Periodic Averaging of Discrete-Time Systems: A Time-Delay Approach

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***Abstract*—This article is concerned with the stability of discrete-time systems with fast-varying coefficients that may be uncertain. Recently, a constructive time-delay approach to averaging was proposed for continuous-time systems. In the present article, we develop, for the first time, this approach to discrete-time case. We first transform the system to a time-delay system with the delay being the period of averaging, which can be regarded as a perturbation of the classical averaged system. The stability of the original system can be guaranteed by the resulting time-delay system. Then under assumption of the classical averaged system being exponentially stable, we derive sufficient stability conditions for the resulting time-delay system, and find a quantitative upper bound on the small parameter that ensures the exponential stability. Moreover, we extend our method to input-to-state stability (ISS) analysis of the perturbed systems. Finally, we apply the approach to the practical stability of discrete-time switched affine systems, where an explicit ultimate bound in terms of the switching period is presented. Two numerical examples are given to illustrate the efficiency of results.**

***Index Terms*—Averaging, Time-delay systems, Discrete-time systems, Switched affine systems, ISS.**

I. Introduction

It is well known that time-varying systems arise in many control systems including rotor-blade system, satellite attitude and hypersonic vehicle flight control systems [2], [4], [17]. Compared to time-invariant systems, the stability analysis for time-varying systems is more challenging since the functions describing the dynamic involve the time as an argument. The research on time-varying systems has received much attention in the control community [9], [15], [18], [22]. One of the effective methods for the stability analysis of such systems is the averaging method, which uses a simpler (averaged) system to approximate the original system [12]. As an asymptotic method, the averaging method has been successfully applied in several fields. For instance, in [3] the averaging of discrete-time systems was used for the adaptive identification. In [5], [13], the averaging method was applied for the stability of extremum seeking systems and in [14] of power electronic systems. However, the classical averaging method cannot provide quantitative upper bounds on the small parameter that ensures the stability.

Recently a new constructive time-delay approach to the continuous-time averaging was presented in [7]. By transforming the original system into a model with time-delays of the length of the small parameter and using the Lyapunov-Krasovskii (L-K) approach, this approach allows, for the first time, to derive efficient linear matrix inequality (LMI)-based conditions for finding the upper bound of the small parameter that ensures the stability. Later on, the time-delay approach to averaging was successfully extended to L_2-gain analysis [20] and applied for the quantitative stability analysis of continuous-time ES algorithms (see [23]) and sampled-data ES algorithms (see [24]).

In this article, we introduce the time-delay approach to averaging of discrete-time systems. Although some arguments (not all) are similar to the continuous-time averaging in [7], it is important to present the discrete-time results where derivation of the time-delay model is not straightforward, whereas appropriate Lyapunov functionals have a novel triple sum terms. Note that results for discrete-time systems are not as readily available as their continuous counterparts [3], [21]. Moreover, the stability analysis of discrete-time switched affine systems is much more involved than the continuous-time case, since the desired equilibrium point cannot be reached, but only a neighborhood of the equilibrium is available [8].

In this article, we first transform the original discrete-time system to a time-delay system. The stability of the original system can be guaranteed by the resulting time-delay system. The latter has the form of the classical averaged system subject to a perturbation. Then by constructing an appropriate Lyapunov functional, we derive the explicit conditions in terms of LMIs to guarantee the exponential stability of the resulting time-delay system (and thus of the original system), and moreover, we also find a quantitative upper bound on the small parameter that ensures the exponential stability. Finally, we successfully extend our method to ISS analysis of the perturbed systems and practical stability analysis of switched affine systems. As already mentioned, the existing results on averaging for discrete-time systems (see, for example, [3], [5]) are qualitative, i.e., the system is stable for small ε if the averaged system is stable, and the only choice of the parameter ε can be done till now by simulations. However, our new time-delay approach to averaging established for discrete-time systems gives the first efficient quantitative bounds on ε making averaging-based control reliable. A conference version of the paper confined to consideration of switched affine systems was submitted to CDC 2022 [19].

Notation: The notation used in this article is fairly standard. For two integers p and q with $p \le q$, the notation $\mathbf{I}[p,q]$ refers to the set $\{p, p+1, \ldots, q\}$. Z_+ is the set of nonnegative integers. The notation $P > 0$ for $P \in \mathbf{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix are denoted by $*$. The notations $|\cdot|$ and $\|\cdot\|$ refer to the usual Euclidean vector norm and the induced matrix 2 norm, respectively. At last, for any integers a and b with $b \ge a$, we let $|w|_{[a,b]} \triangleq \max_{s\in\mathbf{I}[a,b]} |w(s)|$.

We will employ the following extended Jensen's inequality, which is an extension of Lemma 2 in [16]. The proof is similar and thus omitted here.

Lemma 1: Denote

$$\mathcal{G} = \sum_{i=k_1}^{k_2} \sum_{j=i}^{k_2} A(i)x(j),\ \mathcal{Y} = \sum_{i=k_1}^{k_2} \sum_{j=i}^{k_2} x^{\mathrm{T}}(j)A^{\mathrm{T}}(i)QA(i)x(j),$$

where $x(i) : \mathbf{I}[k_1, k_2] \to \mathbf{R}^n$, $k_2 \ge k_1$, are a series of vectors and $A(i) : \mathbf{I}[k_1, k_2] \to \mathbf{R}^{n\times n}$ are a series of matrices. Then for any $n \times n$ matrix $Q > 0$ the following inequalities hold:

$$\mathcal{G}^{\mathrm{T}} Q \mathcal{G} \le \frac{(k_2 - k_1 + 1)(k_2 - k_1 + 2)}{2}\mathcal{Y}. \tag{1}$$

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II. A time-delay approach to stability by averaging

Consider the following discrete-time system with uncertainty:

$$x(k+1) = (I_n + \varepsilon [A(k) + \triangle A(k)])\, x(k), \quad k \in Z_+, \tag{2}$$

where $x(k) \in \mathbf{R}^n$, $A(k) : Z_+ \to \mathbf{R}^{n\times n}$ represents the nominal matrix, $\varepsilon > 0$ is a small parameter, and the time-varying uncertain matrix $\triangle A(k) \in \mathbf{R}^{n\times n}$ satisfies the following inequality

$$\|\triangle A(k)\| \le \kappa, \quad k \in Z_+. \tag{3}$$

Here $\kappa > 0$ is a small enough constant. System (2) can be regarded as the linearized version with uncertainty of difference equation (2.2.1) in [3]. But different from the classical averaging theory in [3] that provides a qualitative stability analysis, in the following we aim to develop a new time-delay approach to periodic averaging, which allows us to derive the explicit conditions and a quantitative upper bound on ε that ensure the stability.

Remark 1: System (2) can be further regarded as the discretization of the linear continuous-time system with fast-varying coefficients (see (2.1) with uncertainty in [7]): $\dot{x}(t) = [\bar{A}(\frac{t}{\varepsilon}) + \triangle\bar{A}(\frac{t}{\varepsilon})]x(t)$, $t \ge 0$. Setting the fast time $\tau = \frac{t}{\varepsilon}$ and denoting $\bar{x}(\tau) = x(t)$, we have $\dot{\bar{x}}(\tau) = \varepsilon[\bar{A}(\tau) + \triangle\bar{A}(\tau)]\bar{x}(\tau)$, $\tau \ge 0$, whose Euler discretization with a small sampling period h is given by $\bar{x}((k+1)h) = \bar{x}(kh) + \varepsilon h[\bar{A}(kh) + \triangle\bar{A}(kh)]\bar{x}(kh)$. The latter has the form of (2) by setting $x(k) = \bar{x}(kh)$, $A(k) = h\bar{A}(kh)$ and $\triangle A(k) = h\triangle\bar{A}(kh)$. In addition, we point out that when the continuous-time system is subject to time-varying parametric uncertainties being unknown, the Euler discretization method is intuitive and efficient.

Similar to the continuous case in [7], we first assume:

A1 There exists a positive integer T such that

$$\frac{1}{T}\sum_{i=k-T+1}^{k} A(i) = A_{\rm av},\ k \ge T-1,\ A_{\rm av} \text{ is Hurwitz.}$$

A2 All entries $a_{pq}(k)$ of $A(k)$ are uniformly bounded for $k \in Z_+$ with the values from some finite intervals $a_{pq}(k) \in [a_{pq}^m, a_{pq}^M]$ for $k \ge T-1$.

Under **A1**, the averaged system of (2) with $\triangle A(k) = 0$ has the form

$$x_{\rm av}(k+1) = (I_n + \varepsilon A_{\rm av})\, x_{\rm av}(k), \quad x_{\rm av} \in \mathbf{R}^n,\ k \in Z_+, \tag{4}$$

which is exponentially stable for small enough ε.

Under **A2**, $A(k)$ can be presented as a convex combination (see e.g., Section 3.3.3 in [6]) of the constant matrices A_i with the entries a_{pq}^m or a_{pq}^M:

$$A(k) = \sum_{i=1}^{N} f_i(k)A_i,\ k \ge T-1,\ f_i(k) \ge 0,\ \sum_{i=1}^{N} f_i(k) = 1, \tag{5}$$

where $1 \le N \le 2^{n^2}$.

Inspired by [7], we apply the time-delay method to averaging of system (2). Denote

$$\bar{x}(j) = x(j+1) - x(j),\ G(k) = \frac{\varepsilon}{T}\sum_{i=k-T+1}^{k-1}\sum_{j=i}^{k-1} A(j)x(j). \tag{6}$$

Summing in k and dividing by T on both sides of (2), we have

$$\begin{aligned}\frac{1}{T}\sum_{i=k-T+1}^{k} x(i+1) =& \frac{1}{T}\sum_{i=k-T+1}^{k} x(i) + \frac{\varepsilon}{T}\sum_{i=k-T+1}^{k} A(i)x(i)\\ &+ \frac{\varepsilon}{T}\sum_{i=k-T+1}^{k} \triangle A(i)x(i).\end{aligned} \tag{7}$$

Note from (2) and (6) that

$$\begin{aligned}&\frac{1}{T}\sum_{i=k-T+1}^{k} x(i+1)\\ =& x(k+1) - \frac{1}{T}\sum_{i=k-T+2}^{k} [x(k+1) - x(i)]\\ =& x(k+1) - \frac{1}{T}\sum_{i=k-T+2}^{k}\sum_{j=i}^{k} \bar{x}(j)\\ \overset{(2)}{=}& x(k+1) - G(k+1) - \frac{\varepsilon}{T}\sum_{i=k-T+2}^{k}\sum_{j=i}^{k} \triangle A(j)x(j).\end{aligned} \tag{8}$$

Similarly,

$$\frac{1}{T}\sum_{i=k-T+1}^{k} x(i) = x(k) - G(k) - \frac{\varepsilon}{T}\sum_{i=k-T+1}^{k-1}\sum_{j=i}^{k-1} \triangle A(j)x(j). \tag{9}$$

Combining (8) and (9) gives

$$\begin{aligned}&\frac{1}{T}\sum_{i=k-T+1}^{k} [x(i+1) - x(i)]\\ =& [x(k+1) - G(k+1)] - [x(k) - G(k)]\\ &+ \frac{\varepsilon}{T}\sum_{i=k-T+1}^{k} \triangle A(i)x(i) - \varepsilon\triangle A(k)x(k).\end{aligned} \tag{10}$$

On the other hand, under **A1** we have

$$\begin{aligned}&\frac{\varepsilon}{T}\sum_{i=k-T+1}^{k} A(i)x(i)\\ =& \frac{\varepsilon}{T}\sum_{i=k-T+1}^{k} A(i)\,[x(i) - x(k) + x(k)]\\ =& \varepsilon A_{\rm av}x(k) - \frac{\varepsilon}{T}\sum_{i=k-T+1}^{k-1}\sum_{j=i}^{k-1} A(i)\bar{x}(j).\end{aligned} \tag{11}$$

Finally, employing (10), (11) and setting

$$z(k) = x(k) - G(k), Y(k) = \frac{1}{T}\sum_{i=k-T+1}^{k-1}\sum_{j=i}^{k-1} A(i)\bar{x}(j), \tag{12}$$

system (7) can be transformed to

$$z(k+1) = z(k) + \varepsilon\,[A_{\rm av} + \triangle A(k)]\, x(k) - \varepsilon Y(k) \tag{13}$$

with $k \ge T-1$. Here we follow the model in [20], where system (13) has only a single $Y(k)$-term to be compensated in the L-K analysis, which can significantly simplify the LMIs and improve the results in the examples.

System (13) is a discrete-time version of the neutral type time-delay system derived in [20], where $z(k+1) - z(k)$ depends on $x(k-i)$, $i \in \mathbf{I}[0, T-1]$. Clearly, the solution $x(k)$ of system (2) is also a solution of system (13). Therefore, the stability of the original non-delayed system (2) can be guaranteed by the stability of the time-delay system (13). In view of the definition $x(k) = z(k) + G(k)$, we can reorganize (13) as

$$\begin{aligned}z(k+1) =& [I_n + \varepsilon\,(A_{\rm av} + \triangle A(k))]\, z(k) - \varepsilon Y(k)\\ &+ \varepsilon\,(A_{\rm av} + \triangle A(k))\, G(k),\ k \ge T-1.\end{aligned} \tag{14}$$

Comparatively to the averaged system (4), system (14) with $\triangle A(k) = 0$ has the additional terms $G(k)$ and $Y(k)$ that are both of the order of $\mathrm{O}(\varepsilon)$ provided $x(k)$ (and thus $z(k)$) is of the order of $\mathrm{O}(1)$. Thus, for small $\varepsilon > 0$ system (14) with $\triangle A(k) = 0$ can be regarded as a perturbation of system (4). In the following, we will derive sufficient stability conditions for (13) (and thus (2)) and also find a quantitative upper bound on the small parameter that ensures the exponential stability. To this end, for some $n\times n$ matrices $P > 0$, $Q > 0$, $R > 0$ and a scalar $\rho \in (0, 1)$, we consider the following L-K functional

$$V(k) = V_P(k) + V_Q(k) + V_R(k),\ k \ge T-1, \tag{15}$$

where

$$V_P(k) = z^{\rm T}(k)Pz(k),$$

$$
\begin{aligned}
V_Q(k) &= \bar{T}\varepsilon^2 \sum_{i=-T+1}^{-1} \sum_{j=i}^{-1} \sum_{s=k+j}^{k-1} \rho^{k-s-1}\phi_1(s),\\
V_R(k) &= \bar{T} \sum_{i=-T+1}^{-1} \sum_{j=i}^{-1} \sum_{s=k+j}^{k-1} \rho^{k-s-1}\phi_2(i,j,s)
\end{aligned}
$$

with $\bar{T} = \frac{T-1}{2T}$ and

$$
\phi_1(s) = |A(s)x(s)|_Q^2\,,\ \phi_2(i,j,s) = |A(s+i-j)\bar{x}(s)|_R^2\,. \quad (16)
$$

The terms $V_Q(k)$ and $V_R(k)$ will compensate the $G(k)$-term and $Y(k)$-term, respectively, in Lyapunov analysis. By using the extended Jensen's inequality (1) and noting the form of $G(k)$ in (6) we have

$$
V_Q(k) \geq \rho^{T-2}\bar{T}\varepsilon^2 \sum_{j=k-T+1}^{k-1} \sum_{s=j}^{k-1} \phi_1(s) \geq \rho^{T-2} G^{\mathrm{T}}(k)QG(k).
$$

It follows that

$$
\begin{aligned}
V(k) &\geq V_P(k) + V_Q(k)\\
&\geq [x(k)-G(k)]^{\mathrm{T}} P\,[x(k)-G(k)] + \rho^{T-2}G^{\mathrm{T}}(k)QG(k)\\
&= \begin{bmatrix} x^{\mathrm{T}}(k) & G^{\mathrm{T}}(k) \end{bmatrix} \begin{bmatrix} P & -P \\ -P & P+\rho^{T-2}Q \end{bmatrix} \begin{bmatrix} x(k) \\ G(k) \end{bmatrix}\\
&\geq \alpha\,|x(k)|^2 \qquad (17)
\end{aligned}
$$

with some $\alpha > 0$.

Theorem 1: Let **A1** and **A2** be satisfied. Given matrices A_{av}, $A_i (i \in \mathbf{I}[1,N])$ and constants $\kappa > 0$, $\theta > 0$ and $\varepsilon^* > 0$ subject to $\theta\varepsilon^* < 1$, let one of the following conditions holds:

(i) there exist $n \times n$ matrices $P > 0$, $Q > 0$, $R > 0$, $\bar{R} > 0$ and a scalar $\zeta > 0$ that satisfy the following LMIs

$$
\Phi_1^i = \begin{bmatrix} -\bar{R} & A_i^{\mathrm{T}} R \\ * & -R \end{bmatrix} < 0,\ i \in \mathbf{I}[1,N], \quad (18)
$$

$$
\Phi_2^i = \left[\begin{array}{c|cc} \Phi & \frac{T-1}{2}A_i^{\mathrm{T}}Q & \frac{T-1}{2}A_i^{\mathrm{T}}\bar{R} \\ & 0_{2n\times n} & 0_{2n\times n} \\ & 0_{n\times n} & \frac{T-1}{2}\bar{R} \\ \hline * & -\frac{1}{\varepsilon^*}Q & 0_{n\times n} \\ * & * & -\frac{1}{\varepsilon^*}\bar{R} \end{array}\right] < 0,\ i \in \mathbf{I}[1,N], \quad (19)
$$

in which

$$
\begin{aligned}
\Phi &= \begin{bmatrix} \Phi_{11} & -A_{\mathrm{av}}^{\mathrm{T}}P - \theta P & -P - \varepsilon^* A_{\mathrm{av}}^{\mathrm{T}}P & P + \varepsilon^* A_{\mathrm{av}}^{\mathrm{T}}P \\ * & \Phi_{22} & P & -P \\ * & * & \Phi_{33} & -\varepsilon^* P \\ * & * & * & -\zeta I_n + \varepsilon^* P \end{bmatrix},\\
\Phi_{11} &= A_{\mathrm{av}}^{\mathrm{T}}P + PA_{\mathrm{av}} + \varepsilon^* A_{\mathrm{av}}^{\mathrm{T}}PA_{\mathrm{av}} + \theta P + \zeta\kappa^2 I_n,\\
\Phi_{22} &= \theta P - \frac{(1-\theta\varepsilon^*)^{T-1}}{\varepsilon^*}Q,\\
\Phi_{33} &= \varepsilon^* P - \frac{(1-\theta\varepsilon^*)^{T-1}}{\varepsilon^*}R;
\end{aligned} \quad (20)
$$

(ii) there exist $n \times n$ matrices $P > 0$, $Q > 0$ and positive scalars r, ζ that satisfy (19) with $R = rI_n$, $\bar{R} = r\Pi_1$ and

$$
\Pi_1 = \frac{2}{T}\sum_{i=0}^{T-1} A^{\mathrm{T}}(i)A(i). \quad (21)
$$

Then system (2) is exponentially stable with a decay rate $\sqrt{1-\theta\varepsilon}$ for all $\varepsilon \in (0,\varepsilon^*]$, namely, there exists a $M > 0$ such that for all $\varepsilon \in (0,\varepsilon^*]$, the solution of (2) initialized by $x(0) \in \mathbf{R}^n$ satisfies for all $k \in Z_+$ the following inequality:

$$
|x(k)|^2 \leq M(1-\theta\varepsilon)^k\,|x(0)|^2\,. \quad (22)
$$

Moreover, if the above LMIs hold with $\theta = 0$, then system (2) is exponentially stable with a decay rate $\sqrt{1-\theta_0\varepsilon}$ with θ_0 being small enough for all $\varepsilon \in (0,\varepsilon^*]$.

Proof: We first verify the case under condition **(i)**. Choose the L-K functional $V(k)$ as shown in (15). The time-shift of $V_P(k)$ along system (13) can be evaluated as

$$
\begin{aligned}
&V_P(k+1) - \rho V_P(k)\\
&= 2\varepsilon\,[x(k)-G(k)]^{\mathrm{T}} P\,[(A_{\mathrm{av}} + \Delta A(k))\,x(k) - Y(k)]\\
&\quad + \varepsilon^2\,[(A_{\mathrm{av}} + \Delta A(k))\,x(k) - Y(k)]^{\mathrm{T}} P\\
&\quad \times [(A_{\mathrm{av}} + \Delta A(k))\,x(k) - Y(k)]\\
&\quad + (1-\rho)\,[x(k)-G(k)]^{\mathrm{T}} P\,[x(k)-G(k)]\,, \qquad (23)
\end{aligned}
$$

where $G(k)$ and $Y(k)$ are given by (6) and (12), respectively. For the term $V_Q(k)$, we have

$$
\begin{aligned}
&V_Q(k+1) - \rho V_Q(k)\\
&= \bar{T}\varepsilon^2 \sum_{i=-T+1}^{-1} \sum_{j=i}^{-1} \Big[\sum_{s=k+j+1}^{k} \rho^{k-s}\phi_1(s) - \sum_{s=k+j}^{k-1} \rho^{k-s}\phi_1(s)\Big]\\
&= \frac{(T-1)^2\varepsilon^2}{4}\phi_1(k) - \bar{T}\varepsilon^2 \sum_{i=k-T+1}^{k-1} \sum_{j=i}^{k-1} \rho^{k-j}\phi_1(j)\\
&\leq \frac{(T-1)^2\varepsilon^2}{4}\phi_1(k) - \bar{T}\rho^{T-1}\varepsilon^2 \sum_{i=k-T+1}^{k-1} \sum_{j=i}^{k-1} \phi_1(j), \qquad (24)
\end{aligned}
$$

where $\bar{T} = \frac{T-1}{2T}$, ϕ_1 satisfies (16) and the last inequality follows from $\rho \in (0,1)$. Moreover, using the inequality (1) gives

$$
G^{\mathrm{T}}(k)QG(k) \leq \bar{T}\varepsilon^2 \sum_{i=k-T+1}^{k-1} \sum_{j=i}^{k-1} \phi_1(j),
$$

by which, inequality (24) can be continued as

$$
\begin{aligned}
&V_Q(k+1) - \rho V_Q(k)\\
&\leq \frac{(T-1)^2\varepsilon^2}{4}\,|A(k)x(k)|_Q^2 - \rho^{T-1}G^{\mathrm{T}}(k)QG(k). \qquad (25)
\end{aligned}
$$

For the term $V_R(k)$, we have

$$
\begin{aligned}
&V_R(k+1) - \rho V_R(k)\\
&= \bar{T} \sum_{i=-T+1}^{-1} \sum_{j=i}^{-1} \big[\phi_2(i,j,k) - \rho^{-j}\phi_2(i,j,k+j)\big]\\
&= \bar{T} \sum_{i=k-T+1}^{k-1} \sum_{j=i}^{k-1} \phi_2(i,j,k) - \bar{T} \sum_{i=k-T+1}^{k-1} \sum_{j=i}^{k-1} \rho^{k-j}\,|A(i)\bar{x}(j)|_R^2\\
&\leq \bar{T} \sum_{i=k-T+1}^{k-1} \sum_{j=i}^{k-1} \phi_2(i,j,k) - \bar{T}\rho^{T-1} \sum_{i=k-T+1}^{k-1} \sum_{j=i}^{k-1} |A(i)\bar{x}(j)|_R^2\,,
\end{aligned} \quad (26)
$$

where $\bar{T} = \frac{T-1}{2T}$, ϕ_2 satisfies (16) and the last inequality follows from $\rho \in (0,1)$. If $\Phi_1^i < 0$ $(i \in \mathbf{I}[1,N])$ in (18), then for all $k \in Z_+$,

$$
\begin{bmatrix} -\bar{R} & \sum_{i=1}^N f_i(k)A_i^{\mathrm{T}}R \\ * & -R \end{bmatrix} \overset{(5)}{=} \begin{bmatrix} -\bar{R} & A^{\mathrm{T}}(k)R \\ * & -R \end{bmatrix} \leq 0. \quad (27)
$$

Applying the Schur complement to (27) we obtain $A^{\mathrm{T}}(k)RA(k) \leq \bar{R}$, $k \in Z_+$, by which and the definition of ϕ_2 in (16), we have

$$
\bar{T} \sum_{i=k-T+1}^{k-1} \sum_{j=i}^{k-1} \phi_2(i,j,k) \leq \frac{(T-1)^2}{4}\bar{x}^{\mathrm{T}}(k)\bar{R}\bar{x}(k). \quad (28)
$$

Moreover, with the extended Jensen's inequality (1), we have

$$
Y^{\mathrm{T}}(k)RY(k) \leq \bar{T} \sum_{i=k-T+1}^{k-1} \sum_{j=i}^{k-1} |A(i)\bar{x}(j)|_R^2\,. \quad (29)
$$

Hence, by using (28) and (29), inequality (26) can be continued as

$$
\begin{aligned}
&V_R(k+1) - \rho V_R(k)\\
&\leq \frac{(T-1)^2}{4}\bar{x}^{\mathrm{T}}(k)\bar{R}\bar{x}(k) - \rho^{T-1}Y^{\mathrm{T}}(k)RY(k). \qquad (30)
\end{aligned}
$$

To compensate $\Delta A(k)x(k)$ in (23) we apply S-procedure: we add to $V(k+1) - \rho V(k)$ the left hand part of

$$
\zeta\varepsilon\left[\kappa^2\,|x(k)|^2 - |\Delta A(k)x(k)|^2\right] \geq 0 \quad (31)
$$

with some $\zeta > 0$. Moreover, the last term in (23) should be of the order of $\mathrm{O}(\varepsilon)$, so we let $\rho = 1 - \theta\varepsilon \in (0, 1)$ with some $\theta > 0$. Then by combining (23), (25), (30) and (31) we find

$$\begin{aligned}
&V(k+1) - (1-\theta\varepsilon)\, V(k) \\
\leq& 2\varepsilon \left[x(k) - G(k)\right]^{\mathrm{T}} P \left[\left(A_{\mathrm{av}} + \Delta A(k)\right) x(k) - Y(k)\right] \\
&+ \varepsilon^* \varepsilon \left[\left(A_{\mathrm{av}} + \Delta A(k)\right) x(k) - Y(k)\right]^{\mathrm{T}} P \\
&\times \left[\left(A_{\mathrm{av}} + \Delta A(k)\right) x(k) - Y(k)\right] \\
&+ \varepsilon\theta \left[x(k) - G(k)\right]^{\mathrm{T}} P \left[x(k) - G\,(k)\right] \\
&- \frac{(1-\theta\varepsilon^*)^{T-1}}{\varepsilon^*} \left[\varepsilon G^{\mathrm{T}}(k) Q G(k) + \varepsilon Y^{\mathrm{T}}(k) R Y(k)\right] \\
&+ \zeta\varepsilon \left[\kappa^2 \left|x(k)\right|^2 - \left|\Delta A(k) x(k)\right|^2\right] \\
&+ \frac{(T-1)^2 \varepsilon^*}{4} \left[\varepsilon \left|A(k)x(k)\right|_Q^2 + \frac{1}{\varepsilon} \left|\bar{x}(k)\right|_{\bar{R}}^2\right] \\
=& \eta^{\mathrm{T}}(k) \Phi \eta(k) + \frac{(T-1)^2 \varepsilon^*}{4} \left[\varepsilon \left|A(k)x(k)\right|_Q^2 + \frac{1}{\varepsilon} \left|\bar{x}(k)\right|_{\bar{R}}^2\right] \quad (32)
\end{aligned}$$

for all $\varepsilon \in (0, \varepsilon^*]$ and $k \geq T-1$, where Φ is given by (20) and

$$\eta^{\mathrm{T}}(k) = \sqrt{\varepsilon}\left[x^{\mathrm{T}}(k), G^{\mathrm{T}}(k), Y^{\mathrm{T}}(k), x^{\mathrm{T}}(k)\Delta A^{\mathrm{T}}(k)\right]. \quad (33)$$

Note from (2) and $\bar{x}(j)$ in (6) that

$$\bar{x}(k) = \varepsilon \left[A(k) + \triangle A(k)\right] x(k), \quad k \in Z_+. \quad (34)$$

Substituting (34) into (32) with $A(k)$ satisfying (5) and applying further Schur complements, we conclude that if

$$\left[\begin{array}{c|cc}
\Phi & \begin{matrix} \frac{T-1}{2}\sum_{i=1}^N f_i(k) A_i^{\mathrm{T}} Q \\ 0_{2n\times n} \\ 0_{n\times n} \end{matrix} & \begin{matrix} \frac{T-1}{2}\sum_{i=1}^N f_i(k) A_i^{\mathrm{T}} \bar{R} \\ 0_{2n\times n} \\ \frac{T-1}{2}\bar{R} \end{matrix} \\
\hline
\begin{matrix} * \\ * \end{matrix} & \begin{matrix} -\frac{1}{\varepsilon^*} Q \\ * \end{matrix} & \begin{matrix} 0_{n\times n} \\ -\frac{1}{\varepsilon^*}\bar{R} \end{matrix}
\end{array}\right] < 0, \quad (35)$$

then for all $\varepsilon \in (0, \varepsilon^*]$ we have

$$V(k+1) - (1-\theta\varepsilon)\, V(k) \leq 0, \;\; k \geq T-1, \quad (36)$$

which with (17) yields, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\alpha \left|x(k)\right|^2 \leq V(k) \leq (1-\theta\varepsilon)^{k-T+1}\, V(T-1), \;\; k \geq T-1. \quad (37)$$

LMI (35) follows from (19) since (35) is affine in $\sum_{i=1}^N f_i(k) A_i^{\mathrm{T}}$.

Under (3) and **A2**, we have

$$\left\|A(k) + \triangle A(k)\right\| \leq \left\|A(k)\right\| + \kappa \leq a, \quad k \in Z_+ \quad (38)$$

for some $a > 0$. For $k \in \mathbf{I}[0, T-1]$, $x(k)$ satisfy (2). Hence, for all $\varepsilon \in (0, \varepsilon^*]$ and $k \in \mathbf{I}[0, T-1]$ we obtain

$$\left|x(k)\right| \leq (1 + a\varepsilon^*)^k \left|x(0)\right|, \;\; \left|\bar{x}(k)\right| \leq a\varepsilon^* (1 + a\varepsilon^*)^k \left|x(0)\right|. \quad (39)$$

Note that for all $\varepsilon \in (0, \varepsilon^*]$, $V(T-1)$ defined by (15) can be upper bounded by

$$V(T-1) \leq c_1 \Big[\left|x(T-1)\right|^2 + \varepsilon^{*2} \sum_{k=0}^{T-2} \left|x(k)\right|^2 + \sum_{k=0}^{T-2} \left|\bar{x}(k)\right|^2 \Big] \quad (40)$$

with ε-independent $c_1 > 0$. Therefore, by employing (39), $V(T-1)$ can be further upper bounded by

$$V(T-1) \leq c_2 (1-\theta\varepsilon)^{T-1} \left|x(0)\right|^2 \quad \forall \varepsilon \in (0, \varepsilon^*] \quad (41)$$

with some ε-independent $c_2 > 0$. Then (22) follows from (37) and (41). Note that the feasibility of the LMIs (19) with $\theta = 0$ implies that the feasibility of (19) with the same decision variables and with a small enough $\theta = \theta_0 > 0$, and thus guarantees exponential stability of system (2) with a decay rate approaching to 1.

Finally, we verify the case under condition **(ii)**. Let $R = rI_n$ with $r > 0$. By the definitions of ϕ_2 in (16) and Π_1 in (21) we have

$$\begin{aligned}
&\bar{T} \sum_{i=k-T+1}^{k-1} \sum_{j=i}^{k-1} \phi_2(i,j,k) \\
\leq& r\bar{T}\bar{x}^{\mathrm{T}}(k) \left[(T-1)\textstyle\sum_{i=k-T+1}^{k} A^{\mathrm{T}}(i) A(i)\right] \bar{x}(k) \\
=& \frac{(T-1)^2}{4} \bar{x}^{\mathrm{T}}(k) r \Pi_1 \bar{x}(k),
\end{aligned}$$

by which, inequality (26) can be continued as

$$\begin{aligned}
&V_R(k+1) - \rho V_R(k) \\
\leq& \frac{(T-1)^2}{4} \bar{x}^{\mathrm{T}}(k) r \Pi_1 \bar{x}(k) - r\rho^{T-1} Y^{\mathrm{T}}(k) Y(k). \quad (42)
\end{aligned}$$

By using (42) instead of (30) in (32), we find that if LMIs (19) with $R = rI_n$ and $\bar{R} = r\Pi_1$ hold for all $\varepsilon \in (0, \varepsilon^*]$, we have (36). The rest proof remains the same. The proof is finished. ■

Remark 2: LMIs (18) and (19) are always feasible for small enough positive ε^*, θ and κ. Note that by the Schur complement, LMIs (18) are reduced to $A_i^{\mathrm{T}} R A_i \leq \bar{R}$ ($i \in \mathbf{I}[1, N]$) that always hold for appropriate $\bar{R}$. Now we check the feasibility of LMIs (19). Since A_{av} is Hurwitz by **A1**, there exists a $n \times n$ matrix $P > 0$ such that for small enough $\theta > 0$, the following inequality holds: $\Theta_0 \triangleq A_{\mathrm{av}}^{\mathrm{T}} P + P A_{\mathrm{av}} + \theta P < 0$. We choose $\zeta = 1/\varepsilon^*$ and $\kappa = \varepsilon^*$. Clearly, for small enough $\theta > 0$ and $\varepsilon^* > 0$, we have $\Phi_{ii} < 0$ ($i \in \mathbf{I}[1, 3]$) in (20). Applying the Schur complement to $\Phi_2^i < 0$ ($i \in \mathbf{I}[1, N]$), we get

$$\begin{aligned}
&\Phi + \varepsilon^* \begin{bmatrix} \frac{T-1}{2} A_i^{\mathrm{T}} Q & \frac{T-1}{2} A_i^{\mathrm{T}} \bar{R} \\ 0_{2n\times n} & 0_{2n\times n} \\ 0_{n\times n} & \frac{T-1}{2}\bar{R} \end{bmatrix} \begin{bmatrix} Q^{-1} & 0_{n\times n} \\ 0_{n\times n} & \bar{R}^{-1} \end{bmatrix} \\
&\times \begin{bmatrix} \frac{T-1}{2} A_i^{\mathrm{T}} Q & \frac{T-1}{2} A_i^{\mathrm{T}} \bar{R} \\ 0_{2n\times n} & 0_{2n\times n} \\ 0_{n\times n} & \frac{T-1}{2}\bar{R} \end{bmatrix}^{\mathrm{T}} \approx \Phi < 0, \; \varepsilon^* \to 0.
\end{aligned}$$

Applying further the Schur complement to $\Phi < 0$, we have

$$\begin{aligned}
&\begin{bmatrix} \Theta_0 + \varepsilon^*\Theta_1 & -A_{\mathrm{av}}^{\mathrm{T}} P - \theta P & -P - \varepsilon^* A_{\mathrm{av}}^{\mathrm{T}} P \\ * & \Phi_{22} & P \\ * & * & \Phi_{33} \end{bmatrix} \\
&+ \varepsilon^* \begin{bmatrix} P + \varepsilon^* A_{\mathrm{av}}^{\mathrm{T}} P \\ -P \\ -\varepsilon^* P \end{bmatrix} \Theta_2^{-1} \begin{bmatrix} P + \varepsilon^* A_{\mathrm{av}}^{\mathrm{T}} P \\ -P \\ -\varepsilon^* P \end{bmatrix}^{\mathrm{T}} \\
\approx& \begin{bmatrix} \Theta_0 + \varepsilon^*\Theta_1 & -A_{\mathrm{av}}^{\mathrm{T}} P - \theta P & -P - \varepsilon^* A_{\mathrm{av}}^{\mathrm{T}} P \\ * & \Phi_{22} & P \\ * & * & \Phi_{33} \end{bmatrix} < 0, \; \varepsilon^* \to 0,
\end{aligned}$$

where $\Theta_1 = A_{\mathrm{av}}^{\mathrm{T}} P A_{\mathrm{av}} + I_n$ and $\Theta_2 = I_n - \varepsilon^{*2} P$. We further apply the Schur complement to the above LMI such that

$$\begin{aligned}
&\begin{bmatrix} \Theta_0 + \varepsilon^*\Theta_1 & -A_{\mathrm{av}}^{\mathrm{T}} P - \theta P \\ * & \Phi_{22} \end{bmatrix} + \varepsilon^* \begin{bmatrix} -P - \varepsilon^* A_{\mathrm{av}}^{\mathrm{T}} P \\ P \end{bmatrix} \Theta_3^{-1} \\
\times& \begin{bmatrix} -P - \varepsilon^* A_{\mathrm{av}}^{\mathrm{T}} P \\ P \end{bmatrix}^{\mathrm{T}} \approx \begin{bmatrix} \Theta_0 + \varepsilon^*\Theta_1 & -A_{\mathrm{av}}^{\mathrm{T}} P - \theta P \\ * & \Phi_{22} \end{bmatrix} < 0, \; \varepsilon^* \to 0,
\end{aligned}$$

where $\Theta_3 = (1-\theta\varepsilon^*)^{T-1} R - \varepsilon^{*2} P$. We finally apply the Schur complement to the above LMI such that

$$\Theta_0 + \varepsilon^*\Theta_1 + \varepsilon^* \left[A_{\mathrm{av}}^{\mathrm{T}} P + \theta P\right] \Theta_4^{-1} \left[A_{\mathrm{av}}^{\mathrm{T}} P + \theta P\right]^{\mathrm{T}} < 0$$

with $\Theta_4 = (1-\theta\varepsilon^*)^{T-1} Q - \varepsilon^*\theta P$, which always holds when $\varepsilon^* \to 0$ as $\Theta_0 < 0$. Finally we point out that, by the similar arguments, the LMIs presented in the following Theorems 2-3 and Corollary 1 are all feasible for small enough positive ε^*, θ and κ (as well as κ_1, κ_2 in Section IV) and large enough b_0 and b in Section IV.

Example 1: Consider the suspended pendulum and assume that the suspension point is subjected to vertical vibrations of small amplitude and high frequency (see Example 10.10 in [12] and Example 2.1 in [7]). The discrete-time version of linearized model

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at the upper equilibrium position ($x_1 = \pi, x_2 = 0$) with a sampling period h can be described as:

$$x(k+1) = x(k) + \varepsilon\left[A(k) + \Delta A(k)\right]x(k), \tag{43}$$

in which

$$A(k) = h\begin{bmatrix} \cos kh & 1 \\ \alpha^2 - \cos^2 kh & -\alpha\beta - \cos kh \end{bmatrix},$$
$$\Delta A(k) = h\begin{bmatrix} 0 & 0 \\ 0 & -\alpha\Delta\beta(kh) \end{bmatrix}$$

with $\alpha, \beta > 0$. Here the uncertainty $\Delta\beta(kh)$ stems from the uncertainties of friction coefficient and satisfies $|\Delta\beta(kh)| \le \beta_1$ with $\beta_1 \ge 0$. For simulation, we choose the sampling period $h = \pi/20$ and $T = 2\pi/h = 40$. Then

$$A_{\rm av} = \frac{\pi}{20}\begin{bmatrix} 0 & 1 \\ \alpha^2 - 0.5 & -\alpha\beta \end{bmatrix},$$
$$\Pi_1 = \frac{\pi^2}{200}\begin{bmatrix} 0.875 + \alpha^4 - \alpha^2 & 0.5\alpha\beta - \alpha^3\beta \\ * & 1.5 + \alpha^2\beta^2 \end{bmatrix}$$

and $\kappa = \pi\alpha\beta_1/20$. When $0 < \alpha < 1/\sqrt{2}$ and $\beta > 0$, $A_{\rm av}$ is Hurwitz. Here we choose $\alpha = 0.2$ and $\beta = 1$. Notice that $\cos kh \in [-1, 1]$ and $\cos^2 kh \in [0, 1]$. Therefore, $A(k)$ can be presented as a convex combination of the constant matrices $A_i (i \in \mathbf{I}[1, 4])$ with

$$A_i = \begin{cases} \frac{\pi}{20}\begin{bmatrix} -1 & 1 \\ -0.46 \pm 0.5 & 0.8 \end{bmatrix}, & i \in \mathbf{I}[1,2], \\ \frac{\pi}{20}\begin{bmatrix} 1 & 1 \\ -0.46 \pm 0.5 & -1.2 \end{bmatrix}, & i \in \mathbf{I}[3,4]. \end{cases}$$

By verifying the feasibility of LMIs (19) with $R = rI_n$ and $\bar{R} = r\Pi_1$, we find the corresponding upper bounds ε^* that guarantee the exponential stability of (43) for all $\varepsilon \in (0, \varepsilon^*]$ with $\theta = 0$ or 0.01 :

$$\begin{array}{lll} \beta_1 = 0, & \theta = 0, & \varepsilon^* = 0.71 \cdot 10^{-2}; \\ & \theta = 0.01, & \varepsilon^* = 0.47 \cdot 10^{-2}; \\ \beta_1 = 0.1, & \theta = 0, & \varepsilon^* = 0.55 \cdot 10^{-2}; \\ & \theta = 0.01, & \varepsilon^* = 0.32 \cdot 10^{-2}. \end{array}$$

Note that in Example 4.1 of [20], when $\alpha = 0$ (corresponding to $\theta = 0$ here), the corresponding results show that

$$\beta_1 = 0, \varepsilon^* = 0.74 \cdot 10^{-2};\ \beta_1 = 0.1, \varepsilon^* = 0.58 \cdot 10^{-2}.$$

By comparing the data, we find that the values obtained in Theorem 1 are close to those for continuous-time case obtained in [20]. In addition, numerical simulations show that system (43) with $\Delta\beta = 0$ is stable for a larger upper bound $\varepsilon^* = 0.47$, which may illustrate essential conservatism of the proposed method.

III. ISS ANALYSIS

In this section, we will extend the stability analysis of the above section to ISS analysis of the perturbed systems. Consider the following discrete-time perturbed system:

$$x(k+1) = x(k) + \varepsilon\left[A(k) + \triangle A(k)\right]x(k) + \varepsilon Bw(k), k \in Z_+ \tag{44}$$

where $x(k) \in \mathbf{R}^n$, $A : Z_+ \to \mathbf{R}^{n\times n}$, $\varepsilon > 0$ is a small parameter, $\triangle A(k) \in \mathbf{R}^{n\times n}$ is the uncertain matrix satisfying (3), $B \in \mathbf{R}^{n\times n_w}$ is a constant matrix, and $w(k) \in \mathbf{R}^{n_w}$ is a disturbance. For tackling the ISS analysis, we also assume that **A1** and **A2** hold. Let $\bar{x}(j)$ and $G(k)$ be defined in (6). Then we can present

$$\begin{aligned} &\frac{1}{T}\sum_{i=k-T+1}^{k}\left[x(i+1) - x(i)\right] \\ =& \left[x(k+1) - G(k+1)\right] - \left[x(k) - G(k)\right] \\ &+ \frac{\varepsilon}{T}\sum_{i=k-T+1}^{k} Bw(i) - \varepsilon Bw(k) - \varepsilon\triangle A(k)x(k). \end{aligned} \tag{45}$$

Summing in k and dividing by T on both sides of system (44), via (11) and (45) we arrive at

$$z(k+1) = z(k) + \varepsilon\left[A_{\rm av} + \Delta A(k)\right]x(k) - \varepsilon Y(k) + \varepsilon Bw(k) \tag{46}$$

with $k \ge T - 1$, where $\{z(k), Y(k)\}$ are given by (12) with $x(k)$ satisfying (44). Note that if the time-delay system (46) is ISS, then the original system (44) is also ISS. Next we present the ISS conditions for system (46) (and thus (44)) in the following theorem.

Theorem 2: Let **A1** and **A2** be satisfied. Given matrices $A_{\rm av}$, $A_i (i \in \mathbf{I}[1, N])$, B and constants $\kappa > 0$, $\theta > 0$ and $\varepsilon^* > 0$ subject to $\theta\varepsilon^* < 1$, let one of the following conditions holds:

(i) there exist $n \times n$ matrices $P > 0$, $Q > 0$, $R > 0$, $\bar{R} > 0$ and positive scalars ζ, b that satisfy (18) and the following LMIs

$$\Phi_2^i = \left[\begin{array}{c|cc} \bar{\Phi} & \begin{array}{c} \frac{T-1}{2}A_i^{\rm T}Q \\ 0_{2n\times n} \\ 0_{n\times n} \\ 0_{n_w\times n} \end{array} & \begin{array}{c} \frac{T-1}{2}A_i^{\rm T}\bar{R} \\ 0_{2n\times n} \\ \frac{T-1}{2}\bar{R} \\ \frac{T-1}{2}B^{\rm T}\bar{R} \end{array} \\ \hline * & -\frac{1}{\varepsilon^*}Q & 0_{n\times n} \\ * & 0_{n\times n} & -\frac{1}{\varepsilon^*}\bar{R} \end{array}\right] < 0,\ i \in \mathbf{I}[1, N], \tag{47}$$

where

$$\bar{\Phi} = \begin{bmatrix} \Phi & \bar{\Phi}_{12} \\ * & \varepsilon^* B^{\rm T}PB - bI_{n_w} \end{bmatrix}, \tag{48}$$
$$\bar{\Phi}_{12} = \begin{bmatrix} I_n + \varepsilon^* A_{\rm av} & -I_n & -\varepsilon^* I_n & \varepsilon^* I_n \end{bmatrix}^{\rm T} PB,$$

in which Φ is given by (20);

(ii) there exist $n \times n$ matrices $P > 0$, $Q > 0$ and positive scalars r, ζ, b that satisfy (47) with $R = rI_n$, $\bar{R} = r\Pi_1$ and Π_1 satisfying (21).

Then system (44) is ISS for all $\varepsilon \in (0, \varepsilon^*]$, meaning that there exists a $M > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$, the solution of (44) initialized by $x(0) \in \mathbf{R}^n$ satisfies for all $k \in Z_+$ the following inequality:

$$|x(k)|^2 \le M(1 - \theta\varepsilon)^k |x(0)|^2 + \left[M(1 - \theta\varepsilon)^k + \frac{b}{\alpha\theta}\right]|w|^2_{[0,k]} \tag{49}$$

with α satisfying (17). Moreover, given $\Delta > 0$ for $\sup_{k\ge 0}|w(k)| \le \Delta$, then for all $\varepsilon \in (0, \varepsilon^*]$ and all $x(0) \in \mathbf{R}^n$ the ellipsoid

$$\Theta = \left\{x \in \mathbf{R}^n : |x|^2 \le \frac{b\Delta^2}{\alpha\theta}\right\}$$

is exponentially attractive with a decay rate $\sqrt{1 - \theta\varepsilon}$.

Proof: Here we just verify the case under condition **(i)**, the case under condition **(ii)** can be achieved by the same arguments. Choose the L-K functional $V(k)$ as shown in (15) with $\rho = 1 - \theta\varepsilon \in (0, 1)$. Then following arguments of Theorem 1, differencing $V(k)$ along (46) we arrive at, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\begin{aligned} &V(k+1) - (1 - \theta\varepsilon)V(k) - \varepsilon b|w(k)|^2 \\ \le& \bar{\eta}^{\rm T}(k)\bar{\Phi}\bar{\eta}(k) + \frac{(T-1)^2\varepsilon^*}{4}\left[\varepsilon|A(k)x(k)|_Q^2 + \frac{1}{\varepsilon}|\bar{x}(k)|_{\bar{R}}^2\right], \end{aligned} \tag{50}$$

where $\bar{\eta}^{\rm T}(k) = [\eta^{\rm T}(k), \sqrt{\varepsilon}w^{\rm T}(k)]$ with $\eta(k)$ given by (33), and $\bar{\Phi}$ is given by (48). Note from $\bar{x}(j)$ in (6) and (44) that

$$\bar{x}(k) = \varepsilon\left[A(k) + \triangle A(k)\right]x(k) + \varepsilon Bw(k), \quad k \in Z_+. \tag{51}$$

Substituting (51) into (50) with $A(k)$ satisfying (5) and applying further Schur complements, we conclude that if LMIs (47) hold, then for all $\varepsilon \in (0, \varepsilon^*]$ we have

$$V(k+1) - (1 - \theta\varepsilon)V(k) - \varepsilon b|w(k)|^2 \le 0,\ k \ge T - 1.$$

By using Lemma 6.2 in [6] and noting (17), we further have, for all $\varepsilon \in (0, \varepsilon^*]$ and $k \ge T - 1$,

$$\alpha|x(k)|^2 \le V(k) \le (1 - \theta\varepsilon)^{k-T+1}V(T-1) + \frac{b}{\theta}|w|^2_{[0,k]}. \tag{52}$$

For $k \in \mathbf{I}[0, T - 1]$, $x(k)$ satisfy (44) with (38). Then

$$\begin{aligned} &|x(k)| \le (1 + a\varepsilon)|x(k-1)| + \varepsilon\|B\||w(k-1)|, \\ &|\bar{x}(k)| \le a\varepsilon|x(k)| + \varepsilon\|B\||w(k)|, \end{aligned}$$

which yields, for all $\varepsilon \in (0, \varepsilon^*]$ and $k \in \mathbf{I}[0, T - 1]$,

$$\begin{aligned} &|x(k)| \le (1 + a\varepsilon^*)^k|x(0)| + \tfrac{\|B\|}{a}\left[(1 + a\varepsilon^*)^k - 1\right]|w|_{[0,k]}, \\ &|\bar{x}(k)| \le a\varepsilon^*(1 + a\varepsilon^*)^k|x(0)| + \varepsilon^*\|B\|(1 + a\varepsilon^*)^k|w|_{[0,k]}. \end{aligned} \tag{53}$$

Note that for all $\varepsilon \in (0,\varepsilon^*]$, $V(T-1)$ satisfies (40). Then with (53), $V(T-1)$ can be further upper bounded by

$$V(T-1) \le c_2 (1-\theta\varepsilon)^{T-1} \left[|x(0)|^2 + |w|^2_{[0,k]} \right] \forall \varepsilon \in (0,\varepsilon^*]$$

with some ε-independent $c_2 > 0$. The latter inequality together with (52) implies (49). ■

IV. Stability Analysis of Switched Affine System

In this section, we discuss the stability analysis of discrete-time switched affine systems by periodic switching. As a special subclass of switched systems, the switched affine systems are very common in practical applications, mainly in the area of power electronics [10]. Due to the affine term, in general the equilibrium point is different from those of any isolated subsystem, and therefore, a high switching frequency is needed to assure that a desired equilibrium point is reached. Compared with the continuous-time systems whose state trajectories may converge to a single point using a continuous-time switching function, the stability analysis of discrete-time switched affine systems is much more involved, since the desired equilibrium point cannot be reached, but only a neighborhood of the equilibrium is available. Our goal is to apply the averaging method established in Section II to assure the practical stability (i.e., ISS with a small ultimate bound) of discrete-time switched affine systems.

Let us consider the following discrete-time switched affine system with uncertainties:

$$x(k+1) = x(k) + \varepsilon \tilde{A}_{\sigma(k)} x(k) + \varepsilon \tilde{B}_{\sigma(k)}, \quad k \in Z_+, \tag{54}$$

where $\tilde{A}_{\sigma(k)} = A_{\sigma(k)} + \Delta A_{\sigma(k)}(k)$, $\tilde{B}_{\sigma(k)} = B_{\sigma(k)} + \Delta B_{\sigma(k)}(k)$, $x(k) \in \mathbf{R}^n$, $\varepsilon > 0$ is a small parameter, $\sigma : Z_+ \to \Upsilon \in \mathbf{I}[1,N]$ is a switching control, $A_i \in \mathbf{R}^{n\times n}$, $B_i \in \mathbf{R}^n$ ($\mathbf{I}[1,N]$) are the nominal matrices, and $\Delta A_i(k) \in \mathbf{R}^{n\times n}$, $\Delta B_i(k) \in \mathbf{R}^n$ ($i \in \mathbf{I}[1,N]$) are the perturbations with respect to the nominal values satisfying

$$\|\Delta A_i(k)\| \le \kappa_1, \ |\Delta B_i(k)| \le \kappa_2, \ i \in \mathbf{I}[1,N], \quad k \in Z_+. \tag{55}$$

Here $\kappa_1, \kappa_2 > 0$ are some small enough constants.

As done in [1], [11], we let $A(\lambda) = \sum_{i=1}^N \lambda_i A_i$ and $B(\lambda) = \sum_{i=1}^N \lambda_i B_i$, $\lambda \in \Lambda$ with

$$\Lambda = \{\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N]^{\mathrm{T}} \in \mathbf{R}^N, \ \lambda_i \ge 0, \ \textstyle\sum_{i=1}^N \lambda_i = 1\}.$$

Moreover, we denote $\Lambda_{\mathrm{h}} \subseteq \Lambda$ composed by all $\lambda \in \Lambda$ such that $A(\lambda)$ is Hurwitz.

Similar to [1], [11], in the absence of uncertainties the set of equilibrium points for (54) is given by $\mathcal{S}_e = \{x_e : x_e = -A^{-1}(\lambda)B(\lambda), \ \lambda \in \Lambda_{\mathrm{h}}\}$. Given an equilibrium point $x_e \ne 0 \in \mathcal{S}_e$ and denote the error $e(k) = x(k) - x_e$. It follows that

$$e(k+1) = e(k) + \varepsilon A_{\sigma(k)} e(k) + \varepsilon \bar{B}_{\sigma(k)}, \quad k \in Z_+,$$

with $\bar{B}_{\sigma(k)} = B_{\sigma(k)} + A_{\sigma(k)} x_e$. Then

$$\textstyle\sum_{i=1}^N \lambda_i \bar{B}_{\sigma(k)} = B(\lambda) + A(\lambda) x_e = 0.$$

Therefore, without loss of generality, we can assume that:

A3 There exists $\lambda \in \Lambda_{\mathrm{h}}$ such that $A(\lambda)$ is Hurwitz and $B(\lambda) = 0$.

Differently from the state-dependent switching law e.g. in [8], [11], we here introduce the time-dependent switching law $\sigma(k)$ which does not need to perform measurements and calculations. Under **A3**, we can choose a positive integer T and design the time-dependent periodic switching law $\sigma(k)$ such that, for $k \ge T-1$,

$$\frac{1}{T}\sum_{i=k-T+1}^{k} A_{\sigma(i)} = A(\lambda), \ \frac{1}{T}\sum_{i=k-T+1}^{k} B_{\sigma(i)} = B(\lambda) = 0. \tag{56}$$

Let $\bar{x}(j)$ and $G(k)$ be defined in (6) with $x(j)$ satisfying (54) and $A(j)x(j)$ replaced by $A_{\sigma(j)}x(j) + B_{\sigma(j)}$. Then summing in k and dividing by T on both sides of system (54) and applying (56), we can present

$$\begin{aligned} z(k+1) =& z(k) + \varepsilon \left[A(\lambda) + \Delta A_{\sigma(k)}(k)\right] x(k) \\ &- \varepsilon Y(k) + \varepsilon \Delta B_{\sigma(k)}(k), \ k \ge T-1, \end{aligned} \tag{57}$$

where $\{z(k), Y(k)\}$ are defined in (12) with $A(i)$ replaced by $A_{\sigma(i)}$. Note that if the time-delay system (57) is practically stable, then the original system (54) under (56) is also practically stable. Next we present the practical stability conditions in the following theorem.

Theorem 3: Let **A3** be satisfied. Given matrices $A(\lambda)$, A_i, B_i ($i \in \mathbf{I}[1,N]$) and constants $\kappa_i > 0$ ($i \in \mathbf{I}[1,2]$), $\theta > 0$ and $\varepsilon^* > 0$ subject to $\theta\varepsilon^* < 1$, let one of the following conditions holds:

(i) there exist $n \times n$ matrices $P > 0$, $Q > 0$, $R > 0$, $\bar{R} > 0$ and positive scalars ζ, b, b_0 that satisfy (18) and the following LMIs

$$\Phi_0 = \begin{bmatrix} P - I & -P \\ * & P + (1-\theta\varepsilon^*)^{T-2} Q \end{bmatrix} > 0, \tag{58}$$

$$\Phi_2^i = \left[\begin{array}{c|cc} & \frac{T-1}{\sqrt{2}} A_i^{\mathrm{T}} Q & \frac{T-1}{\sqrt{2}} A_i^{\mathrm{T}} \bar{R} \\ \tilde{\Phi} & 0_{2n\times n} & 0_{2n\times n} \\ & 0_{n\times n} & \frac{T-1}{\sqrt{2}} \bar{R} \\ & 0_{n\times n} & \frac{T-1}{\sqrt{2}} \bar{R} \\ \hline * & -\frac{1}{\varepsilon^*} Q & 0_{n\times n} \\ * & * & -\frac{1}{\varepsilon^*} \bar{R} \end{array}\right] < 0, \ i \in \mathbf{I}[1,N], \tag{59}$$

$$\Phi_3^i = \begin{bmatrix} b_0 & \frac{T-1}{\sqrt{2}} B_i^{\mathrm{T}} (Q + \bar{R}) \\ * & Q + \bar{R} \end{bmatrix} > 0, \ i \in \mathbf{I}[1,N], \tag{60}$$

where

$$\tilde{\Phi} = \begin{bmatrix} \Phi & \tilde{\Phi}_{12} \\ * & \varepsilon^* P - b I_n \end{bmatrix}, \tag{61}$$

$$\tilde{\Phi}_{12} = \begin{bmatrix} I_n + \varepsilon^* A(\lambda) & -I_n & -\varepsilon^* I_n & \varepsilon^* I_n \end{bmatrix}^{\mathrm{T}} P,$$

in which Φ satisfies (20) with A_{av} and κ replaced by $A(\lambda)$ and κ_1;

(ii) there exist $n \times n$ matrices $P > 0$, $Q > 0$ and positive scalars r, ζ, b, b_0 that satisfy (58)-(60) with $R = rI_n$, $\bar{R} = r\Pi_2$ and

$$\Pi_2 = \frac{2}{T}\sum_{i=0}^{T-1} A_{\sigma(i)}^{\mathrm{T}} A_{\sigma(i)}. \tag{62}$$

Then there exists a $M > 0$ such that for all $\varepsilon \in (0,\varepsilon^*]$, the solution of (54) under (56) initialized by $x(0) \in \mathbf{R}^n$ satisfies for all $k \in Z_+$ the following inequality:

$$|x(k)|^2 \le M(1-\theta\varepsilon)^k |x(0)|^2 + \Big[M(1-\theta\varepsilon)^k + \frac{b\kappa_2^2 + b_0\varepsilon}{\theta}\Big]. \tag{63}$$

Moreover, for all $\varepsilon \in (0,\varepsilon^*]$ and all $x(0) \in \mathbf{R}^n$ the ellipsoid

$$\Theta = \Big\{x \in \mathbf{R}^n : |x|^2 \le \frac{b\kappa_2^2 + b_0\varepsilon}{\theta}\Big\} \tag{64}$$

is exponentially attractive with a decay rate $\sqrt{1-\theta\varepsilon}$.

Proof: Similar to Theorem 2, here we just verify the case under condition **(i)**, since the case under condition **(ii)** can be achieved accordingly. Choose the L-K functional $V(k)$ as shown in (15) with

$$\phi_1(s) = |A_{\sigma(s)} x(s) + B_{\sigma(s)}|^2_Q, \ \phi_2(i,j,s) = |A_{\sigma(s+i-j)} \bar{x}(s)|^2_R.$$

Let $\rho = 1 - \theta\varepsilon$. Then following arguments of Theorems 1-2 we have

$$\begin{aligned} &V(k+1) - (1-\theta\varepsilon) V(k) - \varepsilon b \left|\Delta B_{\sigma(k)}(k)\right|^2 - \varepsilon^2 b_0 \\ \le& \tilde{\eta}^{\mathrm{T}}(k) \tilde{\Phi} \tilde{\eta}(k) - \varepsilon^2 b_0 + \frac{(T-1)^2 \varepsilon^2}{4} |A_{\sigma(k)} x(k) + B_{\sigma(k)}|^2_Q \\ &+ \frac{(T-1)^2}{4} \bar{x}^{\mathrm{T}}(k) \bar{R} \bar{x}(k) \end{aligned} \tag{65}$$

for all $\varepsilon \in (0,\varepsilon^*]$, where $\tilde{\eta}^{\mathrm{T}}(k) = [\eta^{\mathrm{T}}(k), \sqrt{\varepsilon}\Delta B^{\mathrm{T}}_{\sigma(k)}(k)]$, $\eta(k)$ is given by (33) with $\Delta A(k)$ replaced by $\Delta A_{\sigma(k)}(k)$, and $\tilde{\Phi}$ is given by (61). Note from $\bar{x}(j)$ in (6) and (54) that

$$\bar{x}(k) = \varepsilon \left[A_{\sigma(k)} + \Delta A_{\sigma(k)}(k)\right] x(k) + \varepsilon \left[B_{\sigma(k)} + \Delta B_{\sigma(k)}(k)\right]. \tag{66}$$

Substituting (66) into (65) and applying Young's inequality give

$$\begin{aligned}&V(k+1)-(1-\theta\varepsilon)V(k)-\varepsilon b\left|\Delta B_{\sigma(k)}(k)\right|^2-\varepsilon^2 b_0\\ \leq&\tilde{\eta}^{\mathrm{T}}(k)\tilde{\Phi}\tilde{\eta}(k)+\frac{(T-1)^2\varepsilon^*\varepsilon}{2}\left|A_{\sigma(k)}x(k)\right|_Q^2\\ &+\frac{(T-1)^2\varepsilon^*\varepsilon}{2}\left|\left[A_{\sigma(k)}+\Delta A_{\sigma(k)}(k)\right]x(k)+\Delta B_{\sigma(k)}(k)\right|_{\bar{R}}^2\\ &+\varepsilon^2\Big[\frac{(T-1)^2}{2}\left|B_{\sigma(k)}\right|_{Q+\bar{R}}^2-b_0\Big].\end{aligned} \tag{67}$$

Note that the term $\varepsilon^2 b_0$ compensates $B_{\sigma(k)}$. On the one hand, if LMIs (59) hold, then for all $\varepsilon\in(0,\varepsilon^*]$ and $k\geq T-1$ we have

$$\begin{aligned}&\tilde{\eta}^{\mathrm{T}}(k)\tilde{\Phi}\tilde{\eta}(k)+\frac{(T-1)^2\varepsilon^*\varepsilon}{2}\left|A_{\sigma(k)}x(k)\right|_Q^2\\ &+\frac{(T-1)^2\varepsilon^*\varepsilon}{2}\left|\left[A_{\sigma(k)}+\Delta A_{\sigma(k)}(k)\right]x(k)+\Delta B_{\sigma(k)}\right|_{\bar{R}}^2\leq 0.\end{aligned}$$

On the other hand, if LMIs (60) hold, then for all $\varepsilon\in(0,\varepsilon^*]$ and $k\geq T-1$ we have

$$\varepsilon^2\Big[\frac{(T-1)^2}{2}\left|B_{\sigma(k)}\right|_{Q+\bar{R}}^2-b_0\Big]\leq 0,\ \ k\geq T-1.$$

In conclusion, under (59)-(60), we can obtain from (67) that, for all $\varepsilon\in(0,\varepsilon^*]$,

$$V(k+1)-(1-\theta\varepsilon)V(k)-\varepsilon b\left|\Delta B_{\sigma(k)}(k)\right|^2-\varepsilon^2 b_0\leq 0,$$

which yields, for all $\varepsilon\in(0,\varepsilon^*]$ and $k\geq T-1$,

$$V(k)\leq(1-\theta\varepsilon)^{k-T+1}V(T-1)+\frac{b\kappa_2^2+b_0\varepsilon}{\theta}. \tag{68}$$

Here we have used (55) and Lemma 6.2 in [6]. Note that the inequality $\Phi_0>0$ in (58) implies that $V(k)\geq|x(k)|^2$, which with (68) gives, for all $\varepsilon\in(0,\varepsilon^*]$ and $k\geq T-1$,

$$|x(k)|^2\leq V(k)\leq(1-\theta\varepsilon)^{k-T+1}V(T-1)+\frac{b\kappa_2^2+b_0\varepsilon}{\theta}. \tag{69}$$

Now we give a brief explanation for $\Phi_0>0$ in (58). Consider (58) with $-I$ changed by $-\alpha I$ and (18) and (59)-(60). If these LMIs hold with some $\alpha>0$ (see (17)), then by scaling, they hold with $\{-\alpha I,b_0,b\}$ replaced by $\{-I,b_0/\alpha,b/\alpha\}$, which leads to the same bound on $|x(k)|$.

For $k\in\mathbf{I}[0,T-1]$, $x(k)$ satisfy (54). Under (55) we have $\max_{i\in\mathbf{I}[1,N]}\{\|\tilde{A}_i\|\}\leq d_1$ and $\max_{i\in\mathbf{I}[1,N]}\{|\tilde{B}_i|\}\leq d_2$ for some constants $d_1,d_2>0$. Then

$$|x(k)|\leq(1+d_1\varepsilon)|x(k-1)|+d_2\varepsilon,\ \ |\bar{x}(k)|\leq d_1\varepsilon|x(k)|+d_2\varepsilon,$$

which yields, for all $\varepsilon\in(0,\varepsilon^*]$ and $k\in\mathbf{I}[0,T-1]$,

$$\begin{aligned}&|x(k)|\leq(1+d_1\varepsilon^*)^k\left(|x(0)|+\tfrac{d_2}{d_1}\right)-\tfrac{d_2}{d_1},\\ &|\bar{x}(k)|\leq(1+d_1\varepsilon^*)^k\left(d_1\varepsilon^*|x(0)|+d_2\varepsilon^*\right).\end{aligned} \tag{70}$$

Note that for all $\varepsilon\in(0,\varepsilon^*]$, $V(T-1)$ satisfies (40). Then with (70), $V(T-1)$ can be further upper bounded by

$$V(T-1)\leq c_2(1-\theta\varepsilon)^{T-1}\left(|x(0)|^2+1\right)\quad\forall\varepsilon\in(0,\varepsilon^*]$$

with some ε-independent $c_2>0$. The latter inequality together with (69) implies (63). ■

When the uncertainties are absent in (54), namely,

$$x(k+1)=x(k)+\varepsilon A_{\sigma(k)}x(k)+\varepsilon B_{\sigma(k)},\quad k\in Z_+, \tag{71}$$

then under (56), system (57) becomes

$$z(k+1)=z(k)+\varepsilon A(\lambda)x(k)-\varepsilon Y(k),\quad k\geq T-1,$$

where $\{z(k),Y(k)\}$ are defined in (12) with $A(i)$ replaced by $A_{\sigma(i)}$. By using Theorem 3, we have the following corollary.

Corollary 1: Let **A3** be satisfied. Given matrices $A(\lambda)$, A_i, B_i ($i\in\mathbf{I}[1,N]$) and constants $\theta>0$ and $\varepsilon^*>0$ subject to $\theta\varepsilon^*<1$, let one of the following conditions holds:

(i) there exist $n\times n$ matrices $P>0$, $Q>0$, $R>0$, $\bar{R}>0$ and scalar $b_0>0$ that satisfy (18), (58), (60) and the following LMIs:

$$\Phi_2^i=\left[\begin{array}{c|c}\tilde{\Phi} & \begin{array}{c}\frac{T-1}{\sqrt{2}}A_i^{\mathrm{T}}\left(Q+\bar{R}\right)\\ 0_{2n\times n}\end{array}\\ \hline * & -\frac{1}{\varepsilon^*}\left(Q+\bar{R}\right)\end{array}\right]<0, \tag{72}$$

in which

$$\tilde{\Phi}=\begin{bmatrix}\Phi_{11} & -A^{\mathrm{T}}(\lambda)P-\theta P & -P-\varepsilon^*A^{\mathrm{T}}(\lambda)P\\ * & \Phi_{22} & P\\ * & * & \Phi_{33}\end{bmatrix},$$

where $\Phi_{ii}(i\in\mathbf{I}[1,3])$ satisfy (20) with $\kappa=0$ and A_{av} replaced by $A(\lambda)$;

(ii) there exist $n\times n$ matrices $P>0$, $Q>0$ and scalars $r>0$ and $b_0>0$ that satisfy (58), (60) and (72) with $R=rI_n$, $\bar{R}=r\Pi_2$ and Π_2 satisfying (62).

Then there exists a $M>0$ such that for all $\varepsilon\in(0,\varepsilon^*]$, the solution of (71) under (56) initialized by $x(0)\in\mathbf{R}^n$ satisfies for all $k\in Z_+$ the following inequality:

$$|x(k)|^2\leq M(1-\theta\varepsilon)^k|x(0)|^2+\Big[M(1-\theta\varepsilon)^k+\frac{b_0\varepsilon}{\theta}\Big].$$

Moreover, for all $\varepsilon\in(0,\varepsilon^*]$ and all $x(0)\in\mathbf{R}^n$ the ellipsoid

$$\Theta=\Big\{x\in\mathbf{R}^n:|x|^2\leq\frac{b_0\varepsilon}{\theta}\Big\} \tag{73}$$

is exponentially attractive with a decay rate $\sqrt{1-\theta\varepsilon}$.

Remark 3: Although the stability analysis of switched affine system can be regarded as a special ISS analysis in form, due to the special structure and properties of switched affine system, we can obtain the ultimate bound of $|x|$ as a function of ε (see (64) and (73)), which implies that $x(k)$ is exponentially converging to the equilibrium point when $\varepsilon\to 0$ without uncertainties. This is different from that in Theorem 2.

Finally, we give an example from power electronics to show the efficiency of results in Theorem 3 and Corollary 1.

Example 2: Consider the DC-DC converter from [10], [11], the continuous-time model with a fast switching control has the form

$$\dot{\bar{x}}(t)=\bar{A}_{\sigma(t)}\bar{x}(t)+\bar{B}_{\sigma(t)} \tag{74}$$

with $\sigma(t)$ satisfying

$$\sigma(t)=\begin{cases}1, & t\in[n\varepsilon,(n+\beta)\varepsilon),\\ 2, & t\in[(n+\beta)\varepsilon,(n+1)\varepsilon),\end{cases}$$

and

$$\bar{A}_1=\begin{bmatrix}0 & \frac{1}{L}\\ -\frac{1}{C} & -\frac{1}{RC}\end{bmatrix},\ \bar{A}_2=\begin{bmatrix}0 & 0\\ 0 & -\frac{1}{RC}\end{bmatrix},$$

$\bar{B}_1=[0,0]^{\mathrm{T}}$, $\bar{B}_2=[E/L,0]^{\mathrm{T}}$ with $E=6$ V, $R=50\ \Omega$, $L=20$ mH and $C=220\ \mu$F. Choosing the sampling period $\bar{h}>0$, then system (74) can be discretized as

$$\bar{x}((k+1)\bar{h})=\bar{x}(k\bar{h})+\bar{h}\bar{A}_{\sigma(k\bar{h})}\bar{x}(k\bar{h})+\bar{h}\bar{B}_{\sigma(k\bar{h})} \tag{75}$$

with $\sigma(k\bar{h})$ satisfying

$$\sigma(k\bar{h})=\begin{cases}1, & k\bar{h}\in[n\varepsilon,(n+\beta)\varepsilon),\\ 2, & k\bar{h}\in[(n+\beta)\varepsilon,(n+1)\varepsilon).\end{cases}$$

Now we choose the time scale change $h=\bar{h}/\varepsilon$ (the trajectories are invariant with respect to time scaling) and let $x(k)=\bar{x}(k\bar{h})$. Then system (75) can be expressed as

$$x(k+1)=x(k)+\varepsilon h\bar{A}_{\sigma(k)}x(k)+\varepsilon h\bar{B}_{\sigma(k)},$$

with

$$\sigma(k)=\begin{cases}1, & k\in[\frac{n}{h},\frac{n+\beta}{h}),\\ 2, & k\in[\frac{n+\beta}{h},\frac{n+1}{h}).\end{cases}$$

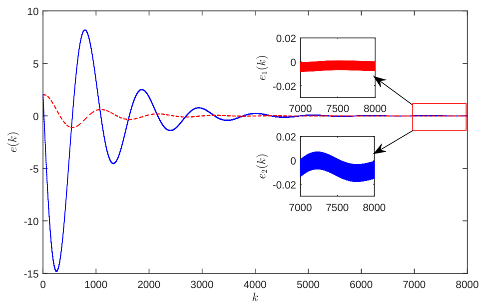

Fig. 1. Evolution of error system states with an initial condition $e(0) = [2; 2]$ and a fixed $\varepsilon = 0.5 \cdot 10^{-4}$ and $\Delta R = \sin(k)$.

Let $e(k) = x(k) - x_e$ with $x_e = -A^{-1}(\lambda)B(\lambda) = [0.24, 6]^{\mathrm{T}}$ for $\lambda = [\lambda_1; \lambda_2] = [0.5; 0.5]$. The dynamic of the error $e(k)$ can be presented as (71) with $A_i = h\bar{A}_i$, $B_i = h\bar{B}_i + h\bar{A}_i x_e$ $(i \in \mathbf{I}[1, 2])$. For simulation, we choose $\beta = \lambda_1 = 0.5$, $h = 0.5$ and $T = 2$, which satisfies (56). By verifying the feasibility of LMIs in **(i)** of Corollary 1, we find the corresponding upper bounds ε^* that guarantee the system's practical stabilization for all $\varepsilon \in (0, \varepsilon^*]$ with $\theta = 0$ or 25 :

$$\theta = 0, \quad \varepsilon^* = 0.334 \cdot 10^{-3}; \quad \theta = 25, \quad \varepsilon^* = 0.149 \cdot 10^{-3}.$$

If we choose $\theta = 25$ and $\varepsilon = 0.5 \cdot 10^{-4}$ (corresponding to $T_{\max} = 0.25 \cdot 10^{-4}$ in [10], [11]), we find the conditions in **(i)** of Corollary 1 guarantee that for $k \to \infty$, $|e(k)| < 0.862$, which is smaller than the values $|e(t)| < 1.9$ for $t \to \infty$ in [11] and $|e(t_k)| < 1.25$ for $k \to \infty$ in [10] by using sampled-data controllers.

To illustrate Theorem 3 for uncertain systems, we assume that the resistor is subject to unknown time-varying uncertainties $\Delta R \in [-1\ \Omega, +1\ \Omega]$. Similar to the above treatment, the error dynamic model can be expressed as (54) with $A_i = h\bar{A}_i$, $B_i = h\bar{B}_i + h\bar{A}_i x_e$, $\Delta B_i = \Delta A_i x_e$ $(i \in \mathbf{I}[1, 2])$ and

$$\Delta A_i = h \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{RC} - \frac{1}{(R+\Delta R)C} \end{bmatrix}.$$

For simulation, we choose $\beta = \lambda_1 = 0.5$, $h = 0.5$ and $T = 2$, which satisfies (56). Moreover, we have $\kappa_1 = 0.9276$ and $\kappa_2 = 5.5659$. By verifying the feasibility of LMIs in **(i)** of Theorem 3, we find the corresponding upper bounds ε^* that guarantee the system's practical stabilization for all $\varepsilon \in (0, \varepsilon^*]$ with $\theta = 0$ or 15 :

$$\theta = 0, \quad \varepsilon^* = 0.27 \cdot 10^{-3}; \quad \theta = 15, \quad \varepsilon^* = 0.159 \cdot 10^{-3}.$$

If we choose $\theta = 15$ and $\varepsilon = 0.5 \cdot 10^{-4}$, we find the conditions in **(i)** of Theorem 3 guarantee that for $k \to \infty$, $|e(k)| < 3.86$. The error system evolution with an initial condition $e(0) = [2; 2]$ and a fixed $\varepsilon = 0.5 \cdot 10^{-4}$ and $\Delta R = \sin(k)$ is shown in Fig.1, from which we can see that the system state does not converge to the equilibrium point but only to a bounded region due to discreteness and to parametric uncertainties. Numerical simulations show that the system's practical stability can be achieved for bigger $\varepsilon = 1.6 \cdot 10^{-3}$, which may illustrate some conservatism of the proposed method.

V. CONCLUSION

This article developed the time-delay approach to averaging for the stability of discrete-time systems. Explicit conditions in terms of LMIs were established to guarantee the stability of the original system by constructing an appropriate Lyapunov functional. The upper bound on the small parameter that ensures the exponential stability can be obtain from the LMIs. Moreover, the established method was extended to ISS analysis of the perturbed systems and practical stability of discrete-time switched affine systems. We finally mention that the proposed method can be further improved and extended in the future to discrete-time time-delay systems [7] and extremum seeking control for discrete-time systems [5].

REFERENCES

[1] C. Albea Sanchez, A. Ventosa-Cutillas, A. Seuret, and F. Gordillo, Robust switching control design for uncertain discrete-time switched affine systems, *International Journal of Robust and Nonlinear Control*, vol. 30, no. 17, pp. 7089-7102, 2020.

[2] H. An, Q. Wu, H. Xia, and C. Wang, Control of a time-varying hypersonic vehicle model subject to inlet un-start condition, *Journal of the Franklin Institute*, vol. 355, no. 10, pp. 4164-4197, 2018.

[3] E.-W. Bai, L.-C. Fu, S. S. Sastry, Averaging analysis for discrete time and sampled data adaptive systems, *IEEE Transactions on Circuits and Systems*, vol. 35, no. 2, pp. 137-148, 1988.

[4] R.H. Christensen and I.F. Santos, Design of active controlled rotor-blade systems based on time-variant modal analysis, *Journal of Sound and Vibration*, vol. 280, pp. 863-882, 2005.

[5] J.Y. Choi, M. Krstic, K.B. Ariyur, and J.S. Lee, Extremum seeking control for discrete-time systems, *IEEE Transactions on Automatic Control*, vol. 47, no. 2, pp. 318 - 323, 2002.

[6] E. Fridman, *Introduction to time-delay systems: Analysis and control*. Birkhauser, 2014.

[7] E. Fridman and J. Zhang, Averaging of linear systems with almost periodic coefficients: A time-delay approach, *Automatica*, vol. 122, p. 109287, 2020.

[8] G.S. Grace and J.C. Geromel, Stability analysis and control design of discrete-time switched affine systems, *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 4058-4065, 2017.

[9] C. J. Harris and J. F. Miles, *Stability of linear systems: some aspects of kinematic similarity*. London: Academic Press, 1980.

[10] P. Hauroigne, P. Riedinger, and C. Iung, Switched affine systems using sampled-data controllers: robust and guaranteed stabilization, *IEEE Transactions on Automatic Control*, vol. 56, no. 12, pp. 2929-2935, 2011.

[11] L. Hetel and E. Fridman, Robust sampled–data control of switched affine systems, *IEEE Transactions on Automatic Control*, vol. 58, no. 11, pp. 2922-2928, 2013.

[12] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River NJ: Prentice Hall, 2002.

[13] M. Krstic and H.-H. Wang, Stability of extremum seeking feedback for general nonlinear dynamic systems, *Automatica*, vol. 36, no. 4, pp. 595-601, 2000.

[14] B. Lehman and R. M. Bass, Extensions of averaging theory for power electronic systems, *IEEE Transactions on Power electronics*, vol. 11, no. 4, pp. 542-553, 1996.

[15] P. Li, J. Lam, R. Lu, and K.W. Kwok, Stability and L_2 synthesis of a class of periodic piecewise time-varying systems, *IEEE Transactions on Automatic Control*, vol. 64, no. 8, pp. 3378-3384, 2019.

[16] K. Liu, E. Fridman, K. Johansson, and Y. Xia, Generalized Jensen inequalities with application to stability analysis of systems with distributed delays over infinite time-horizons, *Automatica*, vol. 69, pp. 222-231, 2016.

[17] R. Wisniewski, Linear time-varying approach to satellite attitude control using only electromagnetic actuation, *Journal of Guidance, Control, and Dynamics*, vol. 23, no. 4, pp. 640-647, 2000.

[18] X. Xie, J. Lam, C. Fan, and K.W. Kwok, A polynomial blossoming approach to stabilization of periodic time-varying systems, *Automatica*, vol. 141, p. 110305, 2022.

[19] X. Yang and E. Fridman, Practical stabilization of affine discrete-time systems by periodic switching via a time-delay approach to averaging, *2022 IEEE Conference on Decision and Control*, Accepted.

[20] J. Zhang and E. Fridman, L_2-gain analysis via time-delay approach to periodic averaging with stochastic extension, *Automatica*, vol. 137, p. 110126, 2022.

[21] Y. Zhang, J. Zhang, and X. Liu, Implicit function based adaptive control of non-canonical form discrete-time nonlinear systems, *Automatica*, vol. 129, p. 109629, 2021.

[22] B. Zhou, Lyapunov differential equations and inequalities for stability and stabilization of linear time-varying systems, *Automatica*, vol. 131, p. 109785, 2021.

[23] Y. Zhu and E. Fridman, Extremum seeking via a time-delay approach to averaging, *Automatica*, vol. 135, p. 109965, 2022.

[24] Y. Zhu, E. Fridman, and T. Oliveira, Sampled-data extremum seeking with constant delay: a time-delay approach, *IEEE Transactions on Automatic Control*, 2022.