \times (1+q)\lambda_P + \frac{1+q}{4}\lambda_P + \frac{1}{3}\lambda_R \left. \right] + \left[\frac{\pi\alpha}{\delta}\lambda_R(l_1 + l_2) \right. \\ &\quad \left. + \frac{\pi\alpha^3k^2\sqrt{\varepsilon^*}}{\delta} \left(\frac{\gamma_1}{9}\Delta_1^2 + \frac{\gamma_2}{36}k^2\Delta_2^2 \right) \right] < \sigma^2, \\ P - \lambda_P I &< 0, \quad R - \lambda_R I < 0, \\ \Phi_4 &= \sigma_0 + \sqrt{2\pi\alpha\varepsilon^*(l_1 + l_2)} < \sigma, \end{aligned} \tag{77}$$

where

$$\begin{aligned} \sigma &= \sqrt{\sigma_1^2 + \sigma_2^2}, \\ \Delta_1 &= \sqrt{(l_1^2 + l_2^2)(|h_{11}|\sqrt{l_1} + |h_{12}|\sqrt{l_2})} \\ &\quad + \sqrt{(l_1 l_2 + l_2^2)(|h_{12}|\sqrt{l_1} + |h_{22}|\sqrt{l_2})}, \\ \Delta_2 &= \left[\sqrt{(l_1^2 + l_2^2)(|h_{11}|\sigma_1 + |h_{12}|\sigma_2)} \right. \\ &\quad \left. + \sqrt{(l_1 l_2 + l_2^2)(|h_{12}|\sigma_1 + |h_{22}|\sigma_2)} \right] \\ &\quad \times \left[(\sigma_1|h_{11}| + \sigma_2|h_{12}|)\sqrt{l_1} + (\sigma_1|h_{12}| + \sigma_2|h_{22}|)\sqrt{l_2} \right]. \end{aligned} \tag{78}$$

Then $\forall \varepsilon \in (0, \varepsilon^*]$ the solution of the closed-loop system (72) satisfies

$$\begin{aligned} |\tilde{\theta}(t)|^2 &\leq (|\tilde{\theta}(0)| + \sqrt{2\pi\alpha\varepsilon(l_1 + l_2)})^2 < \sigma^2, \quad t \in [0, \varepsilon], \\ |\tilde{\theta}(t)|^2 &< \left(1 + \frac{1}{q}\right)^2 \lambda_P e^{-2\delta(t-\varepsilon)} |\tilde{\theta}(0)|^2 \\ &+ 2\pi\alpha\varepsilon(l_1 + l_2)e^{-2\delta(t-\varepsilon)} \left[\left(1 + \frac{1}{q}\right)(1+q)\lambda_P \right. \\ &+ \left. \frac{1+q}{4}\lambda_P + \frac{1}{3}\lambda_R \right] + (1 - e^{-2\delta(t-\varepsilon)}) \left[\frac{\pi\alpha}{\delta}\lambda_R(l_1 + l_2) \right. \\ &+ \left. \frac{\pi\alpha^3 k^2 \sqrt{\varepsilon}}{\delta} \left(\frac{\gamma_1}{9}\Delta_1^2 + \frac{\gamma_2}{36}k^2\Delta_2^2 \right) \right] < \sigma^2, \quad t \geq \varepsilon. \end{aligned} \tag{79}$$

Moreover, for all $\varepsilon \in (0, \varepsilon^*]$ and all initial conditions $|\tilde{\theta}(0)| \leq \sigma_0$ the ball

$$\Theta = \left\{ \tilde{\theta} \in \mathbb{R}^2 : |\tilde{\theta}|^2 < \frac{\pi\alpha}{\delta}\lambda_R(l_1 + l_2) + \frac{\pi\alpha^3 k^2 \sqrt{\varepsilon}}{\delta} \left(\frac{\gamma_1}{9}\Delta_1^2 + \frac{\gamma_2}{36}k^2\Delta_2^2 \right) \right\} \tag{80}$$

is exponentially attractive with a decay rate δ . ■

Proof. See Appendix B. ■

Remark 3. Comparing the classical ES (5) and (32) with the bounded ES (52) and (69), the difference is that the output function being optimized enters the classical ES control system in an affine way, whereas the output function in the update law of bounded ES is confined to the argument of a cosine term. That is to say, the dynamics of classical ES depends upon the output, whereas the update rate of bounded ES is independent of the output function. Accordingly, the ultimate bound of the seeking error for the classical ES (19) and (49) is related to the bound of the output, whereas the ultimate bound of the seeking error for the bounded ES (68) and (80) has no such relation with the output bound. Furthermore, the ultimate bound of classical ES in (19) mainly depends upon ε , whereas the ultimate bound of bounded ES in (68) is based on two parameters α and ε . A possible choice is $\alpha = O(\varepsilon)$, $k = O(\varepsilon^{-\frac{1}{2}})$, which leads to $\delta = O(\varepsilon^{\frac{1}{2}})$ and the ultimate bound is of the order of $O(\sqrt{\varepsilon})$. If the ultimate bound of two approaches are of the same order of $O(\sqrt{\varepsilon})$, the convergence rate of the bounded ES is less than that of the classical ES ($\delta = O(1)$). For the bounded ES, the detailed qualitative analysis of the ultimate bound is not given in Scheinker and Krstic (2014). ■

4. Examples

4.1. Scalar systems

Given the scalar map $J(\theta(t)) = \theta^2(t) + f^*$, we consider the classical ES

$$\begin{aligned} \dot{\theta}(t) &= \hat{\theta}(t) + a \sin(\omega t), \\ \hat{\theta}(t) &= ka \sin(\omega t) J(\theta(t)), \end{aligned} \tag{81}$$

with $k = -1.3$, $a = 0.1$, as well as bounded ES

$$\dot{\theta}(t) = \sqrt{\alpha\omega} \cos(\omega t + kJ(\theta(t))), \tag{82}$$

with $\alpha = 0.0001$, $k = 11$. The LMI solution is shown in Table 1 (where ES refers to classical ES and BES refers to bounded ES). For the map $J(\theta(t)) = \frac{f''}{2}\theta^2(t) + f^*$, referring to (23) and (25), if f'' and f^* are unknown but satisfy $0 < f''_m < |f''| < f''_M$ and $|f^*| < f^*_M$, where $f''_m = 1.9$, $f''_M = 2.1$ and $f^*_M = 0.1$ are known for classical ES or $f^*_M = \infty$ for bounded ES, the LMI solutions for the classical and bounded ES are shown in Table 2.

Table 1
Scalar systems.

	ε	$\omega = \frac{2\pi}{\varepsilon}$	σ_0	σ	Ultimate Bound
ES ($f^* = 0$)	0.22	28.55	$1e^{-2}$	1	0.98
ES ($f^* = 0$)	0.021	299.05	1	$\sqrt{2}$	0.68
BES ($\forall f^*$)	0.043	146.05	$1e^{-2}$	1	0.99
BES ($\forall f^*$)	0.013	483.08	1	2	1.52

Table 2
Scalar systems with uncertainties.

	ε	$\omega = \frac{2\pi}{\varepsilon}$	σ_0	σ	Ultimate Bound
ES ($f^*_M = 0.1$)	0.17	36.94	$1e^{-2}$	1	0.98
ES ($f^*_M = 0.1$)	0.017	369.41	1	$\sqrt{2}$	0.67
BES ($\forall f^*_M$)	0.024	261.67	$1e^{-2}$	1	0.99
BES ($\forall f^*_M$)	0.007	897.14	1	2	1.52

Table 3
Vector systems.

	ε	$\omega = \frac{2\pi}{\varepsilon}$	σ_0	σ	Ultimate Bound
ES ($Q^* = 0$)	3.33	1.89	$1e^{-3}$	$\sqrt{2}$	1.26
ES ($Q^* = 0$)	0.36	17.44	$\sqrt{2}$	2	0.94
BES ($\forall Q^*$)	0.0004	15700	$1e^{-3}$	$\sqrt{2}$	1.41
BES ($\forall Q^*$)	0.0001	62800	$\sqrt{2}$	$2\sqrt{2}$	2.05

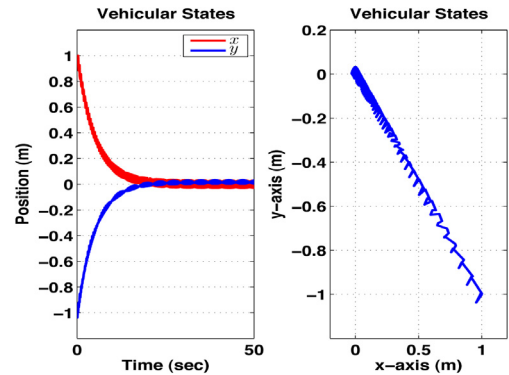


Fig. 4. Classical ES for GPS-denied 2D vehicle control.

4.2. Vector systems: GPS-denied 2D vehicle control

In this section we consider an autonomous vehicle in an environment without GPS orientation (Scheinker & Krstic, 2014, 2017). The goal is to reach the location of the stationary minimum of a measurable function $J(x(t), y(t)) = x^2(t) + y^2(t) + Q^*$. In Scheinker and Krstic (2014, 2017), the bounded ES is considered. We employ the classical ES

$$\begin{aligned} x(t) &= \hat{x}(t) + a_1 \cos(\omega_1 t), \quad y(t) = \hat{y}(t) + a_2 \sin(\omega_2 t) \\ \dot{\hat{x}}(t) &= \frac{2k_1}{a_1} \cos(\omega_1 t) J(t), \quad \dot{\hat{y}}(t) = \frac{2k_2}{a_2} \sin(\omega_2 t) J(t), \end{aligned} \tag{83}$$

with $k_1 = k_2 = -0.001$, $a_1 = a_2 = 2$, $\omega_2 = 2\omega_1$, as well as bounded ES

$$\begin{aligned} \dot{x}(t) &= \sqrt{\alpha_1\omega_1} \cos(\omega_1 t + k_1 J(t)), \\ \dot{y}(t) &= \sqrt{\alpha_2\omega_2} \sin(\omega_2 t + k_2 J(t)), \end{aligned} \tag{84}$$

with $\alpha_1 = \alpha_2 = 0.0001$, $k_1 = k_2 = 11$, $\omega_2 = 2\omega_1$. The LMI solution is shown in Table 3.

For the numerical simulations, under the initial condition $x(0) = 1$, $y(0) = -1$ and $\varepsilon = 0.36$, the results of two methods for $Q^* = 0$ are shown in Figs. 4 and 5, respectively.

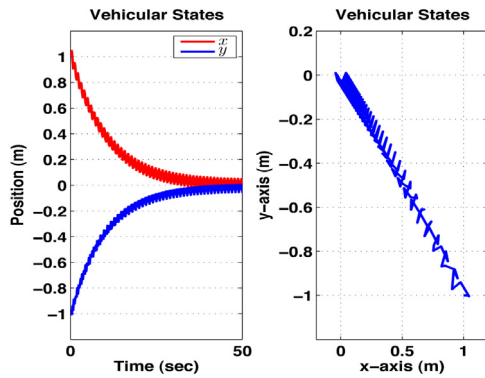


Fig. 5. Bounded ES for GPS-denied 2D vehicle control.

5. Conclusion

This article offers a constructive method based on time-delay approach to averaging to prove the stability of ES control systems. The resulting time-delay method allows to derive a precise computation on the bound of the seeking error, through which users are able to obtain detailed guidelines to choose tuning parameters like the frequency and amplitude of perturbation signals, as well as the adaptation gains.

The time-delay approach perhaps opens a gate for extremum seeking where the static maps have sampled-data and delayed measurements. Other possible topics are ES for dynamic maps and time-varying systems via the time-delay approach to averaging.

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Appendix A. Proof of Theorem 2

For the analysis of the plant (44), we choose

$$V_p(t) = [\tilde{\theta}(t) - G(t)]^T P [\tilde{\theta}(t) - G(t)]. \tag{A.1}$$

Thus we have

$$\begin{aligned} \dot{V}_p(t) + 2\delta V_p(t) &= 2 [\tilde{\theta}(t) - G(t)]^T P \\ &\times [KH\tilde{\theta}(t) - Y_1(t) - Y_2(t)] \\ &+ 2\delta [\tilde{\theta}(t) - G(t)]^T P [\tilde{\theta}(t) - G(t)], \\ &= 2\tilde{\theta}^T(t)P(KH + \delta I)\tilde{\theta}(t) + 2\delta G^T(t)PG(t) \\ &- 2\tilde{\theta}^T(t)(HK + 2\delta I)PG(t) \\ &- 2\tilde{\theta}^T(t)PY_1(t) - 2\tilde{\theta}^T(t)PY_2(t) \\ &+ 2G^T(t)PY_1(t) + 2G^T(t)PY_2(t). \end{aligned} \tag{A.2}$$

To compensate $G(t)$ in (A.2), we utilize

$$V_R(t) = \frac{1}{\varepsilon} \int_{t-\varepsilon}^t e^{-2\delta(t-\tau)} (\tau - t + \varepsilon)^2 \dot{\tilde{\theta}}^T(\tau) R \dot{\tilde{\theta}}(\tau) d\tau. \tag{A.3}$$

Then we get

$$\begin{aligned} \dot{V}_R(t) + 2\delta V_R(t) &= \varepsilon \dot{\tilde{\theta}}^T(t) R \dot{\tilde{\theta}}(t) \\ &- \frac{2}{\varepsilon} \int_{t-\varepsilon}^t e^{-2\delta(t-\tau)} (\tau - t + \varepsilon) \dot{\tilde{\theta}}^T(\tau) R \dot{\tilde{\theta}}(\tau) d\tau, \\ &\leq \varepsilon \dot{\tilde{\theta}}^T(t) R \dot{\tilde{\theta}}(t) - \frac{2}{\varepsilon} e^{-2\delta\varepsilon} \int_{t-\varepsilon}^t (\tau - t + \varepsilon) \dot{\tilde{\theta}}^T(\tau) R \dot{\tilde{\theta}}(\tau) d\tau, \\ &\leq \varepsilon \dot{\tilde{\theta}}^T(t) R \dot{\tilde{\theta}}(t) - \frac{4}{\varepsilon} e^{-2\delta\varepsilon} G^T(t) R G(t). \end{aligned} \tag{A.4}$$

in which the extended Jensen's inequality (2) is employed

$$\begin{aligned} &2G^T(t)RG(t) \\ &= \frac{2}{\varepsilon^2} \int_{t-\varepsilon}^t (\tau - t + \varepsilon) \dot{\tilde{\theta}}^T(\tau) d\tau R \int_{t-\varepsilon}^t (\tau - t + \varepsilon) \dot{\tilde{\theta}}(\tau) d\tau, \\ &\leq \frac{2}{\varepsilon^2} \int_{t-\varepsilon}^t (\tau - t + \varepsilon) d\tau \cdot \int_{t-\varepsilon}^t (\tau - t + \varepsilon) \dot{\tilde{\theta}}^T(\tau) R \dot{\tilde{\theta}}(\tau) d\tau, \\ &= \int_{t-\varepsilon}^t (\tau - t + \varepsilon) \dot{\tilde{\theta}}^T(\tau) R \dot{\tilde{\theta}}(\tau) d\tau. \end{aligned} \tag{A.5}$$

Define the Lyapunov candidate as

$$V(t) = V_p(t) + V_R(t). \tag{A.6}$$

With Jensen's inequality (1), we have

$$\begin{aligned} V(t) &= [\tilde{\theta}(t) - G(t)]^T P [\tilde{\theta}(t) - G(t)] \\ &+ \frac{1}{\varepsilon} \int_{t-\varepsilon}^t e^{-2\delta(t-\tau)} (\tau - t + \varepsilon)^2 \dot{\tilde{\theta}}^T(\tau) R \dot{\tilde{\theta}}(\tau) d\tau \\ &\geq \tilde{\theta}^T(t)P\tilde{\theta}(t) + G^T(t)PG(t) - 2\tilde{\theta}^T(t)PG(t) \\ &+ e^{-2\delta\varepsilon} \frac{1}{\varepsilon^2} \int_{t-\varepsilon}^t (\tau - t + \varepsilon) \dot{\tilde{\theta}}^T(\tau) d\tau R \\ &\times \int_{t-\varepsilon}^t (\tau - t + \varepsilon) \dot{\tilde{\theta}}(\tau) d\tau \\ &= \begin{bmatrix} \tilde{\theta}(t) \\ G(t) \end{bmatrix}^T \begin{bmatrix} P & -P \\ * & P + e^{-2\delta\varepsilon} R \end{bmatrix} \begin{bmatrix} \tilde{\theta}(t) \\ G(t) \end{bmatrix} \geq |\tilde{\theta}(t)|^2, \end{aligned} \tag{A.7}$$

where the above inequality holds due to $\Phi_1 > 0$ in (46).

Taking into account (A.2) and (A.4), we get

$$\begin{aligned} \dot{V}(t) + 2\delta V(t) &- \frac{\gamma_1}{\varepsilon} Y_1^2(t) - \frac{\gamma_2}{\varepsilon} Y_2^2(t) \\ &\leq \xi^T(t)\Phi_2\xi(t) + \varepsilon \dot{\tilde{\theta}}^T(t) R \dot{\tilde{\theta}}(t), \end{aligned} \tag{A.8}$$

where $\xi(t) = [\tilde{\theta}(t), G(t), Y_1(t), Y_2(t)]^T$ and Φ_2 is defined in (46). When the overall bound

$$|\tilde{\theta}(t)| = \sqrt{\tilde{\theta}_1^2(t) + \tilde{\theta}_2^2(t)} < \sqrt{\sigma_1^2 + \sigma_2^2} = \sigma, \forall t \geq 0 \tag{A.9}$$

is supposed, Eqs. (31)–(34) suggest

$$\begin{aligned} |y(t)| &= \left| Q^* + \frac{1}{2} (\tilde{\theta}(t) + S(t))^T H (\tilde{\theta}(t) + S(t)) \right| \\ &\leq |Q^*| + \frac{\lambda(H)}{2} (\tilde{\theta}(t) + S(t))^2, \\ &< |Q^*| + \frac{\lambda(H)}{2} \left(\sigma + \sqrt{a_1^2 + a_2^2} \right)^2 = \bar{\Delta}, \quad t \geq 0 \\ |\dot{\tilde{\theta}}(t)| &= |KM(t)y(t)| < \sqrt{\frac{4k_1^2}{a_1^2} + \frac{4k_2^2}{a_2^2}} \bar{\Delta} = \Delta, \end{aligned} \tag{A.10}$$

where $\lambda(H) = \max\{|\lambda_{\min}(H)|, |\lambda_{\max}(H)|\}$ and

$$\begin{aligned} |\tilde{\theta}(t)| &= \left| \tilde{\theta}(0) + \int_0^t \dot{\tilde{\theta}}(s) ds \right| < |\tilde{\theta}(0)| + \varepsilon \Delta, \quad t \in [0, \varepsilon], \\ |\tilde{\theta}(\varepsilon)|^2 &< \left(1 + \frac{1}{q} \right) |\tilde{\theta}(0)|^2 + (1 + q)\varepsilon^2 \Delta^2. \end{aligned} \tag{A.11}$$

The first inequality in (48) follows from (A.11) since $\Phi_4 < \sigma$ in (46).

From (A.1), we have

$$\begin{aligned}
 V_P(t) &= \left[\tilde{\theta}(t) - \frac{1}{\varepsilon} \int_{t-\varepsilon}^t (\tau - t + \varepsilon) \dot{\tilde{\theta}}(\tau) d\tau \right]^T P \\
 &\quad \times \left[\tilde{\theta}(t) - \frac{1}{\varepsilon} \int_{t-\varepsilon}^t (\tau - t + \varepsilon) \dot{\tilde{\theta}}(\tau) d\tau \right], \\
 &\leq \lambda_P \left[\tilde{\theta}(t) - \frac{1}{\varepsilon} \int_{t-\varepsilon}^t (\tau - t + \varepsilon) \dot{\tilde{\theta}}(\tau) d\tau \right]^2, \\
 &< \left(1 + \frac{1}{q} \right) \lambda_P |\tilde{\theta}(t)|^2 + \frac{(1+q)\lambda_P}{\varepsilon^2} \\
 &\quad \times \left[\int_{t-\varepsilon}^t (\tau - t + \varepsilon) d\tau \right]^2 \cdot \Delta^2, \\
 &= \left(1 + \frac{1}{q} \right) \lambda_P |\tilde{\theta}(t)|^2 + \frac{\varepsilon^2(1+q)}{4} \lambda_P \Delta^2.
 \end{aligned} \tag{A.12}$$

From (A.3), we get

$$\begin{aligned}
 V_R(t) &= \frac{1}{\varepsilon} \int_{t-\varepsilon}^t e^{-2\delta(t-\tau)} (\tau - t + \varepsilon)^2 \dot{\tilde{\theta}}^T(\tau) R \dot{\tilde{\theta}}(\tau) d\tau, \\
 &< \frac{\lambda_R}{\varepsilon} \int_{t-\varepsilon}^t (\tau - t + \varepsilon)^2 d\tau \cdot \Delta^2 = \frac{\varepsilon^2}{3} \lambda_R \Delta^2.
 \end{aligned} \tag{A.13}$$

From (45) we have

$$\begin{aligned}
 |Y_1(t)| &= \left| \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t N_1(\tau) H \dot{\tilde{\theta}}(s) ds d\tau \right|, \\
 &\leq \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t |N_1(\tau) H \dot{\tilde{\theta}}(s)| ds d\tau, \\
 &= \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t \left[\begin{array}{cc} N_{1,11}(\tau) & N_{1,12}(\tau) \\ N_{1,21}(\tau) & N_{1,22}(\tau) \end{array} \right] H \dot{\tilde{\theta}}(s) ds d\tau, \\
 &= \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t \left(\left[\begin{array}{cc} N_{1,11}(\tau) & \\ & N_{1,21}(\tau) \end{array} \right] [1, 0] + \left[\begin{array}{cc} N_{1,12}(\tau) & \\ & N_{1,22}(\tau) \end{array} \right] [0, 1] \right) H \dot{\tilde{\theta}}(s) \\
 &\quad \times ds d\tau, \\
 &\leq \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t \left(\sqrt{4k_1^2 + \frac{4k_2^2 a_1^2}{a_2^2}} |h_{11} \dot{\tilde{\theta}}_1(s) + h_{12} \dot{\tilde{\theta}}_2(s)| \right. \\
 &\quad \left. + \sqrt{\left(\frac{4k_1^2 a_2^2}{a_1^2} + 4k_2^2 \right)} |h_{12} \dot{\tilde{\theta}}_1(s) + h_{22} \dot{\tilde{\theta}}_2(s)| \right) ds d\tau, \\
 &< \varepsilon \Delta_1 \bar{\Delta},
 \end{aligned} \tag{A.14}$$

$$\begin{aligned}
 |Y_2(t)| &= \left| \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t N_2(\tau) \tilde{\theta}^T(s) H \dot{\tilde{\theta}}(s) ds d\tau \right|, \\
 &\leq \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t |N_2(\tau) \tilde{\theta}^T(s) H \dot{\tilde{\theta}}(s)| ds d\tau, \\
 &\leq \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t \left[\begin{array}{c} N_{2,1}(\tau) \\ N_{2,2}(\tau) \end{array} \right] \left| \tilde{\theta}^T(s) H \dot{\tilde{\theta}}(s) \right| ds d\tau, \\
 &\leq \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \int_{\tau}^t \sqrt{\left(\frac{4k_1^2}{a_1^2} + \frac{4k_2^2}{a_2^2} \right)} \left(\tilde{\theta}_1(s) h_{11} + \tilde{\theta}_2(s) h_{12} \right) \dot{\tilde{\theta}}_1(s) \\
 &\quad + \left(\tilde{\theta}_1(s) h_{12} + \tilde{\theta}_2(s) h_{22} \right) \dot{\tilde{\theta}}_2(s) ds d\tau < \varepsilon \Delta_2 \bar{\Delta}.
 \end{aligned} \tag{A.15}$$

Via (A.8), (A.14)–(A.15), and $\Phi_2 < 0$ in (46), we derive

$$\begin{aligned}
 \dot{V}(t) + 2\delta V(t) &\leq \varepsilon \dot{\tilde{\theta}}^T(t) R \dot{\tilde{\theta}}(t) + \frac{\gamma_1}{\varepsilon} Y_1^2(t) + \frac{\gamma_2}{\varepsilon} Y_2^2(t), \\
 &< \varepsilon \left[\lambda_R \left(\frac{4k_1^2}{a_1^2} + \frac{4k_2^2}{a_2^2} \right) + \gamma_1 \Delta_1^2 + \gamma_2 \Delta_2^2 \right] \bar{\Delta}^2.
 \end{aligned} \tag{A.16}$$

Applying the comparison principle to (A.16), we have

$$\begin{aligned}
 V(t) &< V(\varepsilon) e^{-2\delta(t-\varepsilon)} + (1 - e^{-2\delta(t-\varepsilon)}) \frac{\varepsilon}{2\delta} \\
 &\quad \times \left[\lambda_R \left(\frac{4k_1^2}{a_1^2} + \frac{4k_2^2}{a_2^2} \right) + \gamma_1 \Delta_1^2 + \gamma_2 \Delta_2^2 \right] \bar{\Delta}^2, \quad t \geq \varepsilon
 \end{aligned} \tag{A.17}$$

Employing $V(t) \geq |\tilde{\theta}(t)|^2$ that follows from (A.7) and $V(\varepsilon) < \left(1 + \frac{1}{q} \right)^2 \lambda_P |\tilde{\theta}(0)|^2 + \varepsilon^2 \left[\left(1 + \frac{1}{q} \right) (1+q)\lambda_P + \frac{1+q}{4}\lambda_P + \frac{1}{3}\lambda_R \right] \Delta^2$ that follows from (A.11)–(A.13), we arrive at the second inequality in (48) due to $\Phi_3 < \sigma^2$ in (46).

Finally, we prove that the conditions (46) guarantee the overall bound (A.9). Consider first $t \in [0, \varepsilon]$. Since $|\tilde{\theta}(0)| \leq \sigma_0 < \sigma$ and $\tilde{\theta}(t)$ is continuous in time, (A.9) holds for small enough $t > 0$. We assume by contradiction that for some $t \in (0, \varepsilon]$ the formula (A.9) does not hold. Namely, there exists the smallest time instance $t^* \in (0, \varepsilon]$ such that $|\tilde{\theta}(t^*)| = \sigma$ and $|\tilde{\theta}(t)| < \sigma$ when $t \in [0, t^*)$. Thus $|\tilde{\theta}(t)| \leq \sigma$ holds for all $t \in [0, t^*]$ and this leads to the first inequality of (A.11) in its non-strict version such that $|\tilde{\theta}(t)| \leq |\tilde{\theta}(0)| + \varepsilon \Delta \leq \sigma_0 + \varepsilon^* \Delta$ for $0 \leq t \leq t^* \leq \varepsilon$. Furthermore, the feasibility of $\Phi_4 < \sigma$ in (46) ensures that $|\tilde{\theta}(t^*)| \leq \sigma_0 + \varepsilon^* \Delta < \sigma$. This contradicts to the definition of t^* such that $|\tilde{\theta}(t^*)| = \sigma$. Hence $|\tilde{\theta}(t)|^2 < \sigma^2$ for $t \in [0, \varepsilon]$. Next, we prove (A.9) for $t \geq \varepsilon$. Note that since (A.11) is strict and holds for $t = \varepsilon$, it holds also for some $t > \varepsilon$ due to continuity of $\tilde{\theta}(t)$. We assume by contradiction that for some $t > \varepsilon$ the formula (A.11) does not hold. In other words, there exists the smallest time instance $t^* \in (\varepsilon, \infty)$ such that $|\tilde{\theta}(t^*)|^2 = \sigma^2$ and $|\tilde{\theta}(t)|^2 < \sigma^2$ when $t \in [\varepsilon, t^*)$. Thus $|\tilde{\theta}(t)|^2 \leq \sigma^2$ holds for all $t \in [\varepsilon, t^*]$ and this leads to (A.17) in its non-strict version $V(t) \leq V(\varepsilon) e^{-2\delta(t-\varepsilon)} + (1 - e^{-2\delta(t-\varepsilon)}) \frac{\varepsilon}{2\delta} (\gamma_1 k^2 a^4 + \frac{\gamma_2}{4} k^2 a^2 \sigma^2 + R) \Delta^2$, for $t \geq \varepsilon$. Moreover, the feasibility of $\Phi_3 < \sigma^2$ in (46) ensures $|\tilde{\theta}(t)|^2 < \sigma^2$ in the second equality of (48) for any $t \in [\varepsilon, t^*]$. This contradicts to the definition of t^* such that $|\tilde{\theta}(t^*)|^2 = \sigma^2$. Hence $|\tilde{\theta}(t)|^2 \leq \sigma^2$ for $t \geq \varepsilon$. Then Theorem 2 is proved.

Appendix B. Proof of Theorem 4

The LKF are identical with (A.1), (A.3), (A.6). From (69), we get

$$\begin{aligned}
 \left| \dot{\tilde{\theta}}(t) \right| &= \left[\frac{2\pi\alpha l_1}{\varepsilon} \cos^2 \left(\frac{2\pi l_1}{\varepsilon} t + ky(t) \right) \right. \\
 &\quad \left. + \frac{2\pi\alpha l_2}{\varepsilon} \cos^2 \left(\frac{2\pi l_2}{\varepsilon} t + ky(t) \right) \right]^{\frac{1}{2}}, \\
 &\leq \sqrt{\frac{2\pi\alpha}{\varepsilon} (l_1 + l_2)}, \quad \forall t \geq 0,
 \end{aligned} \tag{B.1}$$

$$\begin{aligned}
 |\tilde{\theta}(t)| &= \left| \tilde{\theta}(0) + \int_0^t \dot{\tilde{\theta}}(s) ds \right| \leq |\tilde{\theta}(0)| + \sqrt{2\pi\alpha\varepsilon(l_1 + l_2)}, \\
 &t \in [0, \varepsilon],
 \end{aligned}$$

$$|\tilde{\theta}(\varepsilon)|^2 \leq \left(1 + \frac{1}{q} \right) |\tilde{\theta}(0)|^2 + (1+q)2\pi\alpha\varepsilon(l_1 + l_2).$$

We assume the overall bound

$$|\tilde{\theta}(t)| = \sqrt{\tilde{\theta}_1^2(t) + \tilde{\theta}_2^2(t)} < \sqrt{\sigma_1^2 + \sigma_2^2} = \sigma, \quad \forall t \geq 0. \tag{B.2}$$

From (75), we have

$$\begin{aligned}
 |Y_1(t)| &= \left| \frac{\alpha k}{\varepsilon^2} \int_{t-\varepsilon}^t \int_{\tau}^t \int_s^t M(\tau, s, \xi) H \dot{\tilde{\theta}}(\xi) d\xi ds d\tau \right|, \\
 &\leq \frac{|\alpha k|}{\varepsilon^2} \int_{t-\varepsilon}^t \int_{\tau}^t \int_s^t |M(\tau, s, \xi) H \dot{\tilde{\theta}}(\xi)| d\xi ds d\tau, \\
 &= \frac{|\alpha k|}{\varepsilon^2} \int_{t-\varepsilon}^t \int_{\tau}^t \int_s^t \left(\left[\begin{array}{c} M_{11}(\tau, s, \xi) \\ M_{21}(\tau, s, \xi) \end{array} \right] [1, 0] \right. \\
 &\quad \left. + \left[\begin{array}{c} M_{12}(\tau, s, \xi) \\ M_{22}(\tau, s, \xi) \end{array} \right] [0, 1] \right) |H \dot{\tilde{\theta}}(\xi)| d\xi ds d\tau, \\
 &\leq \frac{|\alpha k|}{3} \left[\sqrt{(l_1^2 + l_2^2)} (|h_{11}| \sqrt{l_1} + |h_{12}| \sqrt{l_2}) \right. \\
 &\quad \left. + \sqrt{(l_1 l_2 + l_2^2)} (|h_{12}| \sqrt{l_1} + |h_{22}| \sqrt{l_2}) \right] \sqrt{2\pi\alpha\varepsilon}
 \end{aligned} \tag{B.3}$$

and

$$\begin{aligned}
 |Y_2(t)| &= \left| \frac{\alpha k^2}{\varepsilon^2} \int_{t-\varepsilon}^t \int_{\tau}^t \int_s^t N(\tau, s, \xi) H \tilde{\theta}(\xi) \right. \\
 &\quad \left. \times \tilde{\theta}^T(\xi) H \dot{\tilde{\theta}}(\xi) d\xi ds d\tau \right|, \\
 &\leq \frac{|\alpha| k^2}{\varepsilon^2} \int_{t-\varepsilon}^t \int_{\tau}^t \int_s^t \left| N(\tau, s, \xi) H \tilde{\theta}(\xi) \tilde{\theta}^T(\xi) H \dot{\tilde{\theta}}(\xi) \right| d\xi ds d\tau, \\
 &= \frac{|\alpha| k^2}{\varepsilon^2} \int_{t-\varepsilon}^t \int_{\tau}^t \int_s^t \left(\begin{bmatrix} N_{11}(\tau, s, \xi) \\ N_{21}(\tau, s, \xi) \end{bmatrix} \begin{bmatrix} 1, 0 \\ 0, 1 \end{bmatrix} \right) \\
 &\quad \left| H \tilde{\theta}(\xi) \tilde{\theta}^T(\xi) H \dot{\tilde{\theta}}(\xi) \right| d\xi ds d\tau, \tag{B.4} \\
 &< \frac{|\alpha| k^2}{6} \left[\sqrt{(l_1^2 + l_1 l_2)(|h_{11}| \sigma_1 + |h_{12}| \sigma_2)} \right. \\
 &\quad \left. + \sqrt{(l_1 l_2 + l_2^2)(|h_{12}| \sigma_1 + |h_{22}| \sigma_2)} \right] \\
 &\quad \times \left[(\sigma_1 |h_{11}| + \sigma_2 |h_{12}|) \sqrt{l_1} \right. \\
 &\quad \left. + (\sigma_1 |h_{12}| + \sigma_2 |h_{22}|) \sqrt{l_2} \right] \sqrt{2\pi \alpha \varepsilon}.
 \end{aligned}$$

The remaining argument is referred to the proof of Theorem 2.

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