Automatica 135 (2022) 109966

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Sampled-data finite-dimensional boundary control of 1D parabolic PDEs under point measurement via a novel ISS Halanay's inequality ☆

Rami Katz, Emilia Fridman *

School of Electrical Engineering, Tel-Aviv University, Tel-Aviv, Israel

ARTICLE INFO

Article history:
Received 17 January 2021
Received in revised form 15 May 2021
Accepted 25 July 2021
Available online 20 October 2021

Keywords:
Distributed parameter systems
Sampled-data control
ISS
Halanay's inequality
Observer-based control

ABSTRACT

Recently, finite-dimensional observer-based controllers were introduced for 1D parabolic PDEs via the modal decomposition method. In the present paper we suggest a sampled-data implementation of a finite-dimensional boundary controller for 1D parabolic PDEs under discrete-time point measurement. We consider the heat equation under boundary actuation and point (either in-domain or boundary) measurement. In order to manage with point measurement, we employ dynamic extension and prove H^1-stability. Due to dynamic extension, which leads to proportional–integral controller, we suggest a sampled-data implementation of the controller via a generalized hold device. We take into account the quantization effect that leads to a disturbed closed-loop system and input-to-state stability (ISS) analysis. We use Wirtinger-based piecewise continuous in time Lyapunov functionals which compensate sampling in the finite-dimensional state and lead to the simplest efficient stability conditions for ODEs. To compensate sampling in the infinite-dimensional tail, we introduce a novel form of Halanay's inequality for ISS, which is appropriate for functions with jump discontinuities that do not grow in the jumps. Numerical examples demonstrate the efficiency of our method.

1. Introduction

Finite-dimensional observer-based control for PDEs is attractive for applications and theoretically challenging. Such controllers for parabolic systems were designed by the modal decomposition approach in Balas (1988), Christofides (2001), Curtain (1982) and Harkort and Deutscher (2011). The latter results were mostly restricted to bounded control and observation operators, whereas efficient bounds on the observer and controller dimensions were missing. In the recent paper (Katz & Fridman, 2020a), the first constructive LMI-based method for finite-dimensional observer-based controller for the 1D heat equation was suggested, where the controller dimension and the resulting exponential decay rate were found from simple LMI conditions. Robustness of finite-dimensional controllers with respect to input and output delays was studied in Katz and Fridman (2021b). The results of Katz and Fridman (2020a) and Katz and Fridman (2021b) are confined to cases where at least one of the observation or control operators is bounded. Sampled-data and delayed boundary control of the 1D heat equation under boundary measurement was studied in Katz, Fridman and Selivanov (2021) by using an infinite-dimensional PDE observer. Finite-dimensional boundary control of a linear 1D Kuramoto–Sivashinsky equation (KSE) under point measurement was studied in Katz and Fridman (2021a) and Katz and Fridman (2020b), where a dynamic extension was employed.

Sampled-data finite-dimensional controllers for parabolic PDEs, implemented by zero-order hold devices, were suggested in Bar Am and Fridman (2014), Fridman and Blighovsky (2012) and Kang and Fridman (2018) for distributed static output-feedback control, in Karafyllis and Krstic (2017) and Karafyllis and Krstic (2018) for boundary state-feedback and in Katz and Fridman (2021b) and Katz, Fridman et al. (2021) for observer-based control. Event-triggered sampled-data control of PDEs has been studied in Espitia (2020), Espitia, Karafyllis, and Krstic (2021) and Selivanov and Fridman (2016a). Recently, input-to-state stability (ISS) of PDEs has regained much interest. ISS for the 1D heat equation with boundary disturbance was studied in Karafyllis and Krstic (2016). State-feedback with ISS analysis of diagonal boundary control systems was considered in Lhachemi, Shorten, and Prieur (2020). Non-coercive Lyapunov functionals for ISS of infinite-dimensional systems were studied in Jacob, Mironchenko, Partington, and Wirth (2019). A survey of ISS results can be found in Mironchenko and Prieur (2020).

☆ Supported by Israel Science Foundation (grant 673/19), the C. and H. Manderman Chair at Tel Aviv University and by the Y. and C. Weinstein Research Institute for Signal Processing. The material in this paper was partially presented at the 2021 European Control Conference, June 29–July 22, 2021, Virtual Conference. This paper was recommended for publication in revised form by Associate Editor Nikolaos Bekiaris-Liberis under the direction of Editor Miroslav Krstic.
* Corresponding author.
E-mail addresses: rami@benis.co.il (R. Katz), emilia@tauex.tau.ac.il (E. Fridman).

https://doi.org/10.1016/j.automatica.2021.109966

For sampled-data and delayed control of parabolic PDEs, combinations of Lyapunov functionals with Halanay's inequality appear to be an efficient tool. This combination was introduced for stabilization via the spatial decomposition method under point measurements in Fridman and Blighovsky (2012) and via modal decomposition in Katz and Fridman (2021b). This tool is also useful for ODEs with delays: for sampled-data control of nonlinear time-delays systems (Pepe & Fridman, 2017), decentralized delayed control of coupled ODE systems with delayed coupling (Zhu & Fridman, 2020) and distributed observers with time-varying delays (Silm et al., 2021).

Wirtinger-based Lyapunov functionals that are piecewise continuous in time lead to the simplest efficient LMI conditions for sampled-data control of ODEs (Liu & Fridman, 2012; Selivanov & Fridman, 2016b). For combination of such functionals with Halanay's inequality, an extension of Halanay's inequality to piecewise continuous in time functions is needed. Note that existing Halanay's inequalities for ISS are confined to continuous functions (Hien, Phat, & Trinh, 2015; Wen, Yu, & Wang, 2008). Moreover, the corresponding ISS bound has an additive constant. Therefore, using this bound between the sampling intervals in the case of piecewise continuous functions leads to an additive accumulation of this constant in the ISS bound as $t \to \infty$. Recently, a relaxed ISS Halanay's inequality for C^1 functions was suggested in Mazenc, Malisoff, and Krstic (2021) with an ISS bound in terms of some constants, whereas the values of these constants were given only implicitly.

In the present paper we suggest a sampled-data implementation of finite-dimensional boundary controllers for 1D parabolic PDEs under discrete-time point measurement. We consider the heat equation under boundary actuation and point (either in-domain or boundary) measurement. In order to manage with point measurement, we employ dynamic extension and prove H^1-stability. We derive a reduced-order closed-loop system. Our analysis leads to reduced-order LMIs that offer both computational and theoretical advantages (essentially simpler proofs of LMIs feasibility for large enough observer dimension N and of the fact that LMI feasibility for N implies feasibility for $N+1$). Such reduced-order conditions were initiated in our recent paper (Katz, Basre & Fridman, 2021) for the case of bounded measurements. Due to dynamic extension, we suggest a sampled-data implementation of the controller via a generalized hold device (see e.g. Mirkin (2016) for ODEs and references therein). We also take into account a quantization effect that leads to a disturbed closed-loop system and ISS analysis. Note that quantized control of PDEs was studied in Bekiaris-Liberis (2020) and Selivanov and Fridman (2016a).

An essential tool for our sampled-data ISS analysis is a novel ISS Halanay's inequality with explicit constants in the bounds, which is appropriate for functions with jump discontinuities that do not grow in the jumps. For sampled-data finite-dimensional control of the heat equation, we use Wirtinger-based Lyapunov functionals which compensate sampling in the finite-dimensional state, and combine them with the novel ISS Halanay's inequality that compensates for measurement sampling in the infinite-dimensional tail. Our Lyapunov-based ISS analysis results in an explicit estimate of the ultimate bound, in terms of the quantization error. Numerical examples show the efficiency of our method.

The article is organized as follows. Section 2 presents new ISS Halanay's inequalities, whose proofs are given in Appendix. As the first basic step for stabilization under unbounded control and observation operator, Section 3 considers finite-dimensional design in the continuous-time case under Dirichlet actuation and point measurement. Results on quantized sampled-data control under point measurement are presented in Sections 4 (Dirichlet actuation) and 5 (Neumann actuation). Numerical examples are given in Section 6 and Conclusions in Section 7. Some preliminary results on finite-dimensional design in the continuous-time case under Dirichlet actuation and point measurement were presented in Katz and Fridman (2021c), where the stability analysis was provided by using the full-order system leading to the full-order LMIs.

Notations and preliminaries: $L^2(0,1)$ is the Hilbert space of Lebesgue measurable and square integrable functions $f:[0,1]\to\mathbb{R}$ with the inner product $\langle f,g\rangle := \int_0^1 f(x)g(x)dx$ and induced norm $\|f\|^2 := \langle f,f\rangle$. $H^k(0,1)$ is the Sobolev space of functions $f:[0,1]\to\mathbb{R}$ having k square integrable weak derivatives, with the norm $\|f\|^2_{H^k} := \sum_{j=0}^k \left\|f^{(j)}\right\|^2$. The Euclidean norm on $\mathbb{R}^n$ is denoted by $|\cdot|$. We write $f\in H_0^1(0,1)$ if $f\in H^1(0,1)$ and $f(0)=f(1)=0$. For $P\in\mathbb{R}^{n\times n}$, $P>0$ means that P is symmetric and positive definite. The sub-diagonal elements of a symmetric matrix will be denoted by $*$. For $0<U\in\mathbb{R}^{n\times n}$ and $x\in\mathbb{R}^n$ we denote $|x|_U^2 = x^TUx$. $\mathbb{Z}_+$ denotes the nonnegative integers.

In this paper we use Wirtinger-based Lyapunov functionals that were introduced for sampled-data control of ODEs in Liu and Fridman (2012). These functionals were extended to ISS analysis in Selivanov and Fridman (2016b). The positivity of such functionals follows from the following extension of Wirtinger's inequality:

Lemma 1.1 (*Selivanov & Fridman, 2016b*). *Let $\delta_0\in\mathbb{R}$ and $X:[a,b]\to\mathbb{R}^n$ be an absolutely continuous function with $\dot X\in L^2(a,b)$ such that $X(a)=0$ or $X(b)=0$. Then for any $0<W\in\mathbb{R}^{n\times n}$, the following inequality holds:*

$$\int_a^b e^{2\delta_0\xi}X^T(\xi)WX(\xi)d\xi \le e^{2|\delta_0|(b-a)}\frac{4(b-a)^2}{\pi^2}\int_a^b e^{2\delta_0\xi}\dot X^T(\xi)W\dot X(\xi)d\xi. \tag{1.1}$$

Consider the Sturm–Liouville eigenvalue problem

$$\phi''+\lambda\phi=0,\quad x\in(0,1) \tag{1.2}$$

with boundary conditions

$$\phi'(0)=\phi(1)=0. \tag{1.3}$$

This problem induces a sequence of eigenvalues with corresponding eigenfunctions. The normalized eigenfunctions form a complete orthonormal system in $L^2(0,1)$. The eigenvalues and corresponding eigenfunctions are given by

$$\phi_n(x)=\sqrt{2}\cos\left(\sqrt{\lambda_n}x\right),\ \lambda_n=(n-0.5)^2\pi^2,\ n\ge 1. \tag{1.4}$$

The following lemma will be used:

Lemma 1.2 (*Katz & Fridman, 2020a*). *Let $h \overset{L^2}{=} \sum_{n=1}^\infty h_n\phi_n$, where $h_n=\langle h,\phi_n\rangle$. Then $h\in H^1(0,1)$ satisfies $h(1)=0$ iff $\sum_{n=1}^\infty\lambda_nh_n^2<\infty$. Moreover,*

$$\left\|h'\right\|^2=\sum_{n=1}^\infty\lambda_nh_n^2. \tag{1.5}$$

In this paper, all functions of interest will belong to $\{h\in H^1(0,1)|h(1)=0\}$. By Wirtinger's inequality, the standard H^1-norm of h is equivalent to $\left\|h'\right\|$. Therefore, in this work we use $\|h\|_{H^1}=\left\|h'\right\|$.

2. ISS Halanay's inequalities for piecewise continuous functions

In this section we introduce novel forms of Halanay's inequalities for ISS. Our formulations allow the function to have jump discontinuities, provided it does not grow in the jumps. The resulting inequalities are applied in the next sections to sampled-data boundary control of the heat equation in the presence of quantization, where two sequences of sampling instances will be introduced: $\{s_k\}_{k=0}^{\infty}$ will be the measurement sampling instances, whereas $\{t_j\}_{j=0}^{\infty}$ will be the controller hold times. Since the sequences are assumed to be independent, $[s_k, s_{k+1}),\ k \in \mathbb{Z}_+$ may contain elements from $\{t_j\}_{j=0}^{\infty}$. Our Lyapunov functional $V(t)$ (see (4.20) and Fig. 1), which compensates sampling in the finite-dimensional part of the closed-loop system (4.17), may be discontinuous at $t = s_k$ and $t = t_j,\ k, j \in \mathbb{Z}_+$, whereas Halanay's inequality will be used to compensate $s_k,\ k \in \mathbb{Z}_+$ in the infinite-dimensional tail. Thus, the presented Lyapunov functional may exhibit jump discontinuities at $s_k,\ k \in \mathbb{Z}_+$ and inside the intervals $[s_k, s_{k+1})$, where we want to apply Halanay's inequality.

Note that in the presence of only one sequence of sampling instances $s_k,\ k \in \mathbb{Z}_+$, where the Lyapunov functional has jump discontinuities and does not grow, our Halanay's inequalities are still novel and useful for many sampled-data control problems for ODEs and PDEs that combine Lyapunov functionals with Halanay's inequality.

For proofs of all claims appearing in this section see the Appendix.

Lemma 2.1. *Let $V : [a, b) \to [0, \infty)$ be a bounded function, where $b - a \le h$ for some $h > 0$. Assume that $V(t)$ is continuous on $[t_i, t_{i+1}),\ i = 0, \ldots, N-1$, where*

$$a =: t_0 < t_1 < \cdots < t_{N-1} < t_N := b, \tag{2.1}$$

and

$$\lim_{t \nearrow t_i} V(t) \ge V(t_i), \quad i = 1, 2, \ldots, N-1. \tag{2.2}$$

Assume further that for some $d \ge 0$ and $\delta_0 > \delta_1 > 0$

$$D^+V(t) \le -2\delta_0 V(t) + 2\delta_1 \sup_{a \le \theta \le t} V(\theta) + d,\ \ t \in [a, b) \tag{2.3}$$

where $D^+V(t)$ is the right upper Dini derivative, defined by

$$D^+V(t) = \limsup_{s \to 0^+} \frac{V(t+s) - V(t)}{s}. \tag{2.4}$$

Then

$$V(t) \le e^{-2\delta_\tau (t-a)} V(a) + d \int_a^t e^{-2\delta(t-s)} ds,\ \ t \in [a, b) \tag{2.5}$$

where $\delta = \delta_0 - \delta_1$ and $\delta_\tau > 0$ is the unique solution of the equation $\delta_\tau = \delta_0 - \delta_1 e^{2\delta_\tau h}$.

Note that by (2.2), the *one-sided* limits exist at $\{t_i\}_{i=1}^{N-1}$. Thus, at $t_i,\ 1 \le i \le N-1$, $V(t)$ may have at most a jump discontinuity. Moreover, if (2.2) holds with equality for some $t_i,\ 1 \le i \le N-1$, then $V(t)$ is continuous at t_i, meaning that our theorem is also valid for $V(t)$ continuous on $[a, b)$. Finally, note also that $\sup_{a \le \theta \le t} V(\theta)$ is well-defined, since the assumptions imply that $V(t)$ is bounded on $[a, c]$ for every $a < c < b$. An example of such a function $V(t)$ is given in Fig. 1, where we separate the points t_i where $V(t)$ is continuous (t_1, t_3 and t_5) and points where $V(t)$ has a jump discontinuity (t_2 and t_4, which we also denote by ξ_1 and ξ_2, respectively.)

Corollary 2.1. *Let $V : [a, b) \to [0, \infty)$ be a bounded function, where $b - a \le h$ for some $h > 0$. Assume that $V(t)$ is absolutely continuous on $[t_i, t_{i+1}),\ i = 0, \ldots, N-1$, where t_i are subject to (2.1), and satisfy (2.2). Assume that for some constants $d \ge 0$ and $\delta_0 > \delta_1 > 0$ the following inequality holds:*

$$\begin{aligned} \dot{V}(t) \quad &\le -2\delta_0 V(t) + 2\delta_1 \sup_{a \le \theta \le t} V(\theta) + d \\ &\text{almost for all } t \in [a, b). \end{aligned} \tag{2.6}$$

Then $V(t)$ satisfies (2.5), where $\delta = \delta_0 - \delta_1$ and $\delta_\tau > 0$ is the unique solution $\delta_\tau = \delta_0 - \delta_1 e^{2\delta_\tau h}$.

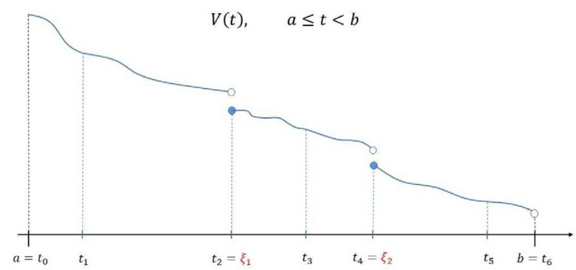

Fig. 1. Example of $V(t)$ in Lemma 2.1.

Using Lemma 2.1 and Corollary 2.1 we have the following:

Proposition 2.1 (*Piecewise Continuous V for Sampled-data Systems*)**.** *Let $s_0 < s_1 < \cdots < s_k < \cdots$ satisfy $\lim_{k\to\infty} s_k = \infty$ and $s_{k+1} - s_k \le h,\ k \in \mathbb{Z}_+$. Let $V : [s_0, +\infty) \to [0, \infty)$ be a bounded function such that*

$$\lim_{t \nearrow s_k} V(t) \ge V(s_k),\ \ k \in \mathbb{Z}_+ \tag{2.7}$$

For any $k \in \mathbb{Z}_+$, let

$$s_k =: t_0^{(k)} < t_1^{(k)} < \cdots < t_{N_k - 1}^{(k)} < t_{N_k}^{(k)} := s_{k+1}. \tag{2.8}$$

Assume that $V(t)$ is absolutely continuous on $[t_j^{(k)}, t_{j+1}^{(k)})$ for all $0 \le j \le N_k - 1$ and satisfies

$$\lim_{t \nearrow t_j^{(k)}} V(t) \ge V\left(t_j^{(k)}\right),\ \ 1 \le j \le N_k - 1. \tag{2.9}$$

Assume further that for any $k = 0, 1, \ldots$

$$\begin{aligned} \dot{V}(t) \quad &\le -2\delta_0 V(t) + 2\delta_1 \sup_{s_k \le \theta \le t} V(\theta) + d \\ &\text{almost for all } t \in [s_k, s_{k+1}). \end{aligned} \tag{2.10}$$

Then

$$V(t) \le e^{-2\delta_\tau (t - s_0)} V(s_0) + d \int_{s_0}^t e^{-2\delta_\tau (t-s)} ds,\ \ t \ge s_0. \tag{2.11}$$

where $\delta_\tau > 0$ is a unique solution of $\delta_\tau = \delta_0 - \delta_1 e^{2\delta_\tau h}$.

We end this section with a novel Halanay's ISS inequality for continuous functions.

Lemma 2.2 (*Continuous V for Time-delay Systems*)**.** *Let $V : [t_0 - h, +\infty) \to [0, \infty)$ be bounded on $[t_0 - h, t_0]$ and continuous on $[t_0, \infty)$. Assume that for some constants $d \ge 0$ and $\delta_0 > \delta_1 > 0$ the following inequality holds for $t \ge t_0$:*

$$D^+V(t) \le -2\delta_0 V(t) + 2\delta_1 \sup_{-h \le \theta \le 0} V(t + \theta) + d. \tag{2.12}$$

Then

$$\begin{aligned} V(t) \quad &\le e^{-2\delta_\tau (t - t_0)} \sup_{-h \le \theta \le 0} V(t_0 + \theta) \\ &+ d \int_{t_0}^t e^{-2\delta(t-s)} ds, \quad t \ge t_0, \end{aligned} \tag{2.13}$$

where $\delta = \delta_0 - \delta_1$ and $\delta_\tau > 0$ is a unique solution of $\delta_\tau = \delta_0 - \delta_1 e^{2\delta_\tau h}$.

Remark 2.1. Halanay's ISS inequalities were derived for differentiable functions in Wen et al. (2008) and for continuous functions in Hien et al. (2015). In the latter work, the authors obtain the estimate

$$V(t) \le e^{2\delta h} e^{-2\delta_\tau (t - t_0)} sup_{-h \le \theta \le 0} V(t_0 + \theta) + \frac{d}{\delta}. \tag{2.14}$$

Note that Lemma 2.2 improves this estimate by removing the factor $e^{2\delta h} > 1$ as well as replacing $\frac{d}{\delta}$ with an integral. In

Lemma 2.1 we improve on Hien et al. (2015) by allowing jump discontinuities of $V(t)$ in the subintervals and by replacing $\frac{d}{\delta}$ with an integral for which summation over the subintervals leads to a finite ISS bound in Proposition 2.1.

Corollary 2.2 (*Absolutely Continuous V for Time-delay Systems*).. *Let $V : [t_0 - h, \infty) \to [0, \infty)$ be continuous on $[t_0 - h, \infty)$ and absolutely continuous on $[t_0, \infty)$. Assume that for some constants $d \geq 0$ and $\delta_0 > \delta_1 > 0$ the following inequality holds:*

$$\dot{V}(t) \leq -2\delta_0 V(t) + 2\delta_1 \sup_{-h\leq\theta\leq 0} V(t+\theta) + d \quad \text{almost for all } t \geq t_0. \tag{2.15}$$

Then $V(t)$ satisfies (2.13), where $\delta = \delta_0 - \delta_1$ and $\delta_\tau > 0$ is the unique solution of $\delta_\tau = \delta_0 - \delta_1 e^{2\delta_\tau h}$.

Remark 2.2. Recently, instead of Halanay's inequality, a small-gain analysis was used in Ahmed-Ali, Karafyllis, and Giri (2021) for ODE-hyperbolic PDE systems, instead of Halanay's inequality. Comparison between the small-gain approach and Halanay's inequality in the control problem presented in Sections 4 and 5 is interesting and may be a topic for future research.

3. Continuous-time control of a heat equation

In this section we consider continuous-time stabilization of the linear 1D heat equation

$$z_t(x,t) = z_{xx}(x,t) + az(x,t), \quad t \geq 0 \tag{3.1}$$

where $x \in [0,1]$, $z(x,t) \in \mathbb{R}$ and $a \in \mathbb{R}$ is the reaction coefficient. We consider Dirichlet actuation given by

$$z_x(0,t) = 0, \quad z(1,t) = u(t) \tag{3.2}$$

where $u(t)$ is a control input to be designed, and point measurement given by

$$y(t) = z(x_*, t), \quad x_* \in [0,1). \tag{3.3}$$

Note that $x_* = 0$ corresponds to boundary measurement.

Remark 3.1. For simplicity, in the present paper we consider a reaction–diffusion PDE with constant diffusion and reaction coefficients. As in Katz and Fridman (2020a), our results can be easily extended to the more general reaction–diffusion PDE

$$z_t = \partial_x\left(p(x)z_x(x,t)\right) + q(x)z(x,t), \quad x \in [0,1], \ t \geq 0,$$

where $p(x)$ and $q(x)$ are sufficiently smooth on $(0,1)$.

Following Curtain and Zwart (1995), Karafyllis (2021), Katz and Fridman (2021b) and Prieur and Trélat (2018), we introduce the change of variables

$$w(x,t) = z(x,t) - u(t) \tag{3.4}$$

to obtain the following equivalent ODE–PDE system

$$\begin{aligned} &\dot{u}(t) = v(t), \\ &w_t(x,t) = w_{xx}(x,t) + aw(x,t) + au(t) - v(t), \quad t \geq 0 \end{aligned} \tag{3.5}$$

with boundary conditions

$$w_x(0,t) = 0, \quad w(1,t) = 0 \tag{3.6}$$

and measurement

$$y(t) = w(x_*, t) + u(t). \tag{3.7}$$

Henceforth we treat $u(t)$ as an additional state variable and $v(t)$ as the control input. Given $v(t)$, $u(t)$ can be computed by integrating $\dot{u}(t) = v(t)$, where we choose $u(0) = 0$. This choice implies $z(\cdot,0) = w(\cdot,0)$. Dynamic extension allows to obtain (3.5) with the state $[u(t), w(\cdot,t)]^T$ and control input $v(t)$, where now the control operator is bounded and the observation operator (3.7) is still unbounded. This approach, where $u(t)$ is obtained from $v(t)$ by direct integration poses no problems due to the fact that the corresponding state $u(t)$ is included in the stability analysis.

We present the solution to (3.5) as

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\phi_n(x), \quad w_n(t) = \langle w(\cdot,t), \phi_n \rangle, \tag{3.8}$$

with $\{\phi_n\}_{n=1}^{\infty}$ defined in (1.4). By differentiating under the integral sign, integrating by parts and using (1.2) and (1.3) we obtain

$$\begin{aligned} &\dot{w}_n(t) = (-\lambda_n + a)w_n(t) + ab_n u(t) - b_n v(t), \quad t \geq 0 \\ &b_n = (-1)^{n+1}\sqrt{\tfrac{2}{\lambda_n}}, \ w_n(0) = \langle w(\cdot,0), \phi_n \rangle, \ n \geq 1. \end{aligned} \tag{3.9}$$

In particular, note that

$$b_n \neq 0, \quad n \geq 1. \tag{3.10}$$

Remark 3.2. Without dynamic extension, modal decomposition of (3.1) with boundary conditions (3.2) results in ODEs similar to (3.9) without $v(t)$ and $|b_n| \approx \lambda_n^{\frac{1}{2}}$. The growth of $\{b_n\}_{n=1}^{\infty}$ poses a problem in compensating cross terms which arise in the Lyapunov stability analysis (see (3.36)). The use of dynamic extension leads to $\{b_n\}_{n=1}^{\infty} \in l^2(\mathbb{N})$.

Let $\delta > 0$ be a desired decay rate and let $N_0 \in \mathbb{N}$ satisfy

$$-\lambda_n + a < -\delta, \quad n > N_0. \tag{3.11}$$

Let $N \in \mathbb{N}$, $N_0 \leq N$. N_0 will define the dimension of the controller and N will define the dimension of the observer.

We construct a finite-dimensional observer of the form

$$\hat{w}(x,t) := \sum_{n=1}^{N} \hat{w}_n(t)\phi_n(x), \tag{3.12}$$

where $\hat{w}_n(t)$ satisfy the ODEs for $t \geq 0$:

$$\begin{aligned} &\dot{\hat{w}}_n(t) = (-\lambda_n + a)\hat{w}_n(t) + ab_n u(t) - b_n v(t) \\ &\qquad - l_n\left[\hat{w}(x_*,t) + u(t) - y(t)\right], \ n \geq 1, \\ &\hat{w}_n(0) = 0, \quad 1 \leq n \leq N. \end{aligned} \tag{3.13}$$

with $y(t)$ in (3.7) and scalar observer gains $\{l_n\}_{n=1}^{N}$. Let

$$\begin{aligned} &A_0 = \operatorname{diag}\left\{-\lambda_1 + a, \ldots, -\lambda_{N_0} + a\right\}, \\ &B_0 = \left[b_1, \ldots, b_{N_0}\right]^T, \ L_0 = \left[l_1, \ldots, l_{N_0}\right]^T, \ c_n = \phi_n(x_*), \\ &C_0 = \left[c_1, \ldots, c_{N_0}\right], \ \tilde{B}_0 = \left[1, -b_1, \ldots, -b_{N_0}\right]^T, \\ &\tilde{A}_0 = \begin{bmatrix} 0 & 0 \\ aB_0 & A_0 \end{bmatrix} \in \mathbb{R}^{(N_0+1)\times(N_0+1)}. \end{aligned} \tag{3.14}$$

Assumption 1. The point $x_* \in [0,1)$ satisfies

$$c_n = \phi_n(x_*) \neq 0, \quad 1 \leq n \leq N_0. \tag{3.15}$$

Note that this assumption is satisfied in the case $x_* = 0$ of boundary measurement. Under Assumption 1, the pair (A_0, C_0) is observable by the Hautus lemma. Let $L_0 = \left[l_1, \ldots, l_{N_0}\right]^T \in \mathbb{R}^{N_0}$ satisfy the Lyapunov inequality

$$P_o(A_0 - L_0C_0) + (A_0 - L_0C_0)^T P_o < -2\delta P_o, \tag{3.16}$$

with $0 < P_o \in \mathbb{R}^{N_0\times N_0}$. We choose $l_n = 0$, $n > N_0$.

Since $b_n \neq 0$, $n \geq 1$ the Hautus lemma implies that $(\tilde{A}_0, \tilde{B}_0)$ is controllable. Let $K_0 \in \mathbb{R}^{1\times(N_0+1)}$ satisfy

$$P_c(\tilde{A}_0 - \tilde{B}_0K_0) + (\tilde{A}_0 - \tilde{B}_0K_0)^T P_c < -2\delta P_c, \tag{3.17}$$

with $0 < P_c \in \mathbb{R}^{(N_0+1)\times(N_0+1)}$. We propose a (N_0+1)-dimensional controller of the form

$$\begin{aligned} &v(t) = -K_0\hat{w}^{N_0}(t),\\ &\hat{w}^{N_0}(t) = \left[u(t), \hat{w}_1(t), \ldots, \hat{w}_{N_0}(t)\right]^T \end{aligned} \tag{3.18}$$

which is based on the N-dimensional observer (3.12).

3.1. Well-posedness of (3.5)

For well-posedness of the closed-loop system (3.5) and (3.13) subject to the control input (3.18) we consider

$$\begin{aligned} &\mathcal{A}_1 : \mathcal{D}(\mathcal{A}_1) \subseteq L^2(0,1) \to L^2(0,1), \quad \mathcal{A}_1 w = -w_{xx},\\ &\mathcal{D}(\mathcal{A}_1) = \left\{w \in H^2(0,1) | w'(0) = w(1) = 0\right\}. \end{aligned} \tag{3.19}$$

Since $\mathcal{A}_1$ is positive, it has a unique positive square root with domain

$$\mathcal{D}\left(\mathcal{A}_1^{\frac{1}{2}}\right) \overset{(1.5)}{=} \left\{w \in H^1(0,1); w(1) = 0\right\}. \tag{3.20}$$

Let $\mathcal{H} = L^2(0,1) \times \mathbb{R}^{N+1}$ be a Hilbert space with the norm $\|\cdot\|_{\mathcal{H}} = \sqrt{\|\cdot\| + |\cdot|}$. Defining the state $\xi(t)$ as

$$\begin{aligned} &\xi(t) = \text{col}\left\{w(\cdot,t), \hat{w}^N(t)\right\},\\ &\hat{w}^N(t) = \text{col}\left\{u(t), \hat{w}_1(t), \ldots, \hat{w}_N(t)\right\} \end{aligned}$$

by arguments of Katz and Fridman (2020a), it can be shown that the closed-loop system (3.5) and (3.13) with control input (3.18) and initial condition $w(\cdot,0) \in \mathcal{D}\left(\mathcal{A}_1^{\frac{1}{2}}\right)$ has a unique classical solution

$$\xi \in C\left([0,\infty); \mathcal{H}\right) \cap C^1\left((0,\infty); \mathcal{H}\right) \tag{3.21}$$

such that

$$\xi(t) \in \mathcal{D}\left(\mathcal{A}_1\right) \times \mathbb{R}^{N+1}, \quad t > 0. \tag{3.22}$$

3.2. H^1-Stability of (3.5)

Let $e_n(t)$ be the estimation error defined by

$$e_n(t) = w_n(t) - \hat{w}_n(t), \;\; 1 \le n \le N. \tag{3.23}$$

By using (3.7), (3.8) and (3.12), the last term on the right-hand side of (3.13) can be written as

$$\hat{w}(x_*,t) + u(t) - y(t) = -\sum_{n=1}^N c_n e_n(t) - \zeta(t), \tag{3.24}$$

where

$$\begin{aligned} &\zeta(t) = w(x_*,t) - \sum_{n=1}^N w_n(t)\phi_n(x_*)\\ &\overset{(1.3),(3.6)}{=} -\int_{x_*}^1 \left[w_x(x,t) - \sum_{n=1}^N w_n(t)\phi_n'(x)\right] dx. \end{aligned} \tag{3.25}$$

Then the error equations have the form

$$\begin{aligned} &\dot{e}_n(t) = (-\lambda_n + a)e_n(t)\\ &\qquad -l_n\left(\sum_{n=1}^N c_n e_n(t) + \zeta(t)\right), \quad t \ge 0. \end{aligned} \tag{3.26}$$

Note that $\zeta(t)$ satisfies the following estimate:

$$\begin{aligned} &\zeta^2(t) \overset{(3.25)}{\le} \left\|w_x(\cdot,t) - \sum_{n=1}^N w_n(t)\phi_n'(\cdot)\right\|^2\\ &\qquad \overset{(1.5)}{=} \sum_{n=N+1}^\infty \lambda_n w_n^2(t). \end{aligned} \tag{3.27}$$

Next, following Katz, Basre et al. (2021), we formulate the *reduced-order* closed-loop system. Let

$$\begin{aligned} &e^{N_0}(t) = \left[e_1(t), \ldots, e_{N_0}(t)\right], \; B_1 = \left[b_{N_0+1}, \ldots, b_N\right]^T,\\ &e^{N-N_0}(t) = \left[e_{N_0+1}(t), \ldots, e_N(t)\right]^T, \; C_1 = \left[c_{N_0+1}, \ldots, c_N\right],\\ &\hat{w}^{N-N_0}(t) = \left[\hat{w}_{N_0+1}(t), \ldots, \hat{w}_N(t)\right]^T, \; \tilde{L}_0 = \text{col}\left\{0_{1\times1}, L_0\right\},\\ &X_0(t) = \text{col}\left\{\hat{w}^{N_0}(t), e^{N_0}(t)\right\}, \; \mathcal{L} = \text{col}\left\{\tilde{L}_0, -L_0\right\},\\ &K_a = K_0 + [a,0], \; \tilde{K}_a = [K_a, 0] \in \mathbb{R}^{1\times(2N_0+1)},\\ &F_0 = \begin{bmatrix} \tilde{A}_0 - \tilde{B}_0K_0 & \tilde{L}_0C_0 \\ 0 & A_0 - L_0C_0 \end{bmatrix}. \end{aligned} \tag{3.28}$$

From (3.9), (3.13), (3.18) and (3.28) we observe that $e^{N-N_0}(t)$ satisfies

$$\begin{aligned} &\dot{e}^{N-N_0}(t) = A_1 e^{N-N_0}(t),\\ &A_1 = diag\{-\lambda_{N_0+1} + a, \ldots, -\lambda_N + a\} \end{aligned} \tag{3.29}$$

and is exponentially decaying, whereas the reduced-order closed-loop system

$$\begin{aligned} &\dot{X}_0(t) = F_0X_0(t) + \mathcal{L}C_1e^{N-N_0}(t) + \mathcal{L}\zeta(t),\\ &\dot{w}_n(t) = (-\lambda_n + a)w_n(t) + b_n\tilde{K}_aX_0(t), \;\; n > N. \end{aligned} \tag{3.30}$$

with $\zeta(t)$ satisfying (3.27) does not depend on $\hat{w}^{N-N_0}(t)$. Moreover, $\hat{w}^{N-N_0}(t)$ satisfies

$$\dot{\hat{w}}^{N-N_0}(t) = A_1\hat{w}^{N-N_0}(t) + B_1\tilde{K}_aX_0(t). \tag{3.31}$$

and is exponentially decaying with a decay rate δ, provided $X_0(t)$ is exponentially decaying with a slightly larger decay rate $\delta + \epsilon$. The latter is guaranteed since the LMI (3.41) is satisfied with strict inequality, and thus with δ substituted by $\delta + \epsilon$. In this case, $X_0(t)$ can be thought of as an exponentially decaying disturbance in (3.31) and using the variation of constants formula, the result follows. Hence, for stability of (3.1) under the control law (3.18) it is enough to show stability of the reduced-order system (3.30). Note that (3.30) can be considered as a singularly perturbed system with slow state $X_0(t)$ and fast infinite state $w_n(t)$, $n > N$. For H^1-stability analysis of the closed-loop system (3.30) we define the Lyapunov function

$$\begin{aligned} &V(t) = V_0(t) + p_e\left|e^{N-N_0}(t)\right|^2,\\ &V_0(t) = |X_0(t)|_{P_0}^2 + \sum_{n=N+1}^\infty \lambda_n w_n^2(t), \end{aligned} \tag{3.32}$$

where $0 < p_e$ and $0 < P_0 \in \mathbb{R}^{(2N_0+1)\times(2N_0+1)}$. $V_0(t)$ is chosen to compensate $\zeta(t)$ using (3.27). Differentiating $V_0(t)$ along the solution to (3.30) gives

$$\begin{aligned} &\dot{V}_0 + 2\delta V_0 = X_0^T(t)\left[P_0F_0 + F_0^TP_0 + 2\delta P_0\right]X_0(t)\\ &+2X_0^T(t)P_0\mathcal{L}\zeta(t) + 2X_0^T(t)P_0\mathcal{L}C_1e^{N-N_0}(t)\\ &+2\sum_{n=N+1}^\infty \left(-\lambda_n + a + \delta\right)\lambda_n w_n^2(t)\\ &+2\sum_{n=N+1}^\infty \lambda_n w_n(t)b_n\tilde{K}_aX_0(t), \quad t \ge 0. \end{aligned} \tag{3.33}$$

Differentiating $p_e\left|e^{N-N_0}(t)\right|^2$ along (3.30) we have

$$\begin{aligned} &\tfrac{d}{dt}p_e\left|e^{N-N_0}(t)\right|^2 + 2\delta p_e\left|e^{N-N_0}(t)\right|^2\\ &= 2p_e\left(e^{N-N_0}(t)\right)^T\left(A_1 + \delta I\right)e^{N-N_0}(t) \end{aligned} \tag{3.34}$$

Using the estimate

$$\sum_{n=N+1}^\infty \lambda_n^{-1} \le \pi^{-2}\int_N^\infty \frac{dx}{(x-0.5)^2} = \frac{1}{(N-0.5)\pi^2}, \tag{3.35}$$

the Young inequality and $|b_n| = \sqrt{\frac{2}{\lambda_n}}$ we have

$$\begin{aligned} &2\sum_{n=N+1}^{\infty}\lambda_n w_n(t)b_n\tilde{K}_aX_0(t) \\ &\leq 2\sum_{n=N+1}^{\infty}\left[\lambda_n\left|w_n(t)\right|\right]\left[\sqrt{2}\lambda_n^{-\frac{1}{2}}\left|\tilde{K}_aX_0(t)\right|\right] \\ &\overset{(3.35)}{\leq} \frac{1}{\alpha_0}\sum_{n=N+1}^{\infty}\lambda_n^2w_n^2(t) + \frac{2\alpha_0}{(N-0.5)\pi^2}\left|\tilde{K}_aX_0(t)\right|^2 \end{aligned} \tag{3.36}$$

where $\alpha_0 > 0$. From monotonicity of λ_n we have

$$\begin{aligned} &2\sum_{n=N+1}^{\infty}\left(-\lambda_n^2+(a+\delta)\lambda_n\right)w_n^2(t) \\ &+2\sum_{n=N+1}^{\infty}\lambda_n w_n(t)b_n\tilde{K}_aX_0(t) \\ &\overset{(3.36)}{\leq} 2\sum_{n=N+1}^{\infty}\left(-\lambda_n^2+\frac{1}{2\alpha_0}\lambda_n^2+(a+\delta)\lambda_n\right)w_n^2(t) \\ &+\frac{2\alpha_0}{(N-0.5)\pi^2}\left|\tilde{K}_aX_0(t)\right|^2 \leq \frac{2\alpha_0}{(N-0.5)\pi^2}\left|\tilde{K}_aX_0(t)\right|^2 \\ &-2\left(\lambda_{N+1}-a-\delta-\frac{1}{2\alpha_0}\lambda_{N+1}\right)\sum_{n=N+1}^{\infty}\lambda_n w_n^2(t) \\ &\overset{(3.27)}{\leq} \frac{2\alpha_0}{(N-0.5)\pi^2}\left|\tilde{K}_aX_0(t)\right|^2 \\ &\quad -2\left(\lambda_{N+1}-a-\delta-\frac{1}{2\alpha_0}\lambda_{N+1}\right)\zeta^2(t). \end{aligned} \tag{3.37}$$

provided $\lambda_{N+1}-a-\delta-\frac{1}{2\alpha_0}\lambda_{N+1} > 0$. Let $\eta(t) = \text{col}\left\{X_0(t), \zeta(t), e^{N-N_0}(t)\right\}$. From (3.33), (3.34) and (3.37) we obtain

$$\dot{V} + 2\delta V \leq \eta^T(t)\Psi^{(1)}\eta(t) \leq 0, \quad t \geq 0 \tag{3.38}$$

if

$$\Psi^{(1)} = \begin{bmatrix} \Phi^{(1)} & \text{col}\{P_0\mathcal{L}C_1, 0\} \\ * & 2p_e(A_1+\delta I)\end{bmatrix} < 0, \tag{3.39}$$

where

$$\begin{aligned} &\Phi^{(1)} = \begin{bmatrix} \phi & P_0\mathcal{L} \\ * & -2\left(\lambda_{N+1}-a-\delta-\frac{1}{2\alpha_0}\lambda_{N+1}\right)\end{bmatrix}, \\ &\phi = P_0F_0 + F_0^TP_0 + 2\delta P_0 + \frac{2\alpha_0}{(N-0.5)\pi^2}\tilde{K}_a^T\tilde{K}_a. \end{aligned} \tag{3.40}$$

By Schur complement $\Phi^{(1)} < 0$ holds iff

$$\begin{bmatrix} \phi & P_0\mathcal{L} & 0 \\ * & -2(\lambda_{N+1}-a-\delta) & 1 \\ * & * & -\alpha_0\lambda_{N+1}^{-1}\end{bmatrix} < 0. \tag{3.41}$$

Note that the LMI (3.41) has N-dependent coefficients whereas its dimension depends only on N_0. Therefore the LMI (3.41) is of reduced-order. Summarizing, we arrive at:

Proposition 3.1. *Consider (3.5) with boundary conditions (3.6), boundary measurement (3.7), control law (3.18) and $w(\cdot,0) \in \mathcal{D}(\mathcal{A}_1^{\frac{1}{2}})$. Let $\delta > 0$ be a desired decay rate, $N_0 \in \mathbb{N}$ satisfy (3.11) and $N \in \mathbb{N}$ satisfy $N_0 \leq N$. Let L_0 and K_0 be obtained using (3.16) and (3.17), respectively. Let there exist $0 < P_0 \in \mathbb{R}^{(2N_0+1)\times(2N_0+1)}$ and a scalar $\alpha_0 > 0$ which satisfy the reduced-order LMI (3.41) with ϕ given in (3.40). Then the solutions $w(x,t)$ and $u(t)$ to (3.5) under the control law (3.18), (3.13) and the corresponding observer $\hat{w}(x,t)$ defined by (3.12) satisfy*

$$\begin{aligned} \|w(\cdot,t)\|_{H^1} + \left\|\hat{w}(\cdot,t)\right\|_{H^1} &+ |u(t)| \\ &\leq Me^{-\delta t}\|w(\cdot,0)\|_{H^1} \end{aligned} \tag{3.42}$$

for some constant $M > 0$. Moreover, (3.41) is always feasible for large enough N and feasibility for N implies feasibility for $N+1$.

Proof. Taking into account (3.11) and applying Schur complement to $\Psi^{(1)}$ given in (3.39), we find that $\Psi^{(1)} < 0$ iff

$$\Phi^{(1)} + \frac{1}{2p_e}\text{col}\{P_0\mathcal{L}C_1, 0\}(A_1+\delta I)\left[C_1^T\mathcal{L}^TP_0, \ 0\right] < 0.$$

By taking $p_e \to \infty$ we find that (3.41) implies $\Psi^{(1)} < 0$. Taking $P_0F_0 + F_0^TP_0 + 2\delta P_0 = -I$, $\alpha_0 = 1$ and $N \to \infty$, we have that (3.41) is feasible for large enough N. Finally, by (3.40) feasibility for N implies feasibility for $N+1$.

The comparison principle, $\Psi^{(1)} < 0$ and (3.38) imply

$$V(t) < e^{-2\delta t}V(0), \quad t > 0, \quad V(0) > 0. \tag{3.43}$$

Since $u(0) = 0$, for some $M_0 > 0$ we have

$$V(0) \overset{(1.5)}{\leq} M_0\|w(\cdot,0)\|_{H^1}^2. \tag{3.44}$$

From monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$ and (3.32) we have

$$\begin{aligned} &V(t) \geq \lambda_{min}(P_0)|X_0(t)|^2 + p_e\left|e^{N-N_0}(t)\right|^2 \\ &+\sum_{n=N+1}^{\infty}\lambda_n w_n^2(t) \geq M_1\left[\left|e^{N-N_0}(t)\right|^2 + \left|e^{N_0}(t)\right|^2\right. \\ &\left.+\left|\hat{w}^{N_0}(t)\right|^2 + \sum_{n=N+1}^{\infty}\lambda_n w_n^2(t)\right] \end{aligned} \tag{3.45}$$

for some constant $M_1 > 0$. Since $\hat{w}^{N-N_0}$ is exponentially decaying with decay rate less than δ provided $X_0(t)$ is exponentially decaying with a slightly larger decay rate, (3.42) follows from Lemma 1.2, (3.44), (3.45) and the presentation

$$w(\cdot,t) - \hat{w}(\cdot,t) = \sum_{n=1}^{N}e_n(t)\phi_n(\cdot) + \sum_{n=N+1}^{\infty}w_n(t)\phi_n(\cdot). \quad \square$$

Corollary 3.1. *Under the conditions of Proposition 3.1, the following holds for $z(x,t)$, satisfying (3.4):*

$$\|z(\cdot,t)\|_{H^1} + \left\|z(\cdot,t)-\hat{w}(\cdot,t)\right\|_{H^1} \leq Me^{-\delta t}\|z(\cdot,0)\|_{H^1} \tag{3.46}$$

for some constant $M > 0$.

Proof. From (3.4) we have

$$\begin{aligned} &\|z(\cdot,t)\|_{H^1} \leq \|w(\cdot,t)\|_{H^1} + |u(t)|, \\ &\left\|z(\cdot,t)-\hat{w}(\cdot,t)\right\|_{H^1} \leq \left\|w(\cdot,t)-\hat{w}(\cdot,t)\right\|_{H^1} \\ &\qquad + |u(t)|. \end{aligned} \tag{3.47}$$

From $u(0) = 0$, (3.42) and (3.47), we obtain (3.46). $\square$

Remark 3.3. Differently from our preliminary result (Katz & Fridman, 2021c), where Young's inequality in (3.36) was employed with fractional powers of λ_n, here the fractional powers are not needed. This is due to the reduced-order LMI formulation, which greatly simplifies the proof of feasibility guarantees.

4. Sampled-data control: Dirichlet actuation

Consider now sampled-data control of the 1D linear heat equation (3.1) under Dirichlet actuation (3.2). We introduce two sequences of sampling instances. For the first sequence, let $0 = s_0 < \cdots < s_k < \ldots$, $\lim_{k\to\infty}s_k = \infty$ be the measurement sampling instances. The sampling is variable and subject to $s_{k+1} - s_k \leq \tau_{M,y}$ for all $k \in \mathbb{Z}_+$ and some constants $\tau_{M,y} > 0$. We consider quantized discrete-time in-domain point measurement

$$y(t) = q\left[z(x_*, s_k)\right], \ x_* \in [0,1), \ t \in [s_k, s_{k+1}). \tag{4.1}$$

Here, $q : \mathbb{R} \to \mathbb{R}$ is a quantizer which satisfies

$$|q[r] - r| \leq \Delta, \quad \text{for all } r \in \mathbb{R} \tag{4.2}$$

where $\Delta > 0$ is the quantization error bound (Ishii & Francis, 2003).

Remark 4.1. In this paper we do not consider constraints on the range of quantizer as defined in Liberzon (2003): there

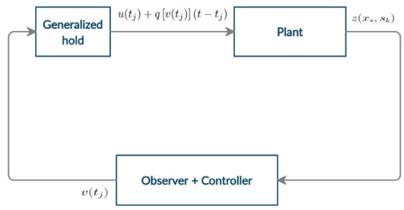

Fig. 2. Sampled-data control of a heat equation.

exists $M_q > 0$ such that (4.2) is applied only in the case $|r| \leq M_q$. Our method can be used in the future to a quantizer with bounded range. Note that since we achieve H^1-ultimate boundedness, $|z(x_*, s_k)|$ can be upper-bounded in terms of $\|z(\cdot, s_k)\|_{H^1}$ by using the Sobolev inequality. However, constraints on the range of quantizer are not in the scope of the current paper.

For the second sequence, let $0 = t_0 < \cdots < t_j < \ldots$, $\lim_{j\to\infty} t_j = \infty$ be the controller hold times. We assume that the sampling is variable and satisfies $t_{j+1} - t_j \leq \tau_{M,u}$ for all $j \in \mathbb{Z}_+$ and some constant $\tau_{M,u} > 0$.

The control signal $u(t)$ is generated by a *generalized hold* device and is of the form

$$\dot{u}(t) = q\left[v(t_j)\right], \quad t \in [t_j, t_{j+1}) \tag{4.3}$$

where the values $\left\{v(t_j)\right\}_{j=1}^{\infty}$ are to be determined. Furthermore, we choose $u(0) = 0$. By a generalized hold we mean the following: given $v(t_j)$, the control signal is computed as (see Fig. 2)

$$u(t) = u(t_j) + q\left[v(t_j)\right](t - t_j), \; t \in [t_j, t_{j+1}), \; j = 0, 1, \ldots \tag{4.4}$$

The considered sampled-data control may correspond to a networked control system with two independent networks (with negligible network-induced delays): from sensor to controller with transmission instances s_k and from controller to actuator with transmission instances t_j. In this case, t_j are the updating times of the generalized hold device on the actuator side.

Introducing (3.4) we obtain the ODE–PDE system

$$\begin{aligned} &\dot{u}(t) = q\left[v(t_j)\right], \quad t \in [t_j, t_{j+1}), \\ &w_t(x,t) = w_{xx}(x,t) + aw(x,t) + au(t) - q\left[v(t_j)\right], \end{aligned} \tag{4.5}$$

with boundary conditions (3.6) and measurement

$$y(t) = q\left[w(x_*, s_k) + u(s_k)\right], \; t \in [s_k, s_{k+1}) \tag{4.6}$$

Note that $y(t)$ is a piecewise constant function. Recall that we treat $u(t)$ as an additional state variable and the values $\left\{v(t_j)\right\}_{j=1}^{\infty}$ as the control inputs to be determined. We choose $u(0) = 0$ which results in $w(\cdot, 0) = z(\cdot, 0)$.

We present the solution to (4.5) as (3.8) with $\{\phi_n\}_{n=1}^{\infty}$ defined in (1.4). By differentiating under the integral sign, integrating by parts and using (1.2) and (1.3) we obtain

$$\begin{aligned} \dot{w}_n(t) \;&= (-\lambda_n + a)w_n(t) + ab_n u(t) \\ &\quad - b_n q\left[v(t_j)\right], t \in [t_j, t_{j+1}) \end{aligned} \tag{4.7}$$

with $\{b_n\}_{n=1}^{\infty}$ given in (3.9). In particular, (3.10) holds.

Using the time-delay approach to sampled-data control (Fridman, 2014), we introduce the following representations of the measurement and input delays

$$\begin{aligned} &\tau_y(t) = t - s_k, \quad t \in [s_k, s_{k+1}), \quad \tau_y(t) \leq \tau_{M,y}, \\ &\tau_u(t) = t - t_j, \quad t \in [t_j, t_{j+1}), \quad \tau_u(t) \leq \tau_{M,u}. \end{aligned} \tag{4.8}$$

Henceforth the dependence of $\tau_y(t)$, $\tau_u(t)$ on t will be suppressed to shorten notations. Note that τ_u will be used starting from (4.18).

Given $\delta > 0$, let $N_0 \in \mathbb{N}$ satisfy (3.11) and $N \in \mathbb{N}$, $N_0 \leq N$. N_0 will define the dimension of the controller, whereas N will define the dimension of the observer. Define a finite-dimensional observer of the form (3.12) where

$$\begin{aligned} &\dot{\hat{w}}_n(t) = (-\lambda_n + a)\hat{w}_n(t) + ab_n u(t) - b_n q\left[v(t_j)\right] \\ &\qquad -l_n\left[\hat{w}(x_*, t - \tau_y) + u(t - \tau_y) - y(t)\right], \quad t \in [t_j, t_{j+1}), \\ &\hat{w}_n(0) = 0, \quad 1 \leq n \leq N \end{aligned} \tag{4.9}$$

with $y(t) =$ and scalar observer gains $\{l_n\}_{n=1}^{N}$.

Under Assumption 1, let the observer and controller gains, L_0 and K_0, satisfy (3.16) and (3.17), respectively. We choose $l_n = 0$ for $n > N_0$. We propose a $(N_0 + 1)$-dimensional controller of the form

$$\begin{aligned} &\dot{u}(t) = q\left[v(t_j)\right], \quad t \in [t_j, t_{j+1}), \\ &v(t_j) = -K_0 \hat{w}^{N_0}(t_j) \end{aligned} \tag{4.10}$$

with $\hat{w}^{N_0}(t) = \left[u(t), \hat{w}_1(t), \ldots, \hat{w}_{N_0}(t)\right]^T$. The proposed controller is found by solving (4.9) on $[t_{j-1}, t_j)$ and choosing by continuity $\hat{w}^{N_0}(t_j) = \lim_{t\nearrow t_j} \hat{w}^{N_0}(t)$.

Well-posedness of the closed-loop system (4.5) and (4.9) with control input (4.10) follows from arguments of Katz and Fridman (2021b), together with the step method (i.e. proving well-posedness step-by-step between consecutive sampling instances). Thus, the closed-loop system (4.5) and (4.9) with control input (4.10) and $w(\cdot, 0) \in \mathcal{D}\left(\mathcal{A}_1^{\frac{1}{2}}\right)$ has a unique classical solution

$$\begin{aligned} &\xi \in C\left([0,\infty); \mathcal{H}\right) \cap C^1\left((0,\infty) \setminus \mathcal{J}; \mathcal{H}\right), \\ &\mathcal{J} = \left\{t_j\right\}_{j=1}^{\infty} \cup \{s_k\}_{k=1}^{\infty} \end{aligned} \tag{4.11}$$

satisfying (3.22).

Recall the estimation error $e_n(t)$ defined in (3.23). By using (3.8), (3.12) and arguments similar to (3.24) the last term on the right-hand side of (4.9) can be written as

$$\begin{aligned} &\hat{w}(x_*, t - \tau_y) + u(t - \tau_y) - y(t) \\ &\quad = -\textstyle\sum_{n=1}^{N} c_n e_n(t - \tau_y) - \zeta(t - \tau_y) - \sigma_y(t) \end{aligned} \tag{4.12}$$

where $\zeta(t)$, given in (3.25), satisfies (3.27) and

$$\begin{aligned} &\sigma_y(t) = q\left[w(x_*, t - \tau_y) + u(t - \tau_y)\right] \\ &\qquad - w(x_*, t - \tau_y) - u(t - \tau_y), \\ &\sigma_y^2(t) \overset{(4.2)}{\leq} \Delta^2. \end{aligned} \tag{4.13}$$

Then, the error equations have the form

$$\begin{aligned} &\dot{e}_n(t) = (-\lambda_n + a)e_n(t) - l_n\left(\textstyle\sum_{n=1}^{N} c_n e_n(t - \tau_y)\right. \\ &\qquad \left. + \zeta(t - \tau_y) + \sigma_y(t)\right), \quad t \geq 0. \end{aligned} \tag{4.14}$$

We formulate further the reduced-order closed-loop system. Recall the notations (3.28) and let

$$\begin{aligned} &\Upsilon_y(t) = X_0(t - \tau_y) - X_0(t), \; \tilde{K}_0 = \left[K_0, \quad 0\right] \in \mathbb{R}^{1\times 2N_0+1}, \\ &\Upsilon_u(t) = X_0(t_j) - X_0(t), \quad t \in [t_j, t_{j+1}), \\ &\mathcal{C} = [0, C_0] \in \mathbb{R}^{1\times(2N_0+1)}, \; \mathcal{B} = \operatorname{col}\left\{\tilde{B}_0, 0\right\} \in \mathbb{R}^{2N_0+1}. \end{aligned} \tag{4.15}$$

Using the notations (3.28) and (4.7), (4.9), (4.14), (4.15) we obtain that $e^{N-N_0}(t)$ satisfies (3.29), which implies

$$e^{N-N_0}(t - \tau_y) = e^{-A_1\tau_y} e^{N-N_0}(t). \tag{4.16}$$

Note that $e^{N-N_0}(t)$ is exponentially decaying. We also have the following reduced-order closed-loop system:

$$\begin{aligned} \dot{X}_0(t) &= F_0X_0(t) + \mathcal{LC}\Upsilon_y(t) - \mathcal{B}\tilde{K}_0\Upsilon_u(t) + \mathcal{B}\sigma_u(t) \\ &\quad + \mathcal{L}C_1e^{-A_1\tau_y}e^{N-N_0}(t) + \mathcal{L}\zeta(t-\tau_y) + \mathcal{L}\sigma_y(t), \\ \dot{w}_n(t) &= (-\lambda_n + a)w_n(t) + b_n\left[\tilde{K}_aX_0(t) + \tilde{K}_0\Upsilon_u(t)\right] \\ &\quad - b_n\sigma_u(t), \ n > N, \quad t \geq 0 \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} &\sigma_u(t) = q\left[-K_0\hat{w}^{N_0}(t_j)\right] + K_0\hat{w}^{N_0}(t_j), \ t \in [t_j, t_{j+1}), \\ &\sigma_u^2(t) \overset{(4.2)}{\leq} \Delta^2. \end{aligned} \tag{4.18}$$

Finally, from (4.9) $\hat{w}^{N-N_0}(t)$ satisfies the following ODEs:

$$\begin{aligned} \dot{\hat{w}}^{N-N_0}(t) &= A_1\hat{w}^{N-N_0}(t) + B_1\tilde{K}_0X_0(t-\tau_u) \\ &\quad - B_1\sigma_u(t) + aB_1u(t), \quad t \geq 0. \end{aligned} \tag{4.19}$$

For H^1-ISS of the closed-loop system (4.17) we fix $\delta_0 > \delta$, $\rho > 0$ and define the Lyapunov functional

$$\bar{V}(t) = V(t) + V_y(t) + V_u(t), \quad t \geq 0 \tag{4.20}$$

where

$$\begin{aligned} &V(t) = V_0(t) + p_e\left|e^{N-N_0}(t)\right|^2, \\ &V_0(t) = |X_0(t)|^2_{P_0} + \rho\textstyle\sum_{n=N+1}^{\infty}\lambda_nw_n^2(t), \\ &V_y(t) = \tau_{M,y}^2e^{2\delta_0\tau_{M,y}}\int_{t-\tau_y}^t e^{-2\delta_0(t-s)}\left|\dot{X}_0(s)\right|^2_{W_1}ds \\ &- \tfrac{\pi^2}{4}\int_{t-\tau_y}^t e^{-2\delta_0(t-s)}\left|\Upsilon_y(s)\right|^2_{W_1}ds, \ W_1 > 0, \\ &V_u(t) = \tau_{M,u}^2e^{2\delta_0\tau_{M,u}}\int_{t-\tau_u}^t e^{-2\delta_0(t-s)}\left|\tilde{K}_0\dot{X}_0(s)\right|^2_{W_2}ds \\ &- \tfrac{\pi^2}{4}\int_{t-\tau_u}^t e^{-2\delta_0(t-s)}\left|\tilde{K}_0\Upsilon_u(s)\right|^2_{W_2}ds, \ W_2 > 0. \end{aligned} \tag{4.21}$$

By Wirtinger's inequality (1.1), $V_y(t), V_u(t) \geq 0$. Moreover, $V_y(s_k) = 0$ and $V_u(t_j) = 0$, $k, j \in \mathbb{Z}_+$, meaning that $\bar{V}(t)$ does not grow in the jumps. Consider $[s_k, s_{k+1})$, $k \in \mathbb{Z}_+$. Since the controller update instances satisfy $\lim_{j\to\infty} t_j = \infty$, there exist at most finitely many controller update instances $t_j^{(k)}$, $1 \leq j \leq N_k - 1$ for which (2.8) and (2.9) hold. Furthermore, $\bar{V}(t)$ defined by (4.20) and (4.21) is continuously differentiable on $[t_j^{(k)}, t_{j+1}^{(k)})$, $0 \leq j \leq N_k - 1$. Our goal is to apply Proposition 2.1 to obtain (2.11). Differentiating $V_0(t)$ on $[t_j^{(k)}, t_{j+1}^{(k)})$, $0 \leq j \leq N_k - 1$ along (4.17) and using arguments similar to (3.36), we have

$$\begin{aligned} &\dot{V}_0 + 2\delta_0V_0 \leq X_0^T(t)\left[P_0F_0 + F_0^TP_0 + 2\delta_0P_0\right. \\ &\left.+ \tfrac{2\alpha_0\rho}{(N-0.5)\pi^2}\tilde{K}_a^T\tilde{K}_a\right]X_0(t) + 2X_0^T(t)P_0\mathcal{LC}\Upsilon_y(t) \\ &- 2X_0^T(t)P_0\mathcal{B}\tilde{K}_0\Upsilon_u(t) + 2X_0^T(t)P_0\mathcal{B}\sigma_u(t) \\ &+ 2X_0^T(t)P_0\mathcal{L}C_1e^{-A_1\tau_y}e^{N-N_0}(t) + 2X_0^T(t)P_0\mathcal{L}\sigma_y(t) \\ &+ 2X_0^T(t)P_0\mathcal{L}\zeta(t-\tau_y) + \tfrac{2\alpha_1\rho}{(N-0.5)\pi^2}\left|\tilde{K}_0\Upsilon_u(t)\right|^2 \\ &+ \tfrac{2\alpha_2\rho}{(N-0.5)\pi^2}\sigma_u^2(t) \\ &+ 2\rho\textstyle\sum_{n=N+1}^{\infty}\left[-\lambda_n + a + \delta_0 + \lambda_n\sum_{i=0}^2\frac{1}{2\alpha_i}\right]\lambda_nw_n^2(t). \end{aligned} \tag{4.22}$$

Differentiation of $p_e\left|e^{N-N_0}(t)\right|^2$ along the solution to (4.17) results in (3.34) with δ replaced by δ_0. Differentiating $V_y(t)$ and $V_u(t)$ along the solution to (4.17) we obtain

$$\begin{aligned} &\dot{V}_y + 2\delta_0V_y = \tau_{M,y}^2e^{2\delta_0\tau_{M,y}}\left|\dot{X}_0(t)\right|^2_{W_1} - \tfrac{\pi^2}{4}\left|\Upsilon_y(t)\right|^2_{W_1}, \\ &\dot{V}_u + 2\delta_0V_u = \tau_{M,u}^2e^{2\delta_0\tau_{M,u}}\left|\tilde{K}_0\dot{X}_0(t)\right|^2_{W_2} \\ &\qquad - \tfrac{\pi^2}{4}\left|\tilde{K}_0\Upsilon_u(t)\right|^2_{W_2}. \end{aligned} \tag{4.23}$$

Taking into account (3.27), (4.15) and (4.16) we will compensate $\zeta(t-\tau_y)$ by employing Halanay's inequality formulated in Proposition 2.1 and the following upper bound:

$$\begin{aligned} &-2\delta_1\sup_{s_k\leq\theta\leq t}\bar{V}(\theta) \overset{(4.8)}{\leq} -2\delta_1V(t-\tau_y) = -2\delta_1\left|\Upsilon_y(t)\right|^2_{P_0} \\ &-2\delta_1|X_0(t)|^2_{P_0} - 2\delta_1\rho\zeta^2(t-\tau_y) - 2\delta_1X_0^T(t)P_0\Upsilon_y(t) \\ &-2\delta_1\Upsilon_y^T(t)P_0X_0(t) - 2\delta_1p_e\left|e^{N-N_0}(t)\right|^2_{e^{-2A_1\tau_y}} \end{aligned} \tag{4.24}$$

where $\delta_0 - \delta_1 = \delta$. Let $\gamma > 0$. By (4.13) and (4.18) we have

$$-2\gamma\Delta^2 \leq -\gamma\sigma_u^2(t) - \gamma\sigma_y^2(t). \tag{4.25}$$

Denote

$$\begin{aligned} \eta(t) = \ &\mathrm{col}\left\{X_0(t), \zeta(t-\tau_y), \Upsilon_y(t), \tilde{K}_0\Upsilon_u(t), \sigma_y(t),\right. \\ &\left.\sigma_u(t), e^{N-N_0}(t)\right\}. \end{aligned}$$

From (4.22), (4.23), (4.24) and (4.25) we have

$$\begin{aligned} &\dot{\bar{V}}(t) + 2\delta_0\bar{V}(t) - 2\delta_1\sup_{s_k\leq\theta\leq t}\bar{V}(\theta) - 2\gamma\Delta^2 \\ &\leq \eta^T(t)\Psi^{(2)}\eta(t) + 2\rho\textstyle\sum_{n=N+1}^{\infty}\mu_n\lambda_nw_n^2(t) \leq 0 \end{aligned} \tag{4.26}$$

provided

$$\mu_n = -\lambda_n + \left[\sum_{i=0}^2\frac{1}{2\alpha_i}\right]\lambda_n + a + \delta_0 < 0, \quad n > N \tag{4.27}$$

and

$$\begin{aligned} \Psi^{(2)} &= \left[\begin{array}{c|ccc} \Phi^{(2)} & \Gamma_1 & \Gamma_2 & \Gamma_3 \\ \hline * & \multicolumn{3}{c}{\mathrm{diag}\left\{\Theta_1, \Theta_2, 2p_e\left(A_1 + \delta_0I - \delta_1e^{-2A_1\tau_y}\right)\right\}} \end{array}\right] \\ &\quad + R^T\left(\varepsilon_yW_1 + \varepsilon_u\tilde{K}_0^TW_2\tilde{K}_0\right)R < 0 \end{aligned} \tag{4.28}$$

where

$$\begin{aligned} &\Phi^{(2)} = \begin{bmatrix} \phi & P_0\mathcal{L} \\ * & -2\delta_1\rho \end{bmatrix}, \ \Gamma_1 = \begin{bmatrix} P_0(\mathcal{LC} - 2\delta_1I) & -P_0\mathcal{B} \\ 0 & 0 \end{bmatrix}, \\ &\phi = P_0F_0 + F_0^TP_0 + 2\delta P_0 + \tfrac{2\alpha_0\rho}{(N-0.5)\pi^2}\tilde{K}_a^T\tilde{K}_a, \\ &\Gamma_2 = \begin{bmatrix} P_0\mathcal{L} & P_0\mathcal{B} \\ 0 & 0 \end{bmatrix}, \ \Gamma_3 = \begin{bmatrix} P_0\mathcal{L}C_1e^{-A_1\tau_y} \\ 0 \end{bmatrix}, \\ &\Theta_1 = \begin{bmatrix} -\frac{\pi^2}{4}W_1 - 2\delta_1P_0 & 0 \\ 0 & -\frac{\pi^2}{4}W_2 + \frac{2\alpha_1\rho}{(N-0.5)\pi^2} \end{bmatrix}, \\ &\Theta_2 = \begin{bmatrix} -\gamma & 0 \\ 0 & -\gamma + \frac{2\alpha_2\rho}{(N-0.5)\pi^2} \end{bmatrix}, \ \varepsilon_y = \tau_{M,y}^2e^{2\delta_0\tau_{M,y}}, \\ &\varepsilon_u = \tau_{M,u}^2e^{2\delta_0\tau_{M,u}}, \ R = [R_1, \mathcal{L}C_1e^{-A_1\tau_y}], \\ &R_1 = [F_0, \mathcal{L}, \mathcal{LC}, -\mathcal{B}, \mathcal{L}, \mathcal{B}]. \end{aligned} \tag{4.29}$$

Note that $\delta = \delta_0 - \delta_1$ and (3.11) imply $A_1 + \delta_0I - \delta_1e^{-2A_1\tau_y} < 0$. Therefore, by applying Schur complement in (4.28) and taking $p_e \to \infty$ we find that (4.28) holds iff the reduced-order LMI

$$\begin{aligned} &\left[\begin{array}{c|cc} \Phi^{(2)} & \Gamma_1 & \Gamma_2 \\ \hline * & \multicolumn{2}{c}{\mathrm{diag}\{\Theta_1, \Theta_2\}} \end{array}\right] \\ &+ R_1^T\left(\varepsilon_yW_1 + \varepsilon_u\tilde{K}_0^TW_2\tilde{K}_0\right)R_1 < 0. \end{aligned} \tag{4.30}$$

is feasible with $\varepsilon_y = \tau_{M,y}^2 e^{2\delta_0 \tau_{M,y}}$ and $\varepsilon_u = \tau_{M,u}^2 e^{2\delta_0 \tau_{M,u}}$. Monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$ and Schur complement imply that $\mu_n < 0$ for all $n > N$ iff

$$\left[\begin{array}{c|ccc} -\lambda_{N+1} + a + \delta_0 & 1 & 1 & 1 \\ \hline * & \multicolumn{3}{c}{-2\,\mathrm{diag}\,\{\alpha_0, \alpha_1, \alpha_2\}\,\lambda_{N+1}^{-1}} \end{array}\right] < 0. \tag{4.31}$$

Strict LMIs (4.30) and (4.31) imply that the conditions of Proposition 2.1 are satisfied with a slightly larger $\bar{\delta}_0 > \delta_0$. Therefore, we obtain

$$\bar{V}(t) \quad \le e^{-2(\delta_\tau + \varepsilon)t}\bar{V}(0) + \frac{\gamma \Delta^2}{\delta_\tau} \quad t \ge 0 \tag{4.32}$$

with small $\varepsilon > 0$, where $\delta_\tau + \varepsilon > 0$ is a unique solution of $\delta_\tau + \varepsilon = \bar{\delta}_0 - \delta_1 e^{2(\delta_\tau + \varepsilon)h}$. Moreover, taking into account that $e^{N-N_0}(t)$ is exponentially decaying and that (4.19) is ISS with respect to $X_0(t-\tau_u)$, $\sigma_u(t)$ and $u(t)$, we obtain ISS with a decay rate δ_τ of the full-order closed-loop system (3.29), (4.17) and (4.19).

Define

$$e(x,t) = w(x,t) - \hat{w}(x,t). \tag{4.33}$$

We also derive an attracting ball in $H^1(0,1)$ for the full-order closed-loop system. We have

$$\begin{aligned} &\bar{V}(t) \ge V_0(t) \ge \lambda_{min}(P_0)\,|X_0(t)|^2 + \rho \textstyle\sum_{n=N+1}^{\infty} \lambda_n w_n^2(t) \\ &\ge M_2 \left(\textstyle\sum_{n=1}^{N_0} \lambda_n \hat{w}_n^2(t) + \sum_{n=1}^{N_0} \lambda_n e_n^2(t) \right. \\ &\left. + u^2(t) + \textstyle\sum_{n=N+1}^{\infty} \lambda_n w_n^2(t) \right), \; M_2 = \min\left(\rho, \frac{\lambda_{min}(P_0)}{\lambda_{N_0}}\right). \end{aligned} \tag{4.34}$$

Therefore, from (4.32) and (4.34) we have

$$\begin{aligned} \limsup_{t\to\infty} \quad & \left(\textstyle\sum_{n=1}^{N_0} \lambda_n \hat{w}_n^2(t) + \sum_{n=1}^{N_0} \lambda_n e_n^2(t) \right. \\ & \left. + \textstyle\sum_{n=N+1}^{\infty} \lambda_n w_n^2(t) + u^2(t) \right) \le \frac{\gamma \Delta^2}{M_2 \delta_\tau}, \\ \limsup_{t\to\infty} \quad & |X_0(t)|^2 \le \frac{\gamma \Delta^2}{\lambda_{min}(P_0)\delta_\tau}. \end{aligned} \tag{4.35}$$

To obtain an ISS bound on $\hat{w}^{N-N_0}$, recall (4.19). Let $D = \mathrm{diag}\left\{\sqrt{\lambda_{N_0+1}}, \ldots, \sqrt{\lambda_N}\right\}$. Since $\hat{w}_n = 0,\ n = 1, 2, \ldots$, by variation of constants we have

$$\begin{aligned} \left| D\hat{w}^{N-N_0}(t) \right| \quad & \le |DB_1| \int_0^t \left| e^{A_1(t-s)} \right| |g(s)|\, ds \\ & \overset{(3.11)}{\le} |DB_1| \int_0^t e^{-(\lambda_{N_0+1} - a)(t-s)} |g(s)|\, ds, \\ g(s) = \tilde{K}_0 X_0(s - \tau_u(s)) & - \sigma_u(s) + a u(s). \end{aligned} \tag{4.36}$$

Using $b_n = (-1)^{n+1}\sqrt{\frac{2}{\lambda_n}}$ we have $|DB_1| = \sqrt{2(N-N_0)}$. Thus, from (4.18), (4.35) and (4.36) we obtain

$$\begin{aligned} & \limsup_{t\to\infty} \left(\textstyle\sum_{n=N_0+1}^{N} \lambda_n \hat{w}_n^2(t) \right) \le M_3 \Delta^2, \\ & M_3 = \frac{2(N-N_0)}{(\lambda_{N_0+1} - a)^2} \left[\left(\left| \tilde{K}_0 \right| + a \right) \sqrt{\frac{\gamma}{\lambda_{min}(P_0)\delta_\tau}} + 1 \right]^2. \end{aligned} \tag{4.37}$$

Finally, note that (3.29) implies

$$\limsup_{t\to\infty} \textstyle\sum_{n=N_0+1}^{N} \lambda_n e_n^2(t) = 0. \tag{4.38}$$

From (1.5), (4.33), (4.35), (4.37) and (4.38) we have

$$\begin{aligned} \limsup_{t\to\infty} & \left(\left\| \hat{w}_x(\cdot,t) \right\|^2 + \|e_x(\cdot,t)\|^2 \right. \\ & \left. + u^2(t) \right) \le \left[\frac{\gamma}{M_2 \delta_\tau} + M_3 \right] \Delta^2, \\ \limsup_{t\to\infty} & \|w_x(\cdot,t)\|^2 \le 2\left[\frac{\gamma}{M_2\delta_\tau} + M_3 \right] \Delta^2 \end{aligned} \tag{4.39}$$

where the latter was obtained using the triangle inequality. Therefore, solutions of the full-order closed-loop system are exponentially converging with decay rate δ_τ to the ball

$$\begin{aligned} & B_\Delta(r) = \left\{ h \in H^1(0,1) \mid \|h\|_{H^1} \le r\Delta \right\}, \\ & r = \sqrt{3\left[\frac{\gamma}{M_2\delta_\tau} + M_3 \right]}. \end{aligned} \tag{4.40}$$

Summarizing, we have:

Theorem 4.1. *Consider (4.5) with boundary conditions (3.6), point measurement (4.6), control law (4.10) and $w(\cdot,0) \in \mathcal{D}(\mathcal{A}_1^{\frac{1}{2}})$. Let $\Delta > 0$ be the quantization error bound. Given $\delta > 0$, let $N_0 \in \mathbb{N}$ satisfy (3.11) and $N \in \mathbb{N}$ satisfy $N_0 \le N$. Let L_0 and K_0 be obtained using (3.16) and (3.17), respectively. Given $\rho, \gamma, \tau_{M,y}, \tau_{M,u} > 0$, $\delta_1 > 0$ and $\delta_0 = \delta_1 + \delta$, let there exist $0 < P_0, W_1 \in \mathbb{R}^{(2N_0+1)\times(2N_0+1)}$ and scalars $0 < \alpha_0, \alpha_1, \alpha_2, W_2$ which satisfy (4.30) and (4.31) with notations (4.29). Then, the full-order closed-loop system (3.29), (4.17) and (4.19) is ISS, meaning that the following inequality is satisfied:*

$$\begin{aligned} & \|w(\cdot,t)\|_{H^1}^2 + \left\| \hat{w}(\cdot,t) \right\|_{H^1}^2 + u^2(t) \\ & \qquad \le M_0 e^{-2\delta_\tau t} \|w(\cdot,0)\|_{H^1}^2 + r^2\Delta^2 \end{aligned} \tag{4.41}$$

with some $M_0 > 0$, r defined in (4.40) (with M_2 and M_3 given in (4.34) and (4.37), respectively). Here $\delta_\tau > 0$ the unique solution of $\delta_\tau = \delta_0 - \delta_1 e^{2\delta_\tau \tau_{M,y}}$. The solutions of the full-order closed-loop system are exponentially converging with a decay rate δ_τ to the attractive ball (4.40). The LMIs (4.30) and (4.31) are always feasible for large enough N and small enough $\tau_{M,y}, \tau_{M,u}$ and their feasibility for N implies feasibility for $N+1$.

Proof. First, we show that feasibility of (4.30) and (4.31) for N implies feasibility for $N+1$. Fix $\rho, \gamma, \tau_{M,y}, \tau_{M,u} > 0$, $\delta_1 > 0$, $\delta_0 = \delta_1 + \delta$, $P_0 > 0$, $W_1 > 0$, $\alpha_i > 0$, $i = 0,1,2$ and $W_2 > 0$ such that (4.30) and (4.31) are feasible for some N. By monotonicity of λ_n, $n = 1, 2 \ldots$ we have that (4.27) implies $\mu_{N+2} < 0$ and feasibility of (4.31) with N replaced by $N+1$. Furthermore, since $\frac{2\alpha_i \rho}{(N-0.5)\pi^2}$, $i \in \{0,1,2\}$ appearing in ϕ, Θ_1 and Θ_2, respectively (see (4.29)), decrease to zero as $N \to \infty$, (4.30) holds with N replaced by $N+1$.

Second, we show that (4.30) and (4.31) are feasible for large enough N and small enough $\tau_{M,y}, \tau_{M,u}$. Fix $\rho = 1$ and $\alpha_i = 2$, $i \in \{0,1,2\}$. From (3.16) and (3.17), there exists some $P_0 > 0$, independent of N, such that $P_0 F_0 + F_0^T P_0 + 2\delta P_0 = -I$. Then, for and large enough N and δ_1 we have

$$\Phi^{(2)} = \begin{bmatrix} -I + \frac{4}{(N-0.5)\pi^2} \tilde{K}_a^T \tilde{K}_a & P_0 \mathcal{L} \\ * & -2\delta_1 \end{bmatrix} < 0 \tag{4.42}$$

with $\Phi^{(2)}$ given in (4.29). Moreover, the eigenvalues of $\Phi^{(2)}$ decrease as $N \to \infty$. Fix δ_1 and $\delta_0 = \delta_1 + \delta$.

Consider (4.30) and let $W_1 = N \cdot I_{2N_0+1}$, $W_2 = N$ and $\gamma = N$ with large enough N. By Schur complement we have

$$\left[\begin{array}{c|cc} \Phi^{(2)} & \Gamma_1 & \Gamma_2 \\ \hline * & \multicolumn{2}{c}{\mathrm{diag}\,\{\Theta_1, \Theta_2\}} \end{array}\right] < 0 \tag{4.43}$$

iff $\Phi^{(2)} - \sum_{i=1}^{2} \Gamma_i \Theta_i^{-1} \Gamma_i^T < 0$. The latter holds for large enough N by (4.42) and our choice of W_1, W_2, γ, ρ and α_i, $i \in \{0,1,2\}$. Choosing $\tau_{M,y} = \tau_{M,u} = \frac{1}{N}$ and using (4.43) we find that (4.30) holds for large enough N. Finally, recall that for $\alpha_i = 2$, $i \in \{0,1,2\}$, (4.31) holds iff

$$\mu_{N+1} = -\lambda_{N+1}\left[\frac{1}{4} - \frac{a+\delta_0}{\lambda_{N+1}} \right] < 0 \tag{4.44}$$

Since δ_1 and δ_0 are fixed, (4.44) holds for large enough N, by monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$. □

Remark 4.2. Note that the estimate (4.40) on $r > 0$, where $r\Delta$ is the radius of the ball of attraction, is only an apriori qualitative bound. In order to obtain a smaller bound on $r > 0$, it is desirable to minimize the quantity $\frac{\gamma}{\lambda_{min}(P_0)}$ given in (4.35) and (4.37). In the examples below, this is done by manually tuning the parameter γ. There exist more advanced methods of incorporating the minimization of $\frac{\gamma}{\lambda_{min}(P_0)}$ into the LMIs (see, e.g, Fridman and Dambrine

(2009)). We leave the development of such methods for future research.

Remark 4.3. The Halanay-based tools developed in this paper can be used for sampled-data ISS analysis of

$$z_t(x,t)=z_{xx}(x,t)+az(x,t)+d_0(x,t),\ t\geq 0$$

under disturbed Dirichlet actuation

$$z_x(0,t)=0,\quad z(1,t)=u(t)+d(t),$$

disturbed sampled-data measurement

$$y(t)=z(x_*,s_k)+\sigma_k,\ x_*\in[0,1),\ t\in[s_k,s_{k+1}) \tag{4.45}$$

and the generalized hold implementation

$$\dot u(t)=v(t_j),\quad t\in[t_j,t_{j+1}). \tag{4.46}$$

Here $\sigma=\{\sigma_k\}_{k=0}^{\infty}$ satisfies $\|\sigma\|_{\ell^\infty}\leq\Delta$, $d\in C^2([0,\infty))$ subject to $\max\left(|d(t)|,\left|\dot d(t)\right|\right)\leq\Delta$ for all $t\geq 0$ and $d_0\in L^2((0,\infty);L^2(0,1))\cap H^1_{\mathrm{loc}}((0,\infty);L^2(0,1))$, with some $\Delta>0$. In this case, the dynamic extension (3.4) leads to the following ODE–PDE system:

$$\begin{aligned}&\dot u(t)=v(t_j),\quad t\in[t_j,t_{j+1}),\\&w_t(x,t)=w_{xx}(x,t)+aw(x,t)+au(t)-v(t_j)+f(x,t),\\&f(x,t)=d_0(x,t)+ad(t)-\dot d(t).\end{aligned}$$

The smoothness assumptions on d_0 and d are needed for well-posedness. In the continuous-time case, L^2-gain and ISS analysis of parabolic PDEs under finite-dimensional observer-based control was initiated in Katz and Fridman (2021a).

5. Sampled-data control: Neumann actuation

In this section we consider sampled-data control of (3.1) under Neumann actuation

$$z_x(0,t)=u(t),\quad z(1,t)=0 \tag{5.1}$$

and quantized point measurement (4.1) with $q:\mathbb{R}\to\mathbb{R}$ satisfying (4.2). The sequences of sampling instances $\{s_k\}_{k=0}^{\infty}$ and $\left\{t_j\right\}_{j=0}^{\infty}$ are the same as in Section 4. The control input $u(t)$ is generated by a generalized hold device of the form (4.3). The derivation of the closed-loop system and practical stability analysis in this section are similar to Section 4. Therefore, we present them succinctly, while emphasizing the main differences.

Introducing the change of variables

$$w(x,t)=z(x,t)-r(x)u(t),\quad r(x)=x-1$$

we obtain the following equivalent ODE–PDE system

$$\begin{aligned}&\dot u(t)=q\left[v(t_j)\right],\quad t\in[t_j,t_{j+1}),\\&w_t(x,t)=w_{xx}(x,t)+aw(x,t)+r(x)\left(au(t)-q\left[v(t_j)\right]\right).\end{aligned} \tag{5.2}$$

with boundary conditions (3.6) and measurement

$$y(t)=q\left[w(x_*,s_k)+r(x_*)u(s_k)\right],\ t\in[s_k,s_{k+1}). \tag{5.3}$$

Recall that we treat $u(t)$ as an additional state variable and the values $\left\{v(t_j)\right\}_{j=1}^{\infty}$ as the control input to be determined. We choose $u(0)=0$ which results in $w(\cdot,0)=z(\cdot,0)$.

We present the solution to (5.2) as (3.8) with $\{\phi_n\}_{n=1}^{\infty}$ defined in (1.4). By differentiating under the integral sign, integrating by parts and using (1.2) and (1.3) we obtain (4.7) where now

$$|b_n|=\left|\int_0^1 r(x)\phi_n(x)dx\right|=\frac{\sqrt2}{\lambda_n},\quad n=1,2,\dots \tag{5.4}$$

satisfy (3.10).

Given $\delta>0$, let $N_0\in\mathbb{N}$ satisfy (3.11) and $N\in\mathbb{N}$, $N_0\leq N$. Define a finite-dimensional observer of the form (3.12) where $\hat w_n(t)$ satisfy (4.9) with innovation term replaced by

$$\hat w(x_*,t-\tau_y)+r(x_*)u(t-\tau_y)-y(t). \tag{5.5}$$

Under Assumption 1 let the observer and controller gains, L_0 and K_0, satisfy (3.16) and (3.17), respectively. We choose $l_n=0$ for $n>N_0$. We propose a (N_0+1)-dimensional controller of the form (4.10) with $\hat w^{N_0}(t)$ given in (3.18).

Recall the estimation error $e_n(t)$ defined in (3.23). By using (3.8), (3.12) and arguments similar to (3.24) the innovation term (5.5) can be written as (4.12) with $\zeta(t)$ and $\sigma_y(t)$ satisfying (3.27) and (4.13), respectively. Then, the error equations have the form (4.14). Using (3.28) and (4.15) we obtain the reduced-order closed-loop system (4.17). Furthermore, note that $e^{N-N_0(t)}$ satisfies (3.29), which implies (4.16), whereas $\hat w^{N-N_0}(t)$ satisfies (4.19). Here, $\sigma_u(t)$ is given by (4.18).

For H^1-ISS of the closed-loop system (4.17) let $\rho>0$, $\delta_0>\delta$ and define the Lyapunov function (4.20), with $V(t)$ given in (3.32) and $V_y(t)$ and $V_u(t)$ given in (4.21). Consider $[s_k,s_{k+1})$, $k=0,1,\dots$, where s_k,s_{k+1} are consecutive measurement sampling instances. There exist at most finitely many controller update instances $t_j^{(k)}$, $0\leq j\leq n_k$ for which (2.9) holds. Moreover, $\bar V(t)$ defined by (4.20) and (4.21) is continuously differentiable on $[t_j^{(k)},t_{j+1}^{(k)})$, $0\leq j\leq n_k-1$. Therefore, we apply Proposition 2.1 to obtain (2.11).

Differentiating $V_0(t)$ on $[t_j^{(k)},t_{j+1}^{(k)})$, $0\leq j\leq n_k-1$ along (4.17) we have (4.22) with the last term replaced by

$$2\rho\sum_{n=N+1}^{\infty}\left[-\lambda_n+a+\delta_0+\sum_{i=0}^{2}\tfrac{1}{2\alpha_i}\right]\lambda_n w_n^2(t).$$

The latter is obtained due to (3.35), (5.4) and the following application of the Young inequality:

$$\begin{aligned}&2\rho\sum_{n=N+1}^{\infty}\lambda_n w_n(t)b_n\left[\tilde K_a X_0(t)+\tilde K_0\Upsilon_u(t)-\sigma_u(t)\right]\\&\leq\left(\tfrac{\rho}{\alpha_0}+\tfrac{\rho}{\alpha_1}+\tfrac{\rho}{\alpha_2}\right)\sum_{n=N+1}^{\infty}\lambda_n w_n^2(t)+\tfrac{2\alpha_2\rho}{\pi^2(N-0.5)}\sigma_u^2(t)\\&+\tfrac{2\alpha_0\rho}{\pi^2(N-0.5)}\left|\tilde K_a X_0(t)\right|^2+\tfrac{2\alpha_1\rho}{\pi^2(N-0.5)}\left|\tilde K_0\Upsilon_u(t)\right|^2.\end{aligned} \tag{5.6}$$

Differentiation of $p_e\left|e^{N-N_0}(t)\right|^2$ along the solution to (4.17) results in (3.34) with δ replaced by δ_0. Differentiating $V_y(t)$ and $V_u(t)$ along (4.17) we obtain (4.23), whereas $\zeta(t-\tau_y)$ is compensated by (4.24) with $\delta_1=\delta_0-\delta$. Let $\gamma>0$ be a scalar. Using (4.13) and (4.18) we have (4.25).

Let

$$\eta(t)=\mathrm{col}\Big\{X_0(t),\zeta(t-\tau_y),\Upsilon_y(t),\tilde K_0\Upsilon_u(t),\sigma_y(t),\sigma_u(t),e^{N-N_0}(t)\Big\}.$$

From (4.23), (4.24), (4.25) and (5.6) we obtain

$$\begin{aligned}&\dot{\bar V}(t)+2\delta_0\bar V(t)-2\delta_1\sup_{s_k\leq\theta\leq t}\bar V(\theta)-2\gamma\Delta^2\\&\leq\eta^T(t)\Psi^{(2)}\eta(t)+2\rho\sum_{n=N+1}^{\infty}\nu_n\lambda_n w_n^2(t)\leq 0\end{aligned} \tag{5.7}$$

if $\Psi^{(2)}<0$ and $\nu_n=-\lambda_n+a+\delta_0+\left[\sum_{i=0}^{2}\frac{1}{2\alpha_i}\right]<0$ for $n>N$, where $\Psi^{(2)}$ is given in (4.28) and (4.29).

Since (3.11) implies $A_1+\delta_0 I-\delta_1 e^{-2A_1\tau_y}<0$, by applying Schur complement to $\Psi^{(2)}$ and taking $p_e\to\infty$ we find that $\Psi^{(2)}<0$ iff (4.30) holds with R_1 given in (4.29). Monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$ and Schur complement imply that $\nu_n<0$ for all $n>N$ iff

$$\left[\begin{array}{c|ccc}-\lambda_{N+1}+a+\delta_0 & 1 & 1 & 1\\\hline * & \multicolumn{3}{c}{-2\,\mathrm{diag}\{\alpha_0,\alpha_1,\alpha_2\}}\end{array}\right]<0. \tag{5.8}$$

The rest of the ISS analysis of the full-order closed-loop system and the estimation of the attracting ball follows arguments identical to (4.32)–(4.40). Summarizing, we have:

Table 1
LMIs of Theorem 4.1 - (N, δ_0) for different $\tau_{M,y}$ and $\tau_{M,u}$.

$\tau_{M_y}/\tau_{M,u}$	0.01	0.03	0.05	0.07
0.01	(2,0.3)	(2,0.3)	(2,0.3)	(2,0.3)
0.03	(2,0.3)	(2,0.3)	(2,0.3)	(2,0.3)
0.05	(2,0.3)	(2,0.3)	(2,0.3)	(2,0.3)
0.07	(2,0.3)	(3,0.4)	(3,0.4)	(3,0.4)

Table 2
LMIs of Theorem 5.1 - (N, δ_0) for different $\tau_{M,y}$ and $\tau_{M,u}$.

$\tau_{M_y}/\tau_{M,u}$	0.03	0.05	0.07	0.09
0.01	(2,0.1)	(2,0.1)	(2,0.2)	(2,0.2)
0.05	(2,0.2)	(2,0.1)	(2,0.1)	(2,0.2)
0.09	(3,0.3)	(3,0.2)	(3,0.1)	(3,0.1)
0.11	(3,0.2)	(4,0.2)	(4,0.3)	(4,0.2)

Theorem 5.1. *Consider (5.2) with boundary conditions (3.6), point measurement (5.3), control law (4.10) and $w(\cdot,0) \in \mathcal{D}(\mathcal{A}_1^{\frac{1}{2}})$. Let $\Delta > 0$ be the quantization error bound. Given $\delta > 0$, let $N_0 \in \mathbb{N}$ satisfy (3.11) and $N \in \mathbb{N}$ satisfy $N_0 \le N$. Let L_0 and K_0 be obtained using (3.16)* and *(3.17), respectively. Given $\rho, \gamma, \delta_1, \tau_{M,y}, \tau_{M,u} > 0$ and $\delta_0 = \delta_1 + \delta$, let there exist $0 < P_0, W_1 \in \mathbb{R}^{(2N_0+1)\times(2N_0+1)}$ and scalars $0 < \rho, \alpha_0, \alpha_1, \alpha_2, W_2$ which satisfy (4.28)* and *(5.8). Then, the full-order closed-loop system (3.29), (4.17)* and *(4.19) is ISS, meaning that inequality (4.41) is satisfied with some $M_0 > 0$, r defined in (4.40) (with M_2 and M_3 given in (4.34)* and *(4.37), respectively) and $\delta_\tau > 0$ the unique solution of $\delta_\tau = \delta_0 - \delta_1 e^{2\delta_\tau \tau_{M,y}}$. Furthermore, the solutions of the full-order closed-loop system are exponentially converging with a decay rate δ_τ to the attractive ball given in (4.40). The LMIs (4.28)* and *(5.8) are always feasible for large enough N and small enough $\tau_{M,y}, \tau_{M,u}$ and their feasibility for N implies feasibility for $N+1$.*

6. Examples

In all numerical examples we choose $a = 3$ which results in an unstable open-loop system. The observer and controller gains, L_0 and K_0 are obtained using (3.16) and (3.17), respectively. All LMIs were verified using the standard Matlab LMI toolbox.

Consider (3.1) under Dirichlet actuation (3.2). Let $\delta = 10^{-4}$, leading to $N_0 = 1$. For in-domain measurement $x_* = \pi^{-1}$ the obtained gains are

$$L_0 = 0.5097, \quad K_0 = [7.8678, 4.2599]. \tag{6.1}$$

Given different values of $N \in \{2,3,4\}$ we verify the LMIs of Theorem 4.1 to guarantee ISS while increasing $\tau_{M,y}$ and $\tau_{M,u}$. We find the values of N, δ_0 and $\delta_1 = \delta_0 - \delta$ for which the LMIs are feasible. The results are presented in Table 1.

Next, we find $r > 0$ (as small as possible) defined in (4.40), where $r\Delta$ is the radius of the ball of attraction. We fix $x_* = 0$ (boundary measurement) or $x_* = \pi^{-1}$ (in-domain measurement) and $\delta = 0.12$, which results in $N_0 = 1$. The observer and controller gains corresponding to $x_* = \pi^{-1}$ are given by (6.1), whereas for $x_* = 0$ we have

$$L_0 = 0.5887, \quad K_0 = [9.9965, 5.4284].$$

For $N = 2$ and $\tau_{M,y} = \tau_{M,u} = 0.01$, we check the LMIs of Theorem 4.1 while tuning $\gamma > 0$ with the above gains to obtain an estimate of r that is as small as possible. The obtained results are

$$\underline{x_* = 0}: \ \delta_0 = 2.15, \ \delta_\tau \approx 0.114, \ \gamma = 0.83, \ r = 148.9,$$
$$\underline{x_* = \pi^{-1}}: \ \delta_0 = 2.47, \ \delta_\tau \approx 0.114, \ \gamma = 0.84, \ r = 169.5.$$

Consider now (3.1) under Neumann actuation (5.1). Let $\delta = 10^{-4}$, leading to $N_0 = 1$. For in-domain measurement $x_* = \pi^{-1}$ the obtained gains are

$$L_0 = 0.5097, \quad K_0 = [4.5, -4.046]. \tag{6.2}$$

Given $N \in \{2,3,4\}$ we verify the LMIs of Theorem 5.1 to guarantee ISS while increasing $\tau_{M,y}$ and $\tau_{M,u}$. The results are presented in Table 2.

Next, we find $r > 0$ (as small as possible) defined in (4.40), where $r\Delta$ is the radius of the ball of attraction. We fix $x_* = 0$ (boundary measurement) or $x_* = \pi^{-1}$ (in-domain measurement) and $\delta = 0.15$, which results in $N_0 = 1$. The observer and controller gains corresponding to $x_* = \pi^{-1}$ are given by (6.2), whereas for $x_* = 0$ we have

$$L_0 = 1.0837, \quad K_0 = [12.6755, -12.7348]. \tag{6.3}$$

For $N = 2$ and $\tau_{M,y} = \tau_{M,u} = 0.01$, we check the LMIs of Theorem 5.1 with the above gain while tuning $\gamma > 0$ to obtain an estimate of r that is as small as possible. The obtained results are

$$\begin{aligned} &\underline{x_* = 0}: \ \delta_0 = 2.86, \ \delta_\tau \approx 0.142, \ \gamma = 0.1246, \ r = 96.9, \\ &\underline{x_* = \pi^{-1}}: \ \delta_0 = 2.78, \ \delta_\tau \approx 0.141, \ \gamma = 0.1354, \ r = 94.3. \end{aligned} \tag{6.4}$$

For simulations of the closed-loop system, we consider Neumann actuation with initial condition

$$z_0(x) = 3(x - x^2)^2, \quad x \in [0,1].$$

Let $x_* = 0$ (boundary measurement) and $\tau_{M,y} = 0.05$ and $\tau_{M,u} = 0.09$. The variable sampling instances were generated by $s_{k+1} = s_k + 0.5(1+U_k)\tau_{M,y}$, where $U_k \sim Unif(0,1)$ was chosen at random. Similarly, the variable controller hold times were generated by $t_{j+1} = t_j + 0.5(1+U_j)\tau_{M,u}$, where $U_j \sim Unif(0,1)$. We consider $\delta = 0.0001$, which results in $N_0 = 1$. The corresponding observer and controller gains are given by (6.2). We further fix $N = 2$ and consider two uniform quantizers: either with the quantization error $\Delta = 0.01$ or with $\Delta = 0.05$. In simulations of (3.6) and (5.2) we use Lemma 1.2 to estimate $|u(t)| + \|w\|_{H_1} \approx |u(t)| + \sum_{n=1}^{40} \lambda_n w_n^2(t)$. The values of $w_n(t)$, $1 \le n \le 2$ were found from simulation of the observer ODEs (4.9) and error ODEs (4.14) and applying $w_n(t) = e_n(t) + \hat{w}_n(t)$, $n \ge 1$. The values of $w_n(t)$, $3 \le n \le 40$ were obtained from simulation of the ODEs (4.7). The value of $\zeta(t)$, given in (3.25), was approximated by $\zeta(t) \approx \sum_{n=3}^{40} w_n(t)\phi_n(x_*)$. The results are presented in Fig. 3 and confirm the theoretical analysis. The maximum values of $\tau_{M,y}$ and $\tau_{M,u}$ for which ISS still holds in simulations were 4–5 times larger than predicted from LMIs. Plots of the boundary control $u(t)$ in (5.1), the values of $q[v(t_j)], j \ge 0$ (see (4.10)) and the quantized measurement (5.3) are presented in Fig. 4. Finally, for the controller hold times $\{t_j\}_{j=1}^{\infty}$, we plot the values of $\lim_{t\to t_j^-} V_u(t)$ (see Fig. 5). Note that by (4.8) and (4.21), we have $V_u(t_j) = 0$, $j \ge 1$. Thus, $\lim_{t\to t_j^-} V_u(t)$ is the size of the jump of $V_u(t)$ at t_j, $j \ge 1$. A similar plot can be obtained for the jumps of $V_y(t)$ at the instances s_k.

7. Conclusions

This paper presented quantized sampled-data finite-dimensional control of a reaction–diffusion PDE under boundary actuation and point (either in-domain or boundary) discrete-time measurement. The design was based on the modal decomposition approach via dynamic extension, which required a generalized hold device for sampled-data implementation. For ISS analysis, we used Wirtinger-based piecewise continuous in time Lyapunov functionals and combined them with appropriate Halanay's inequalities to compensate sampling in the infinite-dimensional

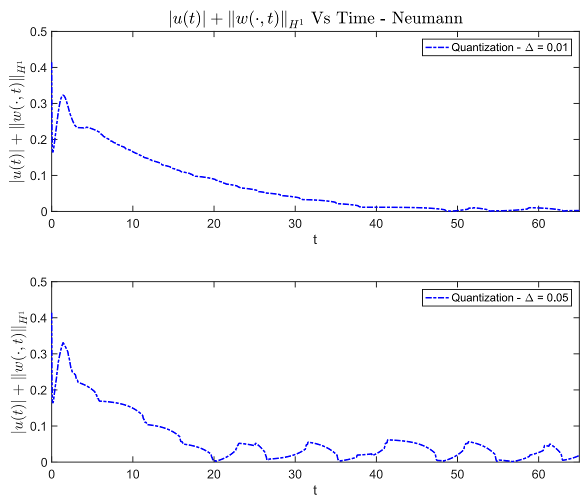

Fig. 3. Closed-loop system simulations.

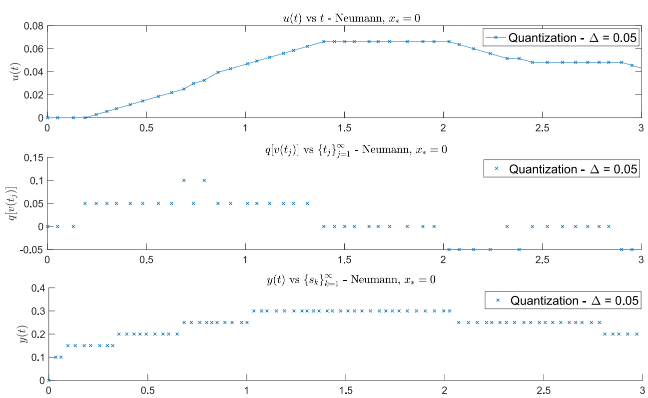

Fig. 4. Input $u(t)$, values of $q[v(t_j)]$ and quantized output.

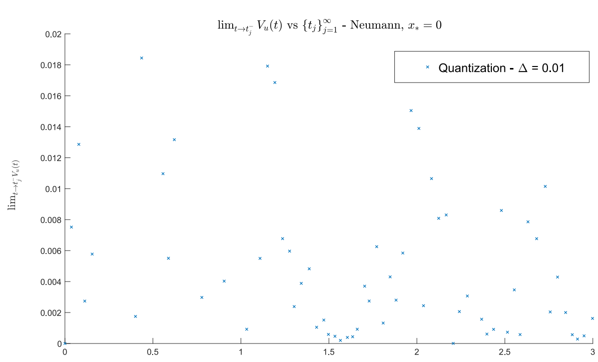

Fig. 5. Values of $\lim_{t\to t_j^-} V_u(t)$, whereas $V_u(t_j) = 0$.

tail. As an additional result, we have derived novel ISS Halanay's inequalities for piecewise continuous functions which do not grow in the jumps.

Novel Halanay's inequalities may be used in the future for various sampled-data control problems for ODEs and PDEs. The presented design method can be extended to other PDEs and to the case of input and output delays. Additional topics for future research may be quantization with saturation, as well as improved methods for finding the radius of the ball of attraction.

Appendix

Proof of Lemma 2.1.

We prove Lemma 2.1 for the case where $V(t)$ has at least one point of jump discontinuity in $[a, b)$. The proof for a continuous $V(t)$ is easier and follows similar arguments.

Denote $\{\xi_i\}_{i=1}^{M} \subseteq \{t_i\}_{i=1}^{N-1}$ to be the points where $V(t)$ has a jump discontinuity (see Fig. 1). Thus, we have:

$$\lim_{t\nearrow\xi_i} V(t) > V(\xi_i), \quad i = 1, 2, \ldots, M. \tag{A.1}$$

A unique solution to $\delta_\tau = \delta_0 - \delta_1 e^{2\delta_\tau h}$ exists by arguments of Lemma 4.2 in Fridman (2014). Let

$$y(t) = e^{-2\delta_\tau(t-a)}V(a) + d\int_a^t e^{-2\delta(t-s)}ds, \ \ t \in [a, b). \tag{A.2}$$

be the right-hand side of (2.5). Differentiating $y(t)$ we have

$$\begin{aligned}\dot{y}(t) + 2\delta_0 y(t) - d &= 2\delta_1 e^{2\delta_\tau h} e^{-2\delta_\tau(t-a)}V(a) \\ &\quad + 2\delta_1 d\int_a^t e^{-2\delta(t-s)}ds,\end{aligned} \tag{A.3}$$

whereas $V(a) + d\int_a^t e^{-2\delta(t-s)}ds \geq \sup_{a\leq\theta\leq t} y(\theta)$. Thus

$$\begin{aligned}&\dot{y}(t) \geq -2\delta_0 y(t) + 2\delta_1 \sup_{a\leq\theta\leq t} y(\theta) + d, \ \ t \in [a, b), \\ &y(a) = V(a).\end{aligned} \tag{A.4}$$

Let $\epsilon_1 > \epsilon_2 > \cdots > 0$ be a sequence of positive scalars such that $\lim_{n\to\infty}\epsilon_n = 0$ and define

$$y_n(t) = y(t) + \frac{\epsilon_n}{2\delta}.$$

Then $y_n(t)$ satisfies the following for $t \in [a, b)$:

$$\begin{aligned}&\dot{y}_n(t) \geq -2\delta_0 y_n(t) + 2\delta_1 \sup_{a\leq\theta\leq t} y_n(\theta) + d + \epsilon_n, \\ &y_n(a) > V(a).\end{aligned} \tag{A.5}$$

It is sufficient to show that $V(t) \leq y_n(t) = y(t) + \frac{\epsilon_n}{2\delta}$ for all $n = 1, 2, \ldots$ and all $t \in [a, b)$ and then take $n \to \infty$. Assume by contradiction that

$$\exists n \geq 1 \ : \ \mathcal{J}_n = \{t \in [a, b) | V(t) > y_n(t)\} \neq \emptyset$$

and denote $t_* = \inf \mathcal{J}_n$. By (A.5) and right continuity of $y_n(t)$ and $V(t)$ on $[a, b)$ we have that $t_* \in (a, b)$. Moreover, by definition of $\mathcal{J}_n$, $V(t) \leq y_n(t)$, $t \in [a, t_*)$ and there exists a sequence τ_k, $k = 1, 2, \ldots$ such that $\tau_k \searrow t_*$ and

$$V(\tau_k) > y_n(\tau_k), \quad k = 1, 2, \ldots$$

which imply $V(t_*) \geq y_n(t_*)$, by right continuity.

Next, we show that t_* is a point of continuity for $V(t)$, by showing that $t_* \neq \xi_i$, $i = 1, \ldots, M$. Assume by contradiction that there exists some $i = 1, \ldots, M$ such that $t_* = \xi_i$. From continuity of $y_n(t)$ on $[a, b)$, $V(t_*) \geq y_n(t_*)$ and (2.2) there exists some $\bar{t} < t_*$ sufficiently close to t_* such that $V(\bar{t}) > y_n(\bar{t})$. Therefore, $\bar{t} \in \mathcal{J}_n$ and $\bar{t} < t_* = \inf \mathcal{J}_n$, which is a contradiction. Now $V(t_*) \geq y_n(t_*)$ and t_* is a point of continuity of both $V(t)$ and $y_n(t)$. Thus, it must be that $V(t_*) = y_n(t_*)$ (otherwise, by continuity, we again have $\bar{t} < t_*$ such that $V(\bar{t}) > y_n(\bar{t})$). We conclude from the previous properties that

$$\sup_{a\leq\theta\leq t_*} V(\theta) \leq \sup_{a\leq\theta\leq t_*} y_n(\theta), \tag{A.6}$$

and

$$\frac{V(\tau_k) - V(t_*)}{\tau_k - t_*} > \frac{y_n(\tau_k) - y_n(t_*)}{\tau_k - t_*}, \ \ k = 1, 2, \ldots \tag{A.7}$$

Then

$$\begin{aligned}D^+V(t_*) &\overset{(2.3)}{\leq} -2\delta_0 V(t_*) + 2\delta_1 \sup_{a\leq\theta\leq t_*} V(\theta) + d \\ &\overset{(A.6)}{\leq} -2\delta_0 y_n(t_*) + 2\delta_1 \sup_{a\leq\theta\leq t_*} y_n(\theta) + d \overset{(A.5)}{<} \dot{y}_n(t_*).\end{aligned} \tag{A.8}$$

On the other hand, since $\tau_k \searrow t_*$, by taking $k \to \infty$ in (A.7) we have

$$D^+V(t_*) \overset{(2.4)}{\geq} \lim_{k\to\infty} \frac{y_n(\tau_k) - y_n(t_*)}{\tau_k - t_*} = \dot{y}_n(t_*). \quad \text{(A.9)}$$

From (A.8) and (A.9) we obtain a contradiction.

Proof of Corollary 2.1.
Let $\mathcal{V}(t) = \sup_{a\leq\theta\leq t} V(\theta)$. Using right continuity of $V(t)$ on $[a, b)$, it can be easily verified that $\mathcal{V}(t)$ is right continuous on $[a, b)$. We show that $V(t)$ satisfies (2.3). Fix $t \geq t_0 = a$. By the assumptions on $V(t)$ there exists some $\epsilon > 0$ such that $V(t)$ is absolutely continuous on $[t, t+\epsilon)$. Let $0 < s < \epsilon$. From (2.6) we have

$$\begin{aligned} &\tfrac{V(t+s)-V(t)}{s} = s^{-1}\int_t^{t+s} \dot{V}(\tau)d\tau \\ &\overset{(2.6)}{\leq} -2\delta_0 s^{-1}\int_t^{t+s} V(\tau)d\tau + 2\delta_1 s^{-1}\int_t^{t+s}\mathcal{V}(\tau)d\tau + d \\ &\leq -2\delta_0 V(t) + 2\delta_1\mathcal{V}(t) - 2\delta_0 s^{-1}\int_t^{t+s}[V(\tau)-V(t)]\,d\tau \\ &\quad +2\delta_1 s^{-1}\int_t^{t+s}[\mathcal{V}(\tau)-\mathcal{V}(t)]\,d\tau + d. \end{aligned} \quad \text{(A.10)}$$

Since $V(t)$ and $\mathcal{V}(t)$ are right continuous on $[a, b)$, we have

$$\begin{aligned} &\left| s^{-1}\int_t^{t+s}[V(\tau)-V(t)]\,d\tau \right| \\ &\qquad \leq \sup_{t\leq\tau\leq t+s}|V(\tau)-V(t)| \overset{s\to0^+}{\longrightarrow} 0, \\ &\left| s^{-1}\int_t^{t+s}[\mathcal{V}(\tau)-\mathcal{V}(t)]\,d\tau \right| \\ &\qquad \leq \sup_{t\leq\tau\leq t+s}|\mathcal{V}(\tau)-\mathcal{V}(t)| \overset{s\to0^+}{\longrightarrow} 0. \end{aligned} \quad \text{(A.11)}$$

By taking $\limsup_{s\to0^+}$ in (A.10) and using (A.11) we have

$$D^+V(t) \overset{(2.4)}{\leq} -2\delta_0 V(t) + 2\delta_1\mathcal{V}(t) + d, \quad t\in[a,b).$$

Proof of Proposition 2.1.
We prove step-by-step on $[s_k, s_{k+1})$, $k \in \mathbb{Z}_+$. For $k = 0$, Corollary 2.1 and $\delta_\tau < \delta_0 - \delta_1$ imply

$$V(t) \leq e^{-2\delta_\tau(t-s_0)}V(s_0) + d\int_{s_0}^t e^{-2\delta_\tau(t-s)}ds, \quad t\in[s_0,s_1). \quad \text{(A.12)}$$

Next, consider $k = 1$. From Corollary 2.1 and (2.9) with $j = 0$ (i.e. $t_0^{(1)} = s_1$) we have

$$\begin{aligned} &V(t) \leq e^{-2\delta_\tau(t-s_1)}V(s_1) + d\int_{s_1}^t e^{-2\delta_\tau(t-s)}ds \\ &\overset{(2.9)}{\leq} e^{-2\delta_\tau(t-s_1)}V(s_1^-) + d\int_{s_1}^t e^{-2\delta_\tau(t-s)}ds \\ &\overset{(A.12)}{\leq} e^{-2\delta_\tau(t-s_0)}V(s_0) + d\int_{s_0}^t e^{-2\delta_\tau(t-s)}ds, \quad t\in[s_1,s_2). \end{aligned} \quad \text{(A.13)}$$

Continuing step-by-step for $k = 2, 3, \ldots$ we are done.

A.1. Proof of Lemma 2.2

Let $\mathbb{1}_{[t_0,\infty)}(t)$ be the indicator function of $[t_0, \infty)$ and

$$\begin{aligned} y(t) &= \kappa e^{-2\delta_\tau(t-t_0)} + d\cdot\mathbb{1}_{[t_0,\infty)}(t)\int_{t_0}^t e^{-2\delta(t-s)}ds, \\ &\kappa = \sup_{-h\leq\theta\leq0} V(t_0+\theta), \quad t \geq t_0 - h. \end{aligned} \quad \text{(A.14)}$$

Note that $y(t) \geq V(t)$, $t \in [t_0 - h, t_0]$ and

$$\dot{y}(t) \geq -2\delta_0 y(t) + 2\delta_1 \sup_{-h\leq\theta\leq0} y(t+\theta) + d, \quad t \geq t_0 \quad \text{(A.15)}$$

where $\dot{y}(t_0)$ is the right derivative at t_0. Let $\epsilon_n \searrow 0$. By arguments of Lemma 2.1 we can obtain the comparison $V(t) \leq y_n(t) = y(t) + \frac{\epsilon_n}{2\delta_2}$ for all $n = 1, 2, \ldots$. The latter inequality finishes the proof.

A.2. Proof of Corollary 2.2

Let $\mathcal{V}(t) = \sup_{-h\leq\theta\leq0} V(t+\theta)$, $t \in [t_0, \infty)$. By continuity of V on $[t_0 - h, \infty)$, it can be easily verified that $\mathcal{V}$ is continuous on $[t_0, \infty)$. Fixing $t \geq t_0$, $s > 0$ and using absolute continuity of $V(t)$ on $[t_0, \infty)$ we see that (A.10) holds. Furthermore, (A.11) holds by continuity of $V(t)$ and $\mathcal{V}(t)$ on $[t_0, \infty)$. Taking $\limsup_{s\to0^+}$ in (A.10) we obtain (2.12) for $t \geq t_0$, which implies (2.13) by Lemma 2.2.

References

Ahmed-Ali, T., Karafyllis, I., & Giri, F. (2021). Sampled-data observers for delay systems and hyperbolic PDE–ODE loops. *Automatica*, *123*, Article 109349.

Balas, M. J. (1988). Finite-dimensional controllers for linear distributed parameter systems: exponential stability using residual mode filters. *Journal of Mathematical Analysis and Applications*, *133*(2), 283–296.

Bar Am, N., & Fridman, E. (2014). Network-based H_∞ filtering of parabolic systems. *Automatica*, *50*, 3139–3146.

Bekiaris-Liberis, N. (2020). Hybrid boundary stabilization of linear first-order hyperbolic PDEs despite almost quantized measurements and control input. *Systems & Control Letters*, *146*, Article 104809.

Christofides, P. (2001). *Nonlinear and robust control of pde systems: methods and applications to transport reaction processes*. Springer.

Curtain, R. (1982). Finite-dimensional compensator design for parabolic distributed systems with point sensors and boundary input. *IEEE Transactions on Automatic Control*, *27*(1), 98–104.

Curtain, R., & Zwart, H. (1995). *An introduction to infinite-dimensional linear systems theory, vol. 21*. Springer.

Espitia, N. (2020). Observer-based event-triggered boundary control of a linear 2× 2 hyperbolic systems. *Systems & Control Letters*, *138*, Article 104668.

Espitia, N., Karafyllis, I., & Krstic, M. (2021). Event-triggered boundary control of constant-parameter reaction–diffusion PDEs: a small-gain approach. *Automatica*, *128*, Article 109562.

Fridman, E. (2014). *Introduction to time-delay systems: analysis and control*. Birkhauser, Systems and Control: Foundations and Applications.

Fridman, E., & Blighovsky, A. (2012). Robust sampled-data control of a class of semilinear parabolic systems. *Automatica*, *48*, 826–836.

Fridman, E., & Dambrine, M. (2009). Control under quantization, saturation and delay: an LMI approach. *Automatica*, *45*, 2258–2264.

Harkort, C., & Deutscher, J. (2011). Finite-dimensional observer-based control of linear distributed parameter systems using cascaded output observers. *International Journal of Control*, *84*(1), 107–122.

Hien, L., Phat, V., & Trinh, H. (2015). New generalized Halanay inequalities with applications to stability of nonlinear non-autonomous time-delay systems. *Nonlinear Dynamics*, *82*(1–2), 563–575.

Ishii, H., & Francis, B. A. (2003). Quadratic stabilization of sampled-data systems with quantization. *Automatica*, *39*(10), 1793–1800.

Jacob, B., Mironchenko, A., Partington, J. R., & Wirth, F. (2019). Non-coercive Lyapunov functions for input-to-state stability of infinite-dimensional systems. arXiv preprint arXiv:1911.01327.

Kang, W., & Fridman, E. (2018). Distributed sampled-data control of Kuramoto-Sivashinsky equation. *Automatica*, *95*, 514–524.

Karafyllis, I. (2021). Lyapunov-based boundary feedback design for parabolic PDEs. *International Journal of Control*, *94*(5), 1247–1260.

Karafyllis, I., & Krstic, M. (2016). ISS With respect to boundary disturbances for 1-D parabolic PDEs. *IEEE Transactions on Automatic Control*, *61*(12), 1–23.

Karafyllis, I., & Krstic, M. (2017). Sampled-data boundary feedback control of 1-D linear transport PDEs with non-local terms. *Systems & Control Letters*, *107*, 68–75.

Karafyllis, I., & Krstic, M. (2018). Sampled-data boundary feedback control of 1-D parabolic PDEs. *Automatica*, *87*, 226–237.

Katz, R., Basre, I., & Fridman, E. (2021). Delayed finite-dimensional observer-based control of 1D heat equation under Neumann actuation. In *2021 European control conference*.

Katz, R., & Fridman, E. (2020a). Constructive method for finite-dimensional observer-based control of 1-D parabolic PDEs. *Automatica*, *122*, Article 109285.

Katz, R., & Fridman, E. (2020b). Finite-dimensional control of the Kuramoto-Sivashinsky equation under point measurement and actuation. In *59th IEEE conference on decision and control*.

Katz, R., & Fridman, E. (2021a). Finite-dimensional boundary control of the linear Kuramoto-Sivashinsky equation under point measurement with guaranteed L^2-gain. *IEEE Transactions on Automatic Control*, (submitted for publication).

Katz, R., & Fridman, E. (2021b). Delayed finite-dimensional observer-based control of 1-D parabolic PDEs. *Automatica*, *123*, Article 109364.

Katz, R., & Fridman, E. (2021c). Finite-dimensional control of the heat equation: Dirichlet actuation and point measurement. *European Journal of Control*.

Katz, R., Fridman, E., & Selivanov, A. (2021). Boundary delayed observer-controller design for reaction-diffusion systems. *IEEE Transactions on Automatic Control*, *66*(1), 275–282.
Lhachemi, H., Shorten, R., & Prieur, C. (2020). Exponential input-to-state stabilization of a class of diagonal boundary control systems with delay boundary control. *Systems & Control Letters*, *138*, Article 104651.
Liberzon, D. (2003). Hybrid feedback stabilization of systems with quantized signals. *Automatica*, *39*(9), 1543–1554.
Liu, K., & Fridman, E. (2012). Wirtinger's inequality and Lyapunov-based sampled-data stabilization. *Automatica*, *48*, 102–108.
Mazenc, F., Malisoff, M., & Krstic, M. (2021). Stability analysis for time-varying systems with asynchronous sampling using contractivity approach. *IEEE Control Systems Letters*.
Mirkin, L. (2016). Intermittent redesign of analog controllers via the youla parameter. *62*(4), 1838–1851.
Mironchenko, A., & Prieur, C. (2020). Input-to-state stability of infinite-dimensional systems: recent results and open questions. *SIAM Review*, *62*(3), 529–614.
Pepe, P., & Fridman, E. (2017). On global exponential stability preservation under sampling for globally Lipschitz time-delay systems. *Automatica*, *82*, 295–300.
Prieur, C., & Trélat, E. (2018). Feedback stabilization of a 1-D linear reaction–diffusion equation with delay boundary control. *IEEE Transactions on Automatic Control*, *64*(4), 1415–1425.
Selivanov, A., & Fridman, E. (2016a). Distributed event-triggered control of diffusion semilinear PDEs. *Automatica*, *68*, 344–351.
Selivanov, A., & Fridman, E. (2016b). Observer-based input-to-state stabilization of networked control systems with large uncertain delays. *Automatica*, *74*, 63–70.
Silm, H., Ushirobira, R., Efimov, D., Fridman, E., Richard, J.-P., & Michiels, W. (2021). Distributed observers with time-varying delays. *IEEE Transactions on Automatic Control*.
Wen, L., Yu, Y., & Wang, W. (2008). Generalized Halanay inequalities for dissipativity of Volterra functional differential equations. *Journal of Mathematical Analysis and Applications*, *347*(1), 169–178.
Zhu, Y., & Fridman, E. (2020). Observer-based decentralized predictor control for large-scale interconnected systems with large delays. *IEEE Transactions on Automatic Control*.

Rami Katz received a B.Sc. degree (summa cum laude) in 2014 and a M.Sc. degree (summa cum laude) in 2016, both in Mathematics, from Tel-Aviv University, Israel. Currently, he is a Ph.D. student at the School of Electrical Engineering, Tel-Aviv University, Israel. His research interests include robust control of time-delay and distributed parameter systems. Rami Katz was a finalist of the Best Student Paper Award at ECC 2021 for the paper "Finite-dimensional control of the heat equation: Dirichlet actuation and point measurement".

Emilia Fridman received the M.Sc. degree from Kuibyshev State University, USSR, in 1981 and the Ph.D. degree from Voronezh State University, USSR, in 1986, all in mathematics. From 1986 to 1992 she was an Assistant and Associate Professor in the Department of Mathematics at Kuibyshev Institute of Railway Engineers, USSR. Since 1993 she has been at Tel Aviv University, where she is currently Professor of Electrical Engineering-Systems. She has held visiting positions at the Weierstrass Institute for Applied Analysis and Stochastic in Berlin (Germany), INRIA in Rocquencourt (France), Ecole Centrale de Lille (France), Valenciennes University (France), Leicester University (UK), Kent University (UK), CINVESTAV (Mexico), Zhejiang University (China), St. Petersburg IPM (Russia), Melbourne University (Australia), Supelec (France), KTH (Sweden).

Her research interests include time-delay systems, networked control systems, distributed parameter systems, robust control, singular perturbations and nonlinear control. She has published two monographs and about 200 articles in international scientific journals. She serves/served as Associate Editor in Automatica, SIAM Journal on Control and Optimization and IMA Journal of Mathematical Control and Information. In 2014 she was Nominated as a Highly Cited Researcher by Thomson ISI. Since 2018, she has been the incumbent for Chana and Heinrich Manderman Chair on System Control at Tel Aviv University. She is IEEE Fellow since 2019. In 2021 she was recipient of IFAC Delay Systems Life Time Achievement Award and of Kadar Award for outstanding research at Tel Aviv University. She is currently a member of the IFAC Council.